The equivariant pair-of-pants product in fixed point Floer cohomology.

Ref: Seidel, "..." & Talk slides at Clay.

(i) Classical story about $\mathbb{Z}/2$-action.

Fix $k = \mathbb{R}$. Let $V$ be a $k$-v.s. over $k$ w/ an $\mathbb{Z}/2$-action

$i : V \to V$ w/ $i^2 = \text{id}.$

Groupwise: $H^i(G, V) = \text{Ext}_{\mathbb{Z}[G]}(Z, V).$ For projective resolution of $Z$ for $G$ cyclic:

$\cdots \to \mathbb{Z}[G] \overset{i \cdot h}{\to} \mathbb{Z}[G] \overset{i}{\to} \mathbb{Z}[G] \overset{\text{id}}{\to} \mathbb{Z} \to 0$

Apply $\text{Hom}_{\mathbb{Z}[G]}(-, V)$, $b/c.$ $\text{Hom}_{\mathbb{Z}[G]}(Z, V) = V$. We get:

$c^* (\mathbb{Z}/2, V) = V[\mathbb{Q}]$ and $d_{\mathbb{Z}/2} = h(\text{id} + i)$ w/ $\mathbb{Q}$.

(2) $H^i (\mathbb{Z}/2, V) \cong V \oplus \text{coker}(\mathbb{Q}[\mathbb{Q}])$ w/ $\mathbb{Q} \cong V/\mathbb{Q} \to V$.

Note $H^0 (\mathbb{Z}/2, V) \cong V^G$ is cokernel $V^G$. One can view $H^i (\mathbb{Z}/2, V)$ as the $i$th-derived functor of the functor of $G$-invariants $i.e.$ $V \to V^G.$

Tate version

$\hat{c}^* (\mathbb{Z}/2; V) = V[\mathbb{Q}].$ (localized at $h$)

$\hat{A} (\mathbb{Z}/2; V) = H^* (\mathbb{Z}/2; V) \otimes \mathbb{Q}[h]$ (kill $h$-torsion).

$\cong \text{coker}(V[h])$

Ex 1: $\mathbb{Z}/2$ acts trivially $\Rightarrow \hat{A}^0 (\mathbb{Z}/2; V) \cong V[\mathbb{Q}].$

Ex 2: $V$ has a basis on which $\mathbb{Z}/2$ acts freely $\Rightarrow H^0 (\mathbb{Z}/2; V) = 0.$

$\cong$ Tate cohomology vanishes.

Now for any vector space over $k$, form $V \otimes V$ w/ $(c \otimes b) = b \otimes c.$ Then $H^* (\mathbb{Z}/2, V \otimes V) \cong V[\mathbb{Q}].$

This decomposition is induced by the Tate map

$V[h] \to c^* (\mathbb{Z}/2; V \otimes V).$

Note: This is NOT additive in $V$, but...
\[(V + \mathbb{W}) \otimes (V + \mathbb{W}) - \mathbb{V} \otimes \mathbb{W} - \mathbb{W} \otimes \mathbb{V} = V \otimes W + W \otimes V = \mathcal{Z}_2 (V \otimes W)\]

i.e. After removing \[K(X) \otimes [K(X)]\], it becomes coordinate in \[V\]

and induces \[(\ast)\].

For chain complexes \[w\] with \[w\] smooth on \[M\]

let \[M\] be a smooth manifold \[w\] with \[\mathbb{Z}/2\] action.

Denote \[C^*(M)\] the cochains \[w\] with \[\mathbb{C}\]-coefficients.

Then \[H^*_w(M) := H^*(\mathbb{Z}/2, C^*(M))\] i.e. the homology of

\[(C^*(M) \otimes \mathbb{C}, \mathcal{O} + h (id + \partial))\]

spectral sequence (filtration by powers of \(h\))

\[E^1 = H^*(M) \otimes \mathbb{C}\]

and differential on \[E^1\] is \[id + d^w\]

\[\Rightarrow\] converges to \[H^*_w(M)\]

we conclude that:

\[\dim_{\mathbb{C}} H^*(M, \mathbb{Z}/2) \geq \dim_{\mathbb{C}} H^*_w(M)\]

Together with the localization theorem for \[\mathbb{Z}/2\] action

the inclusion \[M^{\mathbb{Z}/2} \to M\] induces the restriction map

\[H^*_w(M) \to H^*_w(M^{\mathbb{Z}/2})\]

which is an isomorphism, after removing \[K(X)\] i.e. on Tate cohomology.

\[(\ast) :\] \[\dim H^*(M, \mathbb{Z}/2) \geq \dim H^*_w(M)\] Smith inequality.

Steinberg Squares consider \[\Delta \subset M \times M\] which is the fixed point set.

of the involution on \[M \times M\], \[H^*_w(\Delta) = H^*_w(M) \otimes \mathbb{C}\].

\[H^1(M) \xrightarrow{\text{Tate}} H^2(\mathbb{Z}/2; C^*(M) \otimes C^*(M))\]

\[
\begin{align*}
& \xrightarrow{\text{Total Steinberg Operation}} \\
& \xrightarrow{\text{Sq.}} \\
& \xrightarrow{\text{Eilenberg-Zilber}} \\
& \xrightarrow{\text{restrict to \(\Delta\)}} \\
\end{align*}
\]

\[\text{Sq}^i(x) = x^2 + h \text{Sq}^{i-1}(x) + H^{i+1}(M) \otimes H^{i-1}(M) \otimes \cdots\]
Setup: Given a Liouville domain \((M, \theta)\), i.e. a compact \(w\)-bumpy \(s.t. \, W = \partial M\) and the \(\omega\)-twist of \(\theta\), denoted as \(\tilde{\theta}\), possesses no outwardly ray \(\partial M\). Then consider an equivariant automorphism \(\Phi\) s.t. \(\Phi^* \theta - \theta = d\phi \) for \(\Phi \in C^0_c(M)\).

Also require \( \phi_1, \phi_2 \) have no fixed points on \(\partial M\).

We can define a \(Z_2\)-graded \(k\)-v.s. \(H^k(\Psi)\), fixed pts \(\Phi\) Floer cohomology.

(variational)

Twisted loop space:

\[ L := \{ x : \mathbb{R} \to M \mid x(t) = \Phi(x(t + 1)) \} \]

Action functional:

\[ A_\phi(L) = \int_0^1 \int_0^1 x(t) \theta + \Phi(x(t)) \]

Critical pts of \( A_\phi \) \( \iff \frac{dx}{dt} = 0 \) \( \iff \) fixed pts of \( \Phi \).

Assume the fixed pts are non-deg. i.e. \( \det(I - D\Phi|x) \neq 0 \).

Pick \( J_t \) parameterized by \( \mathbb{R} \):

\[ J_t = \Phi(x(t)) \]

\( \Theta \) periodicity condition. \( J_t = \Phi(x(t + 1)) \).

\( L^2 \)-metric on \( L^2 \): \( \langle \Phi, \Psi \rangle = \int_0^1 \omega^2(\Phi(J_t x), \Psi(J_t x)) dt \).

\( \nabla \) negative gradient flow lines. Connecting \( x \) \& \( y \).

\[ \nabla \phi(x) = \Phi(U(s, t + 1)) \]

\[ \frac{ds}{dt} + J_t \frac{du}{dt} = 0 \]

\[ u(t + 1) = x \quad \Phi(u(s, 0)) = y \]

\( \Phi \) and \( \Gamma \)-translations \( \sim (\Phi \gamma, x) \).

Remark: Alternatively, form the symplectic mapping torus. \( M_\Phi = C[x,y]/(\Theta, \omega)^{(\text{Stich}(\theta) \cdot \Psi)} \).

Fix pts of \( \Phi \) \( \iff \) flat sections of \( M_\Phi \) over \( s = 0 \).

Gradient flow lines \( u(s, t) \) \( \iff \) holomorphic sections of \( M_\Phi \) converging to \( x \) \& \( y \).
Equivariant Floer cohomology of $\phi^2$.

Now for $\phi^2$, let

$$\mathcal{L}_{\phi^2} = \{ x \in C^0(\mathbb{R}, M) \mid x(t) = \phi^2(x(t+1)) \}$$

be the invariant on $\mathcal{L}_{\phi^2}$ by the action $\phi^2$. Let

$$P: \mathcal{L}_{\phi^2} \rightarrow \mathcal{L}_{\phi^2} \quad (P)(x(t)) = \phi^2(x(t+1))$$

Note that the fixed points are exactly $\mathcal{L}_\phi$, and $A_{\phi}|_{\mathcal{L}_\phi} = 2A_{\phi}$.

Also, on $\mathcal{L}_\phi = \{ (s,t) \mid \phi^2(x(s)) = x(t+1) \}$, $P$ acts as

$$(P \cdot J)^t = \phi^2 \cdot J_{t+1}.$$ This induction induces an action on $HF^*(\phi^2)$ via

$$\left( CF^*(\phi^2), d_{\phi^2} \right) \xrightarrow{\phi^2} \left( CF^*(\phi^2), d_{\phi^2} \right)$$

Continuation $\phi^2 \mapsto \phi^2$ as complexes

$$HF^*(\phi^2) \rightarrow HF^*(\phi^2)$$

Warning: A strict $\mathbb{Z}/2\mathbb{Z}$-action on the chain level (due to periodicity of equivariant transversality), only a homotopy $\mathbb{Z}/2\mathbb{Z}$-action.

To construct $\mathbb{Z}/2\mathbb{Z}$-equivariant Floer coho, Consider the Borel construction $\mathbb{R}P^\infty$.

In view of family Floer coho, one needs to pick $a.e.$ $\Lambda \xrightarrow{2/3} \mathbb{R}P^\infty$ (parametrized by $\mathbb{R}P^\infty$)

i.e. for each $v \in S^m$, we pick $T_v$ s.t. $J_{\phi^2} v \in \mathcal{L}_\phi$ and

a) $J_v \cdot v = P \cdot J_{\phi^2} v$

b) $J_{\phi^2} v = \phi^2 \cdot v$ if $v$ lies in a nbhd of $v(t)$, for any $t$

c) $T_v \cdot J_{\phi^2} v \in J_{\phi^2} v$, where $T: S^m \rightarrow S^m$ $T(v_0, v_1, \ldots) = (0, v_0, V_1, \ldots)$
The standard Morse for $\pi^1 = \frac{s}{k} V U v^1$ in $S^0$ has critical pts $c(\xi, t, m, o)$.

Given a flow line $w: I \rightarrow S^0$ connecting $V_0$ and $V_1^{+}/V_1^{-}$, consider

$$\psi(t) = \frac{1}{2}(\cos t + \cos^2 t)$$

Then $\psi(t) = I \rightarrow S^0$ with $\psi(0) = y$ if $\xi > 0$, $\psi(\infty) = y$ if $\xi < 0$.

$$\dim M_{eq}^y(y, x) = \frac{1}{2} - \frac{1}{2} + 1 + 1 \quad \text{mod} \quad 2.$$  

$$d_{eq} \psi (x) = \sum_{y} M_{eq}^y(y, x) \quad \text{deg} \quad 1 - r \text{ operation}.$$  

$$d_{eq}^+ = d_{eq}^- + \sum_{y} h^i \cdot d_{eq}^i.$$  

Compactness $x$ of $M_{eq}^y(y, x) \rightarrow 0 \Rightarrow d_{eq}^2 = 0 \Rightarrow d_{eq}^+ d_{eq}^- = 0 \Rightarrow d_{eq}^+ d_{eq}^- = 0.$

Glimpse

Note: $d_{eq}^+$ is $(C \mathcal{H}^*(\mathcal{M}))$, $d_{eq}^-$ is chain homotopy to $d_{eq}$.

$$d_{eq}^+ : (C \mathcal{H}^*(\mathcal{M}), d_{eq}^-) \rightarrow (C \mathcal{H}^*(\mathcal{M}), d_{eq}^-),$$

the concatenation map inducing $I$ on $\mathcal{H}_i(\mathcal{M}).$

Note that $\Rightarrow$ 2 filtrations on $(C \mathcal{H}^*(\mathcal{M}), d_{eq})$

(i) $h$-adic filtration.

On $E_1$-page we have $(H^*(\mathcal{M}), h^{\cdot \cdot \cdot i} + \ell')$

$$E_2 \text{ page } H^*(\mathcal{M}, H^*(\mathcal{M})) \Rightarrow H^*(\mathcal{M})$$

hence $\dim H^*(\mathcal{M}) \\lgeq \text{rank } h^{\cdot \cdot \cdot i} h^* H^*(\mathcal{M}) \Rightarrow H_{eq}^*(\mathcal{M})$ as before

(ii) Filtration by action.

Note $A_{eq}^*(\mathcal{M})$ unless $x$ is constant (regular for all $T_x e$).

i.e. $d_{eq} = h^{\cdot \cdot \cdot i} + \ell'$ (terms which increase the action)

$$E_1 = H^*(\mathcal{M}, C \mathcal{H}^*(\mathcal{M}))$$
To get "Smith-like" ineqality, we need the missing ingredient is
\[ HF^k_{\text{eq}}(\phi^3) \cong k(\text{Morse}) = HF^k(\phi)(1\text{h}) \]
Seidel defines the equivariant pair of pants \( \mathcal{P} \): \( HF^k(\mathcal{P}) \cong HF^k(\mathcal{P}) \)

Theorem 14 [Seidel, 14]: \( \phi \) becomes an isomorphism after tensoring with \( L(\mathcal{M}) \)
Together w/ the Tate map \( HF^k(\phi) \to HF^k(\mathcal{P}) \), \( H^F(\phi) \to HF^k(\phi) \)
we obtain \( \dim HF^k(\phi^3) = \dim HF^k(\phi) \)

Remark: Take \( \phi = \) small perturbation of \( \alpha \), s.t. \( \phi, \phi^3 \) has no fixed pts at \( \partial \mathcal{M} \).
we have \( HF^k(\phi) = H^k(\mathcal{M}), \quad HF^k(\mathcal{P}) = H^k(\mathcal{M}) \)
reverses the classical case.

Applications:
1) Quantum Steenrod Squares. (2 approaches)
   Approach 1 (via spectra) Build a Floer stable homotopy type \( (\text{-Cohen-Jones-Segal}) \)
   Steenrod squares
   Requirement: Need to assume \( TM \) is a stable manifold as a sym vs (a)
   \( \mathcal{D}(\phi) : M \to \text{Sp}(\alpha) \) is nullhomotopic. (Construction depends on (a) & (b))

   Approach 2: (Fukaya, Bette-Cohen-Norbury)
   \( HF^k(\phi) \xrightarrow{\text{Tate}} H^k(\mathcal{P}/2) \oplus CF^k(\phi) \)
   \( \downarrow \text{equiv product} \)
   Steenrod sq defined by Floer homotopy type \( HF^k(\phi) \cong k(\mathcal{M}) \)
   Localization Theorem defined by Hendricks (requires condition (a) & (b) for different reasons)

2) Hendricks. Let \( M \) be a sym manifold and \( \phi \) be an sym auto s.t. \( \phi \) is symd
Suppose that \( \dim HF^k(\phi) \geq \dim HF^k(1\text{h}) \) and Conditions (a) & (b) hold
then \( \phi^2 \in \text{Sym}(\text{Sym}(\mathcal{M})) \) are noncentral in.
Use: \( \dim HF^k(\phi^3) > \dim HF^k(\phi) > \dim HF^k(1\text{h}) \)
Construction of the equipair of points product.

Consider $\pi: M^p \to M^q$, where $\pi: S \to \mathbb{R} \times S'$ is the double branched cover of $\mathbb{R} \times S'$ branched at $(0,0)$.

We first define the product $P^0: H^k(M^p) \otimes H^k(M^q) \to H^k(M^p \times M^q)$

$M^p_{x^+}((y,x^+), x^-) = \{ y \}$ holomorphic sections of $\tau_x M^p$

Assume $\exists$

$\dim M^p_{x^+}((y,x^+), x^-) = |y| - |x^+| - |x^-| \to$ defines a degree zero operator $\mathcal{X}(X^+) \otimes \mathcal{X}(X^-) \to \mathcal{X}(X)$

Similarly, for a given flaw line $Z: \mathbb{R} \to S^2$, we count holomorphic sections over $Z: \mathbb{R} \to S^2$

Define $\mathcal{P}^i: H^k(M^p) \otimes H^k(M^q) \to H^k(M^p \times M^q)$, let $P^i = P^i_+ + P^i_-$

Since $\dim M^p_{x^+}((y,x^+), x^-) = \sum \dim P^i_+(x^+, x^-)$

Note that transversality fails. For example, for constant maps.

We use inhomogeneous term to perturb the equation.

Choose $H^i \in \mathcal{F}(\phi_\infty) = \{ H^i : \mathbb{R} \times M \to [0,1] \}$, $H^i \to 0 \Rightarrow H^i = \phi \circ H^i + \phi_\infty \circ H^i + \phi_\infty \circ H^i (\phi_\infty (x))$
(4) Proof of the Main Theorem.

Approach 1: \( A_{\Phi^{*}(y)} - A_{\Phi(x)} - A_{\Phi(x')} = 0 \), but due to the small Ham parallelism \( A_{\Phi^{*}(y)} - A_{\Phi(x)} \in (0, 2\varepsilon) \) for \( y, y' \in F_{0}^{*} \).

One can choose \( \varepsilon > 0 \) such that \( A_{\Phi^{*}(y)} - A_{\Phi(x)} - A_{\Phi(x')} \in (0, 2\varepsilon) \) unless \( y - x^{\pm} x' \). \( A_{\Phi(x)} - A_{\Phi(x')} \in (0, 2\varepsilon) \) for \( x, x' \in F_{0}^{*} \).

Define \( \mathcal{F}^{d} C_{F_{0}^{*}(\Phi)} \circ C_{F_{0}^{*}(\Phi)} (h, h') = \int_{x^{\pm} x'} h^{\pm} | A_{\Phi^{*}(y)} - A_{\Phi(x)} - A_{\Phi(x')} = 2(2d) \).

Claim: On \( E_{0} \) page, \( P \) induces a map
\[
(C_{F_{0}^{*}(\Phi)} \circ C_{F_{0}^{*}(\Phi)} (h, h'), (h, (\varepsilon + \varepsilon))) \rightarrow (C_{F_{0}^{*}(\Phi)} (h, h'), (h, (\varepsilon + \varepsilon)))
\]
\[
\times \times \times \rightarrow (h, -N(\Phi x, x))
\]
where \( n = K(D \Phi, x) = 2M(D \Phi, x) - M(D \Phi, x) \) and \( N(D \Phi, x) \) is the Klien index.

This induces an \( \alpha \in H^{r}(Z/2, C_{F_{0}^{*}(\Phi)} \circ C_{F_{0}^{*}(\Phi)}) \rightarrow H_{1}(Z/2, C_{F_{0}^{*}(\Phi)}) \) on \( E \).

By comparison of Eqs, we conclude that \( P \) induces

\[
H^{i}(Z/2, C_{F_{0}^{*}(\Phi)} \circ C_{F_{0}^{*}(\Phi)}) \rightarrow H^{i}_{E_{0}}(\Phi^{2}) \text{ after tensoring } K(h, h').
\]

Approach 2: (Sketch) Show that composition of product & coproduct are isomorphisms after tensoring \( L \) (\( K(h, h') \)).

This composition (in either order) is the equivariant class, \( S_{A} = \sum_{i} h^{n-i} \cdot w_{i} \). Since \( W_{0} = 1 \), \( S_{A} \) is invertible in \( K(h, h') \).

\( S \in H_{Z/2}(\text{MxM}) \) which is P to the \( \Delta \) of \( K(h, h') \) to the \( \Delta \) of \( \text{MxM}, \) when tensored to \( \Delta \).