Final Exam

Linear Algebra
Summer 2011
Math S2010X (3)
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Solutions

Name:

Instructions:

• This is a closed book exam. You may not use the textbook, notes or a calculator.
• You will have 90 minutes to complete this exam.
• Unless otherwise stated you must show your work in order to receive full credit for your solution. Partial credit will be given for incomplete solutions.

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1. (2 points each) Determine which of the following statements are true. Circle T for true and F for false. You do not need to justify your conclusions.

**T F** If \( \vec{v} \) is an eigenvector of a matrix \( A \), and \( B \) is similar to \( A \), then \( \vec{v} \) is also an eigenvector of \( B \).

False, similar matrices have the same eigenvalues, but may not have the same eigenvectors, for example \[
\begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
1 & 0 \\
0 & -1 \\
\end{bmatrix}
\].

**T F** If the linear system \( A\vec{x} = \vec{b} \) has a unique solutions for every vector \( \vec{b} \) in the range of \( A \), then \( A \) must be a square matrix.

True, \( A \) must be an invertible matrix.

**T F** If a matrix \( A \) is diagonalizable, then its transpose \( A^T \) must be diagonalizable as well.

True, \( A \) diagonalizable means \( A \) is similar to a diagonal matrix \( D \). It follows that
\[
A^T = (S^{-1}DS)^T = S^T D^T (S^{-1})^T = S^T D (S^T)^{-1}
\]
Thus \( A^T \) is similar to \( D \).

**T F** There exists a linear transformation \( T \) from \( P_2 \) to \( P_2 \) such that the kernel of \( T \) is isomorphic to the image of \( T \).

False, \( P_2 \) is three dimensional, so by the rank nullity theorem the rank of \( T \) can not equal the nullity of \( T \).

**T F** The matrix \[
\begin{bmatrix}
5 & 6 \\
-6 & 5 \\
\end{bmatrix}
\]
represents a rotation combined with a scaling.
True, the columns $\begin{bmatrix} 5 \\ -6 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 5 \end{bmatrix}$ are orthogonal and have the same norm. Also, the determinant $\det\left(\begin{bmatrix} 5 & 6 \\ -6 & 5 \end{bmatrix}\right) = 61$ is positive.

T F If $A$ is a $3 \times 3$ matrix, then $\det(3A) = 3 \det(A)$.

False, $\det(3A) = 3^3 \det(A)$.

T F If $V$ and $W$ are linear spaces and $T : V \to W$ is a linear map such that both $\text{im}(T)$ and $\ker(T)$ are finite dimensional, then $V$ must be finite dimensional.

True, one can build a finite basis for $V$ by taking a basis for the kernel and appending elements that map to a basis of the image.

T F The image of a cube in $\mathbb{R}^3$ under an orthogonal map $T : \mathbb{R}^3 \to \mathbb{R}^3$ is another cube.

True, orthogonal maps preserve lengths and angles.
2. (14 points) Consider the matrix
\[ A = \begin{bmatrix} 3 & 0 & 4 \\ 0 & 5 & 0 \\ -1 & 0 & -1 \end{bmatrix} \]

(a) Find the eigenvalues of \( T \) and their algebraic and geometric multiplicities

**Solution:** The characteristic polynomial of \( A \) is
\[
f_A(\lambda) = \det(A - \lambda I_3) = \det \begin{bmatrix} 3 - \lambda & 0 & 4 \\ 0 & 5 - \lambda & 0 \\ -1 & 0 & -1 - \lambda \end{bmatrix}
\]
\[
= (3 - \lambda)(5 - \lambda)(-1 - \lambda) - 4(5 - \lambda)(-1)
\]
\[
= (5 - \lambda)(\lambda^2 - 2\lambda + 1)
\]
\[
= (5 - \lambda)(1 - \lambda)^2
\]

Therefore the eigenvalues of \( A \) are 5 and 1. Their algebraic multiplicities are 1 and 2 respectively. The geometric multiplicity of 5 must be one, because the geometric multiplicity is bounded above by the algebraic multiplicity. To determine the geometric multiplicity of 1, consider the matrix
\[
A - I_3 = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 4 & 0 \\ -1 & 0 & -2 \end{bmatrix}
\]

This matrix has rank 2 and nullity 1. It follows that the geometric multiplicity of 1 is 1.

(b) Find an eigenvector for each eigenvalue.

**Solution:** \[ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \] is an eigenvector with eigenvalue 5 and \[ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \] is an eigenvector with eigenvalue 1.
(c) Is $A$ diagonalizable?

Solution: No, $A$ is not diagonalizable, because the geometric multiplicity of 1 is not the same as its algebraic multiplicity.

(d) Compute the determinant of $A$.

Solution: The determinant of $A$ is 5, the product of its eigenvalues.
3. (10 points) Let $A$ be an skew symmetric $n \times n$ matrix. Prove by induction that $A^k$ is symmetric for $k$ even and antisymmetric for $k$ odd.

Proof. First, I will induct on $k$ to show that $(A^k)^T = (A^T)^k$. The claim is trivial when $k$ is one. Assuming $(A^k)^T = (A^T)^k$ for some $k$, we must prove that $(A^{k+1})^T = (A^T)^{k+1}$.

\[
(A^{k+1})^T = (A^k A)^T = A^T (A^k)^T = A^T (A^T)^k = (A^T)^{k+1}
\]

Therefore $(A^k)^T = (A^T)^k$ for all $k$.

A skew symmetric means that $A^T = -A$. It follows that

\[
(A^T)^k = (-A)^k = (-1)^k A^k = \begin{cases} 
A^k & \text{if } k \text{ is even} \\
-A^k & \text{if } k \text{ is odd}
\end{cases}
\]

Thus $A^k$ is symmetric if $k$ is odd and antisymmetric if $k$ is even. □
4. (10 points) This is a short answer problem; you do not need to show your work.

(a) Give an example a $3 \times 3$ matrix $A$ such that
\[
\begin{bmatrix}
1 \\
3 \\
5
\end{bmatrix}
\]
is in the kernel of $A$ and
\[
\begin{bmatrix}
-1 \\
0 \\
1
\end{bmatrix}
is in the image of $A$.

Solution:
\[
\begin{bmatrix}
-1 & \frac{1}{3} & 0 \\
0 & 0 & 0 \\
1 & 0 & -\frac{1}{5}
\end{bmatrix}
\]

(b) Give an example of a matrix $A$ such that $\text{im}(A) \neq \text{im}(\text{rref}(A))$.

Solution:
\[
\begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}
\]

(c) Give two different examples of $4 \times 4$ matrices with characteristic polynomial $(3 - \lambda)^4$ such that the geometric multiplicity of the eigenvalue 3 is two.

Solution:
\[
\begin{bmatrix}
3 & 1 & 0 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & 3 & 1 \\
0 & 0 & 0 & 3
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
3 & 1 & 0 & 0 \\
0 & 3 & 1 & 0 \\
0 & 0 & 3 & 1 \\
0 & 0 & 0 & 3
\end{bmatrix}
\]

(d) Give an example of an infinite dimensional linear space.

Solution: Sequences of real numbers, continuous function from $\mathbb{R}$ to $\mathbb{R}$, polynomials $\mathbb{R}[x]$. 
5. (6 points) Let $C[\pi, \pi]$ be the space of continuous functions on the interval $[\pi, \pi]$ equipped with its usual inner product, $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)g(t) \, dt$.

Consider the functions $f(t) = 5 \sin(t) + 3 \cos(5t) + 2 \sin(5t)$ and $g(t) = 9 + 2 \sin(t) + \cos(2t) - 5 \sin(5t)$

Compute the norm of $f$ and determine whether or not $f$ and $g$ are orthogonal.

**Solution:** Recall that the functions $\sin(nt)$ and $\cos(mt)$ form an orthonormal set in $C[-\pi, \pi]$. Thus we can compute the necessary inner products by considering the coefficients of the sines and cosines in $f$ and $g.$

$$\|f\| = \sqrt{25 + 9 + 4} = \sqrt{38}$$ and $$\|g\| = \sqrt{81 + 4 + 1 + 25} = \sqrt{111}$$

They are orthogonal, because

$$\langle f, g \rangle = 0 + 10 + 0 + 0 - 10 = 0$$
6. (10 points) Fit a linear function of the form $f(x) = mx + b$ to the data $(0, 2), (1, 2), (0, 1)$. Graph your solution with the data points and explain why it is the best fit.

Solution: The three data points give the linear system:

$$
\begin{align*}
2 & = m(0) + b \\
2 & = m(1) + b \\
1 & = m(0) + b
\end{align*}
$$

Thus we are looking for a vector

$$\vec{v} = \begin{bmatrix} m \\ b \end{bmatrix} \in \mathbb{R}^2 \text{ such that } \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{v} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}
$$

The least squares solution to this linear system is

$$\vec{v} = (A^T A)^{-1} A^T \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \text{ where } A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}
$$

Thus we can compute $\vec{v}$ as follows

$$\begin{align*}
\vec{v} & = \left( \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \\
& = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \\
& = \frac{1}{5} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} \\
& = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}
\end{align*}
$$

We conclude that the best fit linear function is $f(x) = \frac{1}{2}x + \frac{3}{2}$. 


7. (14 points) Consider the function \( F : P_2 \to \mathbb{R}^3 \) that maps a polynomial \( P_2 \) to the vector \( \begin{bmatrix} f(3) \\ f'(3) \\ f''(3) \end{bmatrix} \).

(a) Show that \( F \) is a linear transformation.

Solution: This function is linear, because evaluation a polynomial at a point and taking derivatives are both linear operation. To check this, prove that \( F \) satisfies the conditions for being a linear function:

i. \( F(f + g) = F(f) + F(g) \) for all polynomials \( f \) and \( g \).

ii. \( F(af) = aF(f) \) for all polynomials \( f \) and scalars \( a \in \mathbb{R} \).

(b) Compute the determinant of \( F \).

Solution: First evaluate \( F \) on each of the elements of the basis \( \{1, t, t^2\} \)

\[
F(1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad F(t) = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad F(t^2) = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}
\]

The determinant of \( F \) is then

\[
\det F = \det \begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{bmatrix} = 2
\]

(c) Determine whether or not \( F \) is an isomorphism.

Solution: \( F \) is invertible, because it has a nonzero determinant.