1 Homological Algebra continued

**Definition 1** (Convolution) Let \( P \in (A, B)^{\text{mod}}. \) \( P \) gives rise to the convolution functor

\[
\Phi_P : A^{\text{mod}} \to B^{\text{mod}}
\]

\[
M \mapsto "M \otimes_A P"
\]

where

\[
M \otimes P(X) = \bigoplus_{Y_0 \in \text{Ob}(A)} M(Y_0) \otimes_K \mathcal{P}(X, Y_0) \oplus \bigoplus_{Y_0, Y_1 \in \text{Ob}(A)} M(Y_1) \otimes_K \text{hom}(Y_0, Y_1) \otimes_K \mathcal{P}(X, Y_0)[1]
\]

Remarks

1. \( \Phi_A \cong \text{Id}_{A^{\text{mod}}} \) (\( A \) the diagonal bimodule.)

2. There is a diagram, commutative up to quasi-isomorphism of functors

\[
\begin{array}{ccc}
A & \xrightarrow{\mathcal{F}} & B \\
Yon & \downarrow & Yon \\
A^{\text{mod}} & \xrightarrow{\Phi_{\text{Graph}(\mathcal{F})}} & B^{\text{mod}}
\end{array}
\]

3. If \( P = (X^{\text{opp}})^{\text{Yon}} \otimes Y^{\text{Yon}} \) then \( \Phi_P(M) = M(X) \otimes Y^{\text{Yon}}. \)

4. Consequence: If \( A \) is proper and \( P \) is perfect, then \( \Phi_P \) maps proper modules to perfect ones.

**Definition 2** \( A \) is smooth if \( (A, A)^{\text{mod}} \) is perfect.

**Lemma 1** \( A = K[X], X \) an algebraic variety \( \implies \) the two notions of smoothness coincide.

**Lemma 2** \( A \) smooth and proper \( \implies A^{\text{prop}} = A^{\text{Perf}}. \)

**Proof:** Apply \( \Phi_A \) to proper \( M \) and use Remarks 1 and 4.

1.1 Quotient categories

\( B \subset A \) a full subcategory. Want to define \( \mathcal{C} = A/B. \) \( \mathcal{C} \) is equipped with a map \( Q : A \to \mathcal{C} \) such that the composition \( B \hookrightarrow A \to \mathcal{C} \) is essentially \( 0; \) and such that, if \( D \) is an \( A_{\infty} \) category, then \( \text{Fun}(\mathcal{C}, D) \to \text{Fun}(A, D) \) is cohomologically full and faithful, with image precisely those functors which kill \( B. \)

**Theorem 1** These exist.
2 Hochschild Homology

2.1 Motivation

$X$ closed, symplectic.

\[
\begin{array}{c}
\text{QH}^* \\
\downarrow \\
\text{HH}_*(\text{Fuk}(X)) \\
\downarrow \\
\text{HH}_*(A)
\end{array}
\]

Suppose $A \hookrightarrow \text{Fuk}(X)$. If $\text{Id}$ lies in the image then "we can study $A$ instead of $\text{Fuk}(X)$."

2.2 Construction

2.2.1

Suppose $A$ is an algebra over a field $\mathbb{K}$. For an $(A, A)$-bimodule $M$ (which is a right $A^e = A \otimes_\mathbb{K} A^{\text{opp}}$-module), we define

\[
\text{HH}_n(A, M) := \text{Tor}^A_n(M, \mathbb{K}).
\]

To compute, take the standard bar resolution:

\[
\begin{array}{cccccccc}
... & \rightarrow & A^e \otimes^4 & \rightarrow & A^e \otimes^3 & \rightarrow & A^e \otimes^2 & \rightarrow & 0 \\
\downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha & & \\
... & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & A & \rightarrow & 0
\end{array}
\]

where, for example, $\alpha: a \otimes b \otimes c \otimes d \mapsto ab \otimes c \otimes d - a \otimes bc \otimes d + a \otimes b \otimes cd$. Tensor with $M$ to get

\[
\begin{array}{cccccccc}
... & \rightarrow & M \otimes_{A^e} A^e \otimes^4 & \rightarrow & M \otimes_{A^e} A^e \otimes^3 & \rightarrow & M \otimes_{A^e} A^e \otimes^2 & \rightarrow & 0 \\
\downarrow \alpha(\cong) & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
... & \rightarrow & M \otimes_{\mathbb{K}} A^e \otimes^2 & \rightarrow & M \otimes_{\mathbb{K}} A & \rightarrow & M & \rightarrow & 0
\end{array}
\]

where $\alpha: m \otimes a \otimes b \otimes c \otimes d \mapsto d\text{ma} \otimes c - d\text{ma} \otimes bc + cd\text{ma} \otimes b$. Thus, we may define $\text{HC}_n(A, M) := M \otimes_{\mathbb{K}} A^e \otimes^n$ with differential

\[
\partial(m \otimes a_1 \otimes \ldots \otimes a_n) = ma_1 \otimes a_2 \otimes \ldots \otimes a_n + \sum_{1 \leq i < n} \pm m \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \ldots \otimes a_n \pm a_n m \otimes a_1 \otimes \ldots \otimes a_{n-1}
\]

and set

\[
\text{HH}_n(A, M) = H_n(\text{HC}_*(A, M), \partial).
\]
2.2.2

Suppose $A$ is now an $A_\infty$ algebra over $K$. Let $M$ be an $(A, A)$-bimodule. We again have

$$HH_\ast(A, M) := \text{Tor}_\ast^A(M, A).$$

The differential is modified according to Figure 2.

2.2.3

Now let $A$ be an $A_\infty$-category and $Q$ an $(A, A)$ bimodule. The Hochschild chain complex is

$$HC_\ast(A, Q) := \bigoplus Q(X_d, X_0) \otimes K \text{hom}_A(X_{d-1}, X_d) \otimes \ldots \otimes \text{hom}_A(X_0, X_1)[d].$$

Less concretely,

(1) \hspace{1cm} HH_\ast(A, Q) = H^\ast(Q \otimes_{A_\infty} A)

2.3 Key Properties

1. Covariant functoriality in $Q$ (obvious from Eq.1).

2. Given an $A_\infty$-functor $\mathcal{F} : A \to B$ and a $(B, B)$-bimodule $Q$, we get a map $HH_\ast(A, \mathcal{F}^\ast(Q)) \to HH_\ast(B, Q)$, where $\mathcal{F}^\ast$ is the pullback on both sides.
2.4 Consequences of the key properties

1. Let $K \to A^{perf}$ be the functor associated to $P \in \text{Ob}(A^{perf})$. Then the image of 1 under

$$K \xrightarrow{\text{norm}} \text{HH}_0(K) \to \text{HH}_0(A^{perf}) \xrightarrow{\text{Morita}} \text{HH}_0(A)$$

is denoted $[P]_{\text{HH}}$. By functoriality, this is invariant of the quasi-isomorphism class of $P$. By normalization,

$$[P[k]]_{\text{HH}} = (-1)^k[P]_{\text{HH}}.$$

2. Let $M \in \text{Ob}(A^{prop})$. $M$ is (by definition) a functor $A^{opp} \to K^{prop} \cong K^{perf}$. We can then consider

$$\text{HH}_0(A) \xrightarrow{\text{opp}} \text{HH}_0(A^{opp}) \to \text{HH}_0(K^{prop}) \cong \text{HH}_0(K^{perf}) \cong \text{HH}_0(K) \cong K.$$

This associates to $M$ the class $[M]^\vee_{\text{HH}} \in \text{HH}_0(A)^\vee$. 

(a) Notation: $\text{HH}_*(A) \coloneqq \text{HH}_*(A,A)$ (the latter $A$ refers to the diagonal bimodule).

(b) 1 and 2 imply $\text{HH}_*(A)$ is covariantly functorial.

3. Morita invariance: The Yoneda embedding $A \to A^{perf}$ induces an isomorphism $\text{HH}_*(A) \xrightarrow{\sim} \text{HH}_*(A^{perf})$.

**Proof:** $A \to A^{perf}$ induces a restriction map $A^{perf}$-bimodules $\to A$-bimodules, which is a quasi-equivalence sending the diagonal to the diagonal

4. Künneth formula: $\text{HH}_*(A \otimes_K B) \cong \text{HH}_*(A) \otimes \text{HH}_*(B)$.

To see this, use Eq. 1 and pass to d.g. categories.

5. Opposite property: $\text{HH}_*(A^{opp}) \cong \text{HH}_*(A)$.

This follows from the 'concrete' definition.

6. Normalization: $\text{HH}_*(K) = \begin{cases} K & \text{if } * = 0 \\ 0 & \text{else} \end{cases}$

Moreover, $\forall P \in \text{Ob}(K^{perf})$, there is a map $K \to K^{perf}$. This induces a map

$$K \cong \text{HH}_*(K) \to \text{HH}_*(K^{perf}) \xrightarrow{\text{Morita}} \text{HH}_0(K) \cong K.$$

This map is multiplication by $\chi$.

7. Exactness: Let $B \hookrightarrow A$ be a full $A_\infty$ subcategory and $A/B$ the quotient. Then there exists a long exact sequence

$$\cdots \to \text{HH}_*(B) \to \text{HH}_*(A) \to \text{HH}_*(A/B) \to \text{HH}_{*+1}(B) \to \cdots$$

8. (unproperty) $\text{HH}_*$ does not satisfy homotopy invariance, which is when the inclusion $K \to K[t]$ induces an isomorphism in $\text{HH}_*$. It is doubtful that there exists a homology theory with all of the above properties.
3. Recall that if $P$ is a perfect module and $M$ is a proper module, then $H^\ast(\text{hom}_{\text{A-mod}}(P,M))$ is finite dimensional (i.e. $\text{hom}_{\text{A-mod}}(P,M) \in \text{Ob}(K^\text{prop})$).

Let $\mathcal{F}_P : K \to A^{\text{perf}}$ be the obvious functor, and similarly for $\mathcal{F}_M : A^{\text{opp}} \to K^{\text{perf}}$. Also define

$$G_M : A^{\text{perf}} \to (K^\text{prop})^{\text{opp}}$$

$$Q \mapsto \text{hom}_{A \text{-mod}}(Q,M).$$

Then $G_M \circ \mathcal{F}_P : K \to (K^\text{prop})^{\text{opp}}$ induces a map

$$HH_0(K) \to HH_0((K^\text{prop})^{\text{opp}}) \to HH_0(K),$$

and this map is precisely multiplication by $\chi(\text{hom}_{A \text{-mod}}(P,M))$. We obtain the Cardy relation

$$\langle [M]_{\text{HH}}, [P]_{\text{HH}} \rangle = \chi(H^\ast(\text{hom}_{A \text{-mod}}(P,M))).$$