ALGEBRAIC NUMBER THEORY W4043

1. Homework, week 3, due October 1

1. Let $\mathcal{O}$ denote the ring of integers in $K = \mathbb{Q}(\sqrt{-14})$.

(a) Show that $3 + \sqrt{-14}$ is an irreducible element in $\mathcal{O}$.

(b) Show that 3 is not equal to $N_{K/\mathbb{Q}}(x)$ for any $x \in \mathcal{O}$.

(c) Show that 3 is an irreducible element in $\mathcal{O}$.

(d) Show that the principal ideal (3) is not a prime ideal and compute its factorization as a product of prime ideals.

2. Hindry's book, Exercise 6.16, p. 120. You can use Corollary 3-5.10 from Hindry's book; it will be proved later in the semester.

3. Let K/k be a cubic extension of fields of characteristic 0, of the form $K = k(\sqrt[3]{d})$ for some $d \in k$ that is not a cube in k. We assume $k \supset \zeta_3$, a primitive 3-rd root of 1.

(a). Show that $Gal(K/k)$ is cyclic of order e.

Let $s : K \to K$ be a generator of $Gal(K/k)$,

$$s(\sqrt[3]{d}) = \zeta_3(\sqrt[3]{d}),$$

and let $Tr : K \to k$ be the k-linear trace map, $Tr(\alpha) = a + s(a) + s^2(a)$.

(b) Find a basis for $\ker Tr$.

Let $V = K$, viewed as a 3-dimensional vector space over k. Define a bilinear form $B : V \times V \to k$ by

$$B(\alpha, \beta) = Tr(\alpha\beta).$$

A non-zero vector $v \in V$ is *isotropic* if $B(v, v) = 0$.

(c) Show that the subset of isotropic vectors is a k-subspace of V, and find a basis for this subspace.