MODERN ALGEBRA II W4042

Homework, week 3, due October 1

1. Let $R = \mathbb{Q}[X]$. In each of the following exercises, f and g are polynomials in R, and you are asked to apply the division algorithm: find $q, r \in R$, with q *monic* and $\deg r < \deg g$, such that

$$f = qg + r.$$

(a) $f = x^5 + x + 1$, $g = x - 2$.
(b) $f = x^4 + 2x^3 + 4x^2 + x + 3$, $g = 2x^2 + x + 3$.
(c) In each case, find the GCD of f and g.

2. Let $R = \mathbb{Z}[X]$. Let p be a prime number. Let I and J be ideals of R: $I = (X, p^2)$, $J = (p, X^2)$.

(a) Show that the rings R/I and R/J have the same number of elements.

(b) Are the rings R/I and R/J isomorphic?

(c)(optional) Let $N_1 \subset R/I$ and $N_2 \subset R/J$ be the nilradicals. Show that N_1 and N_2 are principal ideals. Are $(R/I)/N_1$ and $(R/J)/N_2$ isomorphic?

3. Let p be an odd prime number. Let R be the ring $\mathbb{Z}/(p^2)$.

(a) Let [2] be the image of the integer 2 in the ring R. Show that [2] is a unit and denote its inverse by u.

(b) Find an explicit formula for u.

(c) Let f be the polynomial $X^2 - (1+p)$ in $R[X]$. Show that f has a root in R.

(d) Find an explicit formula for the roots of f in R.

4. Rotman's book, exercises 42, 43, pp. 30-31.