KNOT CONTACT HOMOLOGY (KCH) AND TOPOLOGICAL STRINGS

Goal: Give some background for recent results by Aganagic–Ekholm–Ng–Vafa (arxiv:1304.5778), relating KCH to string theory. Ekholm has some notes on the subject (arxiv:1312.0800).

Plan:
1. Review of knot contact homology
2. Augmentation variety
3. Conifold transition
4. Relation to augmentation variety
5. Mirror symmetry

Disclaimer: Most of what I will say is not rigorously proved, especially the relations to Physics.

1. KCH

$K \subset S^3$ knot gives

$L_K := \{(x, \alpha) \in T^*S^3|\alpha(v) = 0 \forall v \in T_xK\} \subset T^*S^3$

(Lagrangian) conormal bundle to $K$. Given a metric on $S^3$, have also

$\Lambda_K \subset S^*S^3 = \{(q, p) \in T^*S^3||p| = 1\}, pdq$

(Legendrian) unit sphere conormal bundle.

Note: there are analogues of what I will talk about for links, but we will not go into that today.

For generic $K$, $\Lambda_K$ has nondegenerate Reeb cords.

$\mathcal{A}_K = \text{tensor algebra over } R \text{ generated by Reeb cords of } \Lambda_K$.

Ring: $R = \mathbb{C}[H_2(S^*S^3, \Lambda_K)] = \mathbb{C}[\lambda^{s1}, \mu^{s1}, Q^{s1}]$ where $\lambda = e^x$, $\mu = e^y$ and $Q = e^t$.

Note: $H_2(S^*S^3, \Lambda_K) = H_2(S^2) \oplus H_1(\Lambda_K)$, $x$ is a longitude of $\Lambda_K$ and $p$ is a meridian.

Grading: of cords given by Maslov index, zero on $R$.

Differential: counts pseudo-holomorphic maps to

$u : (D^2, \partial D^2 \setminus P) \rightarrow (\mathbb{R} \times S^*S^3, \mathbb{R} \times \Lambda_K)$

where $P = \{+,-1,\ldots,-k\} \subset \partial D^2$ are punctures, at which $u$ is asymptotic to Reeb cords.

PICTURE 1.

$\mathcal{A}_K$ is the $KCH dga$ (non-commutative).
Note: Mohammed points out that if $S^*S^3$ has closed Reeb orbits of low index, then may also need to augment these disks by planes in some filling, for the same reason that the differential in (closed string) contact homology is sometimes obstructed.

2. Augmentation variety

Definition. An augmentation to a ring $S$ is a dga map $\epsilon : A_K \to S$.

The dga condition implies
- $\epsilon \circ \partial = \partial \circ \epsilon = 0$;
- $\epsilon = 0$ on elements of degree $\neq 0$;
- $\epsilon : R \to \mathbb{C}$ is a (unital) ring map.

Augmentations are useful because they can be used to obtain a (linearized) differential on the free module (not algebra) over $R$ generated by Reeb cords (like a bounding cochain). Such chain complexes are much smaller than $A_K$.

Definition. The augmentation variety $V_K$ of $A_K$ is the (highest dimensional part of the) Zariski-closure in $(\mathbb{C}^*)^3$ of

$$\{(\epsilon(\lambda), \epsilon(\mu), \epsilon(Q)) | \epsilon : A_K \to \mathbb{C} \text{ augmentation} \} \subset (\mathbb{C}^*)^3$$

This is a knot invariant (actually, a link invariant). $\mathbb{C}^*$, because $\epsilon : R \to \mathbb{C}$ is unital.

$V_K = V(\text{Aug}_K(\lambda, \mu, Q))$, where $\text{Aug}_K$ is the augmentation polynomial of $K$.

Example. unknot $O$: can arrange to have only two cords, $e$ and $c$, with $|e| = 2$ and $|c| = 1$. Differential is $\partial(e) = c - c = 0$ and $\partial(c) = Q - \lambda - \mu + \lambda \mu$. Thus,

$$V_O = V(Q - \lambda - \mu + \lambda \mu) \quad \text{and} \quad \text{Aug}_K(\lambda, \mu, Q) = Q - \lambda - \mu + \lambda \mu$$

Conjectures, based on computations and physical evidence:
- fixing $Q = 1$, $V_K \subset ((\mathbb{C}^*)^2, dx \wedge dp)$ is Lagrangian;
- can pass between irreducible components of $V_K$ via intersections of codim 1

Q: How to produce augmentations?

An exact filling of $\Lambda_K$ is a pair $(W, L)$, where $W$ is a Liouville manifold (hence exact symplectic) whose boundary at $\infty$ is $(S^*S^3, pdq)$ and $L \subset W$ is an exact Lagrangian that agrees with $\mathbb{R} \times \Lambda_K$ outside a compact subset. An exact filling produces an augmentation $\epsilon_L : \mathbb{C}[H_2(S^*S^3, \Lambda_K)] \to \mathbb{C}[H_2(W, L)]$

PICTURE 2.

$$\epsilon_L(a) = \sum_{u \in \mathcal{M}(a)} \pm e[u]$$
Exactness precludes the bubbling of holomorphic disks in $L$:

**Example.** Every $\Lambda_K$ has at least two augmentations in $T^*S^3$: one is $L_K$. The fiber class $t$ vanishes on $T^*S^3$ and the meridian $p$ vanishes on $L_K$. This contributes a factor $(1 - \lambda)$ to $\text{Aug}_K(\lambda, \mu, Q = 1)$.

To construct the other augmentation, observe that $\Lambda \cap S^3 = K$ is a clean intersection. A surgical procedure produces a new exact filling $M_K$ for $\Lambda_K$, with the topology of $S^3 \setminus K$. Could also take the graph of $df$ in $T^*S^3$ for a function $f : S^3 \to \mathbb{R}$ with pole along $K$. The longitude $x$ vanishes on $M_K$. This contributes a factor $(1 - \mu)$ to $\text{Aug}_K(\lambda, \mu, Q = 1)$.

In the case of the unknot, restricting to $Q = 1$, the augmentation polynomial becomes $1 - \lambda - \mu + \lambda\mu Q = (1 - \lambda)(1 - \mu)$, obtained precisely from these two augmentations.

Certain non-exact Lagrangians filling also give rise to augmentations. Start with the conormal $L_K$ as before ($K$ not planar, to be able to bound low energy holomorphic disks). Take a push of $L_K$ away from the zero section $S^3$ (using 1-form $d\theta$ on tubular neighborhood of $K \subset S^3$). Get a non-exact Lagrangian $\tilde{L}_K$, which fills a Legendrian nearby $\Lambda_K$.

**Picture 4. of a cotangent fiber**

To continue, we need the conifold transition.

3. **Conifold transition**

Smoothing and resolution:

$$T^*S^3 \xrightarrow{\text{smoothing}} \text{nodal space} \xrightarrow{\text{resolution}} Y := \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{C}P^1$$

$$\{xy - zw = \varepsilon\} \subset \mathbb{C}^4 \hspace{1cm} xy - zw = 0 \hspace{1cm} \text{partial blow-up of origin}$$

**Picture 5. (piramidal)**

‘The Lagrangian $S^3 \subset T^*S^3$ gets replaced by a $\mathbb{C}P^1 \subset Y$.’

Both $T^*S^3$ and $Y$ have $c_1 = 0$ (are CY). Since $\tilde{L}_K \cap S^3 = \emptyset$, get non-exact $\tilde{L}_K \subset Y$. This has Maslov class 0. Hence, moduli spaces of holomorphic disks with boundary in $\tilde{L}_K$ have virtual dimension 0.

**Note:** symplectic structure in $Y$ determined by area of $\mathbb{C}P^1$. Not sure if there is any constraint on this area.
Since \((Y, \tilde{L}_K \subset Y)\) is not exact, it may not produce an augmentation:

**PICTURE 6.** Of disk bubble in \(\tilde{L}_K\).

Will correct this by introducing *bounding chains*. Assume there is a setting in which holo disks in \((Y, L_K)\) are transverse. Recall that \(H_1(L_K) = \mathbb{Z}(x)\), \(x\) the longitude class in \(\Lambda_K\). Fix a representative \(\xi\) of \(x\).

For every rigid holo disk \(u : D^2 \to \tilde{L}_K\), choose a 2-chain \(\sigma_u\) in \(\tilde{L}_K\) from \(u(\partial D)\) to \(k\xi\), for some \(k \in \mathbb{Z}\). Denote the collection of these chains by \(\sigma\).

**PICTURE 7.**

Given a Reeb cord \(a\) in \(\Lambda_K\), denote by \(\mathcal{M}(a, \sigma)\) the moduli space of *quantum corrected disks* \(u : (D, \partial D \setminus \{1\}) \to (Y, L_K)\) asymptotic to \(a\) at 1 and with marked points \(z_i \in \partial D\) such that \(u(z_i) \in \sigma_{u_i}\) for some holo disk \(u_i\). Allow the disks \(u_i\) themselves to have further quantum corrections by other disks.

**PICTURE 8.**

**Note:** Should index \(\mathcal{M}(a, \sigma)\) by the homology class \([u] + \sum_i [u_i] + \ldots \in H_2(Y, L_K)\).

In these moduli spaces, the bad breaking above no longer occurs at the boundary:

**PICTURE 9.** Of moduli space continuing, then boundary

Similarly, let \(\mathcal{M}(\tilde{L}_K, \sigma)\) be the moduli space of trees of holo disks with boundary in \(\tilde{L}_K\), connected by chains in \(\sigma\):

**PICTURE 10.**

These configurations come with weights \(w\), which are the products of the weights at the vertices and at the edges. The weight at a vertex is the given by the fact that the moduli spaces of disks are in principle weighted manifolds (if use virtual techniques). The weight of an edge connecting the disks \(v_1\) and \(v_2\) is the intersection number between \(v_1(\partial D)\) and \(\sigma_{v_2}\).

This can be thought of as a linking number between \(v_1(\partial D)\) and \(v_2(\partial D)\), and it measures the number of times a configuration with disk \(v_1\) intersection 2-chain \(\sigma_{v_2}\) should be counted.

Define the *open GW potential* of \(L_K\) as

\[
W_K(x, Q) = \sum_{v \in \mathcal{M}(L_K, \sigma)} w(v) e^{[v]}
\]
We define the map $\epsilon_{L_K} : A_K \to \mathbb{C}[[e^{\pm x}, Q^{\pm 1}]]$, which is the identity on $x$ and $Q$, such that

$$\epsilon_{L_K}(p) = \frac{\partial W_K}{\partial x}$$

and

$$\epsilon_{L_K}(a) = \sum_{u \in M(a; \sigma)} \pm e^{[u]}$$

**Theorem 1.** $\epsilon_{L_K}$ is an augmentation.

**Proof.** The term $\frac{\partial W}{\partial x}$ is associated with intersections that appear when we quantum correct the punctured holo disks in $\mathbb{R} \times S^* S^3$ (which are not corrected in the differential of $A_K$).

**Corollary 1.** The equation

$$p = \frac{\partial W_K}{\partial x}$$

gives a local parametrization of a branch of $V_K$.

**Note:** A similar construction can be made with other Lagrangian fillings of $\Lambda_K$ (e.g. with $M_K$, if $K$ is a fibered knot).

5. **Mirror symmetry**

Aganagic–Vafa have a description of the mirror to $Y$ given by equations

$$A_K(e^x, e^p, Q) = uv$$

where $A$ is the $Q$-deformed $A$ polynomial. They further claim that

$$A_K(e^x, e^p, Q) = 0$$

is given locally by the equation $p = \frac{\partial W_K}{\partial x}$, just like we saw in (1) for $V_K$ (see the papers for more details on the Physics, which involve Chern–Simons theory and topological strings).

So, the mirror to $Y$ has a (Zariski) open set that coincides with a (Zariski) open set of $V_K$. In some cases, $V_K$ is irreducible, and then Aug$_K$ coincides with $A_K$. There is a relation between $A_K$ and (generating relations for the) colored HOMFLY polynomial, which can be used to compute $A_K$. In all known computations, $A_K$ agrees with Aug$_K$. This was actually one of the starting points of their collaboration.

Dusa asked: in the irreducible case, can see the piece $(1 - \lambda)(1 - \mu)$, $Q = 1$ of the augmentation variety also from this filling?

Mohammed asked: Can Ng-Rutherford-Shende-Sivek-Zaslow produce the same augmentations using sheaf theory?