REPRESENTATION THEORY W4044

1. Homework, week 2, due February 4

This week's homework is (mostly) a review of finite group theory.

1. Let V be an n-dimensional vector space over a field k of characteristic 0, and let $Id : V \to V$ be the identity automorphism of V. Let $f : V \to V$ be an automorphism with $f^2 = I$ but $f \neq Id$ and $f \neq -Id$.

(a) Show that f is diagonalizable and determine its eigenvalues.

(b) Show that there exist two subspaces $V^+, V^- \subset V$ such that $V = V^+ \oplus V^-$ and such that, if $x \in V$ is of the form $x = x^+ + x^-$, with $x^+ \in V^+$ and $x^- \in V^-$, then

$$f(x) = x^+ - x^-.$$

Show that the subspaces V^+ and V^- are uniquely determined.

(c) Suppose V is as above, except that now k is of characteristic 2. Show by an example that (b) is not necessarily true. (You can take $n = \dim V = 2$.)

2. James and Liebeck book, chapter 1, exercises 3, 4, 6, 9, pp. 12, 13.

3. (a) List the conjugacy classes of the symmetric group S_3 (the group of permutations of three letters).

(b) List the conjugacy classes of the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ with relations (some of them redundant)

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

4. Let p be a prime number, $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$ the field with p elements. Let $G = GL(2, \mathbb{F})$ be the group of 2×2 invertible matrices with coefficients in $\mathbb{F}$.

(a) Let $V = \mathbb{F}^2$ be the 2-dimensional vector space over $\mathbb{F}$ (column vectors). Show that G acts transitively on the subset $V \setminus \{0\} \subset V$ by matrix multiplication. Determine the stabilizer in G of the vector $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(b) Use (a) to determine the order of G.

(c) Find a p-Sylow subgroup of G.

(d) (Bonus question) What about the group $GL(n, \mathbb{F})$ of invertible $n \times n$ matrices?