1. Let \( f(z) = x^2 + iy^2 \). Evaluate \( \int_C f(z) \, dz \), where \( C \) is:
(a) the straight line joining 1 to \( 2+i \);
(b) the curve \((1+t) + t^2i, 0 \leq t \leq 1\).
Are the results the same? Why might you expect this?

**Solution.** (a) The straight line joining 1 and \( 2+i \) may be parametrized as \( z(t) = 1(1-t) + (2+i)t = (1+t) + ti \) where \( 0 \leq t \leq 1 \). We have
\[
\begin{align*}
z'(t) & = 1 + i \\
f(z(t)) & = (1+t)^2 + i(t^2) = (1 + 2t + t^2) + i^2 = 1 + 2t + t^2 + t^2 i \\
f(z(t))z'(t) & = (1 + 2t) + (1 + 2t + t^2 i)
\end{align*}
\]
Hence
\[
\int_C f(z) \, dz = \int_0^1 1 + 2t \, dt + i \int_0^1 (1 + 2t + t^2 i) \, dt \\
= \left[ t + t^2 \right]_0^1 + i \left[ t + t^2 + \frac{2}{3} t^3 \right]_0^1 \\
= 2 + \frac{8}{3} i.
\]
(b) We have
\[
\begin{align*}
z'(t) & = 1 + 2ti \\
f(z(t)) & = (1+t)^2 + t^4 i = (1 + 2t + t^2) + t^4 i \\
f(z(t))z'(t) & = (1 + 2t + t^2 - 2t^5) + (2t + 4t^2 + 2t^3 + t^4)
\end{align*}
\]
Hence
\[
\int_C f(z) \, dz = \int_0^1 1 + 2t + t^2 - 2t^5 \, dt + i \int_0^1 2t + 4t^2 + 2t^3 + t^4 \, dt \\
= \left[ t + t^2 + \frac{1}{3} t^3 - \frac{1}{3} t^6 \right]_0^1 + i \left[ t^2 + \frac{4}{3} t^3 + \frac{1}{2} t^4 + \frac{1}{5} t^5 \right]_0^1 \\
= 2 + \frac{91}{30} i.
\]
Since the function \( f(z) = x^2 + iy^2 \) is not analytic, it is not surprising that we get two different answers.

2. Let \( f(z) = -2xy + i(x^2 - y^2) \). Evaluate \( \int_C f(z) \, dz \), where \( C \) is:
(a) the straight line from 0 to \( 1+i \);
(b) the curve \( t^3 + ti, 0 \leq t \leq 1 \).
Are the results the same? Why might you expect this?

**Solution.** (a) Let \( z(t) = t + ti, \ 0 \leq t \leq 1 \) be the straight line joining 0 to 1 + i. We have

\[
\begin{align*}
  z'(t) & = 1 + i \\
  f(z(t)) & = -2t^2 \\
  f(z(t)) & = -2t^2 - 2t^2i
\end{align*}
\]

Hence,

\[
\int_C f(z) \, dz = \int_0^1 -2t^2 \, dt + i \int_0^1 -2t^2 \, dt
\]

\[
= \left[ -\frac{2}{3} t^3 \right]_0^1 + i \left[ \frac{2}{3} t^3 \right]_0^1
\]

\[
= -\frac{2}{3} - \frac{2}{3} i.
\]

(b) We have

\[
\begin{align*}
  z'(t) & = 3t^2 + i \\
  f(z(t)) & = -2t^4 + (t^6 - t^2)i \\
  f(z(t))z'(t) & = (-7t^6 + t^4) + (3t^8 - 5t^4)i
\end{align*}
\]

Hence,

\[
\int_C f(z) \, dz = \int_0^1 -7t^6 + t^4 \, dt + i \int_0^1 3t^8 - 5t^4 \, dt
\]

\[
= \left[ -t^7 + \frac{1}{3} t^3 \right]_0^1 + i \left[ \frac{1}{3} t^9 - t^5 \right]_0^1
\]

\[
= -\frac{2}{3} - \frac{2}{3} i.
\]

In this case, the function \( f(z) \) is analytic (in fact, we have \( f(z) = iz^2 \)), so it is not surprising that the value is the same (Cauchy’s Theorem implies the integral is path independent).

3. Compute

\[
\int_C z e^{z^2} \, dz,
\]

where \( C \) is the curve parametrized by \( z(t) = t - t^3i, \ 0 \leq t \leq 1 \).

**Solution.** The function \( f(z) = z e^{z^2} \) is analytic on all of \( \mathbb{C} \), and has antiderivative \( F(z) = \frac{1}{2} e^{z^2} \), which is also analytic on all of \( \mathbb{C} \). Therefore,

\[
\int_C z e^{z^2} \, dz = F(z(1)) - F(z(0)) = \frac{1}{2} (e^{-2i} - 1).
\]

4. Is the complex function

\[
f(z) = \overline{z} e^z
\]
holomorphic or not? Be sure to explain your reasoning (saying “it has a \( z \) in it” is not sufficient!).

**Solution.** We write \( f(z) \) in terms of its real and imaginary parts:

\[
f(z) = ze^{z} = (x - iy)(e^{x} \cos(y) + ie^{x} \sin(y)) = (xe^{x} \cos(y) + ye^{x} \sin(y)) + i(xe^{x} \sin(y) - ye^{x} \cos(y))
\]

We now have

\[
\frac{\partial u}{\partial x} = (xe^{x} + e^{x}) \cos(y) + ye^{x} \sin(y) \quad \frac{\partial v}{\partial y} = xe^{x} \cos(y) - e^{x}(\cos(y) - y \sin(y))
\]

\[
\frac{\partial u}{\partial y} = -xe^{x} \sin(y) + e^{x}(\sin(y) + y \cos(y)) \quad \frac{\partial v}{\partial x} = -(e^{x} + xe^{x}) \sin(y) + ye^{x} \cos(y)
\]

Since the Cauchy-Riemann equations are not satisfied, the function cannot be holomorphic.

5. Let \( C \) be a circle centered at \( 4+i \) of radius 1. Without any calculation, explain why \( \int_{C} \frac{1}{z} \, dz = 0 \).

**Solution.** The function \( \frac{1}{z} \) only has a pole at \( z = 0 \), and is analytic otherwise. Since the curve \( C \) does not enclose the singularity, Cauchy’s Theorem implies

\[
\int_{C} \frac{1}{z} \, dz = 0.
\]

6. Let \( C \) be the curve defined parametrically as follows:

\[
z(t) = t(1 - t)e^{t} + \cos(2\pi t^{3})i, \quad 0 \leq t \leq 1.
\]

Evaluate the integral \( \int_{C} e^{z^{2}} \, dz \). Be sure to explain your reasoning!

**Solution.** Note that \( z(1) = z(0) \). Since the curve is closed and the function \( e^{z^{2}} \) is analytic on all of \( C \), Cauchy’s theorem again implies

\[
\int_{C} e^{z^{2}} \, dz = 0.
\]

7. Use Cauchy’s integral formula to evaluate

\[
\int_{C} \frac{e^{z}}{z - 1} \, dz,
\]

where \( C \) is the circle of radius 4 centered at 0, oriented counterclockwise.

**Solution.** We apply Cauchy’s Integral Formula with \( f(z) = e^{z}, z_{0} = 1 \) (this is valid since \( z_{0} \) is contained inside \( C \)). Hence

\[
\int_{C} \frac{e^{z}}{z - 1} \, dz = 2\pi i e^{1} = 2\pi ie.
\]

8. Let \( C \) be the unit circle centered at 0 in \( \mathbb{C} \), oriented counterclockwise. Evaluate each of the following integrals, and be sure that you can justify your answer by a calculation or a clear and concise explanation if you use any theorem.

\[
(a) \int_{C} z^{4} \, dz; \quad (b) \int_{C} \frac{e^{-z^{2}}}{z - i/2} \, dz; \quad (c) \int_{C} z^{-5} \, dz;
\]
\( (d) \int_C \frac{z^2 - 1/3}{z + 5} \, dz; \quad (e) \int_C \frac{1}{(12z - 5)^2} \, dz; \quad (f) \int_C \frac{e^{-2z}}{3z + 2} \, dz. \)

**Solution.** We will mostly use Cauchy’s Theorem (CT) or the Cauchy Integral Formula (CIF) to evaluate. Note that sometimes we have to manipulate the integrand in order to get it to a form where we can apply the Cauchy Integral Formula. We indicate where we use these theorems (be sure you know why they are being used!).

(a) \( \int_C z^4 \, dz \, \text{CT} = 0 \)

(b) \( \int_C \frac{e^{-z^2}}{z - i/2} \, dz \, \text{CIF} = 2\pi i e^{-i(2/2)^2} = 2\pi i e^{1/4} \)

(c) The function \( f(z) = z^{-5} \) is analytic on all of \( \mathbb{C} \) without the origin, and has antiderivative \( F(z) = -\frac{1}{4}z^{-4} \), which is analytic on the same set. Therefore \( \int_C z^{-5} \, dz = F(1) - F(1) = 0. \)

(d) \( \int_C \frac{z^2 - 1/3}{z + 5} \, dz \, \text{CT} = 0 \)

(e) The function \( f(z) = \frac{1}{(12z - 5)^2} \) is analytic on all of \( \mathbb{C} \) without the point \( \frac{5}{12} \), and has antiderivative \( F(z) = -\frac{1}{12(12z - 5)}. \) Therefore \( \int_C \frac{1}{(12z - 5)^2} \, dz = F(1) - F(1) = 0. \)

(f) \( \int_C \frac{e^{-2z}}{3z + 2} \, dz = \frac{1}{3} \int_C \frac{e^{-2z}}{z + \frac{2}{3}} \, dz \, \text{CIF} = \frac{2\pi i}{3} e^{4/3} \)