Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. No calculators are permitted. Good luck, have fun!

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1. (a) (4 points) Let \( \vec{v} \) and \( \vec{w} \) be two vectors in \( \mathbb{R}^3 \) with \( |\vec{w}| = 4 \) and \( \vec{v} \cdot \vec{w} = 2|\vec{v}| \). Find the angle between \( \vec{v} \) and \( \vec{w} \).

**Solution:** Since
\[
\cos \theta = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|}
\]
we see that
\[
\cos \theta = \frac{2|\vec{v}|}{|\vec{v}||\vec{w}|} = \frac{2}{|\vec{w}|} = \frac{1}{2}
\]
and so \( \theta = \frac{\pi}{3} \).

(b) (4 points) Show that, for any two vectors \( \vec{a} \) and \( \vec{b} \), we have \((\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = 2(\vec{b} \times \vec{a})\).

**Solution:**
Since the cross product has a distributive law, we get
\[
(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \vec{a} \times \vec{a} + \vec{b} \times \vec{a} - \vec{a} \times \vec{b} - \vec{b} \times \vec{b}
\]
\[
= \vec{b} \times \vec{a} - \vec{a} \times \vec{b}
\]
\[
= \vec{b} \times \vec{a} + \vec{b} \times \vec{a}
\]
\[
= 2 \vec{b} \times \vec{a}
\]

2. (10 points) Let \( \vec{r}(t) = \langle t+1, t^2+2, 2t \rangle \). Find the equation of the osculating plane at time \( t = 1 \).

**Solution:**
We first find
\[
\vec{r}'(t) = \langle 1, 2t, 2 \rangle
\]
so
\[
|\vec{r}'(t)| = \sqrt{5 + 4t^2}
\]
giving
\[
T(t) = \left\langle \frac{1}{\sqrt{5 + 4t^2}}, \frac{2t}{\sqrt{5 + 4t^2}}, \frac{2}{\sqrt{5 + 4t^2}} \right\rangle
\]
So
\[
T'(t) = \left\langle \frac{4t}{(5 + 4t^2)^{3/2}}, \frac{10}{(5 + 4t^2)^{3/2}}, -\frac{8t}{(5 + 4t^2)^{3/2}} \right\rangle
\]
and

\[
|T'(t)| = \sqrt{\frac{16t^2}{(5 + 4t^2)^3} + \frac{100}{(5 + 4t^2)^3} + \frac{64t^2}{(5 + 4t^2)^3}}
\]

\[
= \sqrt{\frac{80t^2 + 100}{(5 + 4t^2)^3}}
\]

\[
= \frac{\sqrt{80t^2 + 100}}{(5 + 4t^2)^{3/2}}
\]

which means

\[
N(t) = \frac{T'(t)}{|T'(t)|} = \frac{(5 + 4t^2)^{3/2}}{\sqrt{80t^2 + 100}} \left\langle -\frac{4t}{(5 + 4t^2)^{3/2}}, \frac{10}{(5 + 4t^2)^{3/2}}, -\frac{8t}{(5 + 4t^2)^{3/2}} \right\rangle
\]

\[
= \left\langle -\frac{4t}{\sqrt{80t^2 + 100}}, \frac{10}{\sqrt{80t^2 + 100}}, -\frac{8t}{\sqrt{80t^2 + 100}} \right\rangle
\]

So we can now compute

\[
B(t) = T(t) \times N(t)
\]

\[
= \left\langle \frac{1}{\sqrt{5 + 4t^2}}, \frac{2t}{\sqrt{5 + 4t^2}}, \frac{2}{\sqrt{5 + 4t^2}} \right\rangle
\]

\[
\times \left\langle -\frac{4t}{\sqrt{80t^2 + 100}}, \frac{10}{\sqrt{80t^2 + 100}}, -\frac{8t}{\sqrt{80t^2 + 100}} \right\rangle
\]

\[
= \frac{1}{\sqrt{5 + 4t^2} \sqrt{80t^2 + 100}} \langle (1, 2t, 2) \times (-4t, 10, -8t) \rangle
\]

\[
= \frac{1}{\sqrt{5 + 4t^2} \sqrt{80t^2 + 100}} \langle (1, 2t, 2) \times (-4t, 10, -8t) \rangle
\]

\[
= \frac{1}{\sqrt{5 + 4t^2} \sqrt{80t^2 + 100}} \left| \begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2t & 2 \\ -4t & 10 & -8t \end{array} \right|
\]

\[
= \frac{1}{\sqrt{5 + 4t^2} \sqrt{80t^2 + 100}} \left( \begin{array}{c} 2t \\ 10 \end{array} -8t \right) \vec{i} - \left( \begin{array}{c} 1 \\ -4t \end{array} -8t \right) \vec{j} + \left( \begin{array}{c} 1 \\ -4t \end{array} \right) \vec{k}
\]

\[
= \frac{1}{\sqrt{5 + 4t^2} \sqrt{80t^2 + 100}} \langle -16t^2 - 20, 0, 8t^2 + 10 \rangle
\]

And at \( t = 1 \), we get

\[
B(1) = \frac{1}{3\sqrt{180}} \langle -36, 0, 18 \rangle = \frac{1}{\sqrt{5}} \langle -2, 0, 1 \rangle = \langle -\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}} \rangle
\]
and since
\[ \vec{r}(1) = \langle 2, 3, 4 \rangle \]
the equation of our plane is
\[ \frac{-2}{\sqrt{5}}(x - 2) + \frac{1}{\sqrt{5}}(z - 4) = 0 \]
(And in case you’re wondering, something this long will absolutely not appear on the final!)

3. (10 points) Find the equation of the osculating circle to the ellipse \(4x^2 + y^2 = 16\) at the point \((2, 0)\).

**Solution:**

We first have to compute the curvature at this point, for which we need a parameterization; the standard one is
\[ \vec{r}(t) = \langle 2\cos \theta, 4\sin \theta \rangle \]
Then
\[ \vec{r}'(t) = \langle -2\sin \theta, 4\cos \theta \rangle \]
\[ \vec{r}''(t) = \langle -2\cos \theta, -4\sin \theta \rangle \]
\[ |\vec{r}' \times \vec{r}''| = 8\sin^2 \theta + 8\cos^2 \theta = 8 \]
\[ |\vec{r}'| = \sqrt{4\sin^2 \theta + 16\cos^2 \theta} \]
So now
\[ \kappa(t) = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{8}{(4\sin^2 \theta + 16\cos^2 \theta)^{3/2}} \]

Now to get the point \((2, 0)\), we can take \(t = 0\). Then
\[ \kappa(0) = \frac{8}{(16)^{3/2}} = \frac{1}{8} \]
So the osculating circle has radius which is \(\frac{1}{\kappa(\pi)} = 8\). By the symmetry of the ellipse, we see that the circle will have its center along the \(x\)-axis. Since
it passes through the point $(2, 0)$ and has radius 8, the center must be at $(-6, 0)$. This gives us the equation

$$(x + 6)^2 + y^2 = 64$$

4. Evaluate the following limits, or explain why they do not exist.

(a) (5 points)  \( \lim_{(x,y) \to (0,0)} \frac{x^3y}{x^4 + y^2} \)

**Solution:**

We compute the limit when restricted to the line $y = mx$, which admits the parameterization $\langle t, mt \rangle$. The limit becomes

$$\lim_{t \to 0} \frac{t^3 \cdot mt}{t^4 + (mt)^2} = \lim_{t \to 0} \frac{mt^2}{t^2 + m^2} = 0$$

So our first check of whether the limit exists tells us that it’s possible. Even though the degrees of the numerator and denominator match, it turns out that this limit exists. We can see this from a Squeeze Theorem argument.

To start, let’s recall the inequality $\frac{uv}{u^2 + v^2} \leq \frac{1}{2}$ (remember this came out of manipulating the obvious inequality $(u - v)^2 \geq 0$). If we set $u = x^2$ and $v = y$, we will get the inequality $\frac{x^2y}{x^4 + y^2} \leq \frac{1}{2}$. Let’s first assume that $x, y \geq 0$. Then we get

$$0 \leq \frac{x^2y}{x^4 + y^2} \leq \frac{1}{2}$$

$$0 \leq \frac{x^3y}{x^4 + y^2} \leq \frac{x}{2}$$

If we take the limit as $(x, y) \to (0, 0)$, we get $\lim_{(x,y) \to (0,0)} \frac{x^3y}{x^4 + y^2} = 0$.

We need to check that this is also the case for the other possible sign combinations for $x$ and $y$, but we will get similar inequalities in the other cases.

(b) (5 points)  \( \lim_{(x,y) \to (0,0)} \frac{(x^2 + y^2) \ln(1 + y^2)}{x^2 + 2y^2} \)

**Solution:**

Since the numerator looks to go to zero faster than the denominator, let’s try to show that the limit exists. Firstly, since $y^2 \geq 0$, we can add
\[ x^2 + y^2 \text{ to each side to get} \]
\[ x^2 + 2y^2 \geq x^2 + y^2 \quad \text{so} \quad \frac{x^2 + y^2}{x^2 + 2y^2} \leq 1 \]

Now since everything involved is positive, we can also see that the result is positive, giving
\[ 0 \leq \frac{x^2 + y^2}{x^2 + 2y^2} \leq 1 \]

We now multiply by ln\((1 + y^2)\) to get
\[ 0 \leq \frac{(x^2 + y^2) \ln(1 + y^2)}{x^2 + 2y^2} \leq \ln(1 + y^2) \]

We can now take limits as \((x, y) \to (0, 0)\), and the Squeeze Theorem will give
\[ \lim_{(x, y) \to (0, 0)} \frac{(x^2 + y^2) \ln(1 + y^2)}{x^2 + 2y^2} = 0 \]

5. (10 points) Find the equation of the tangent plane to the function \(f(x, y) = x^2y - xy^2\) at the point \((2, 2)\).

**Solution:**

We have
\[ \frac{\partial f}{\partial x} = 2xy - y^2 \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2 - 2xy \]
so that
\[ \frac{\partial f}{\partial x}(2, 2) = 4 \quad \text{and} \quad \frac{\partial f}{\partial y}(2, 2) = -4 \]

Then the equation of the tangent plane is
\[ z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) \]
\[ z - 0 = 4(x - 2) - 4(y - 2) \quad \text{or} \quad 4x - 4y - z = 0 \]

6. (8 points) Recall that the wave equation is the partial differential equation
\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \]

Show that the function \(u(x, t) = \cos(x - ct)\) satisfies the wave equation.
Solution:

We first compute both second derivatives that show up:

\[
\frac{\partial u}{\partial t} = c \sin(x - ct)
\]

\[
\frac{\partial^2 u}{\partial t^2} = -c^2 \cos(x - ct)
\]

\[
\frac{\partial u}{\partial x} = -\sin(x - ct)
\]

\[
\frac{\partial^2 u}{\partial x^2} = -\cos(x - ct)
\]

We can now verify that

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}
\]

since this becomes

\[-c^2 \cos(x - ct) = c^2(-\cos(x - ct))\]

7. (a) (5 points) Find the derivative of the function \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) given by \( f(x, y) = (2xy, \sin(xy)) \).

Solution:

We can write out the matrix of the derivative:

\[
Df = \begin{bmatrix}
\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\
\frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}
\end{bmatrix}
= \begin{bmatrix}
2y & 2x \\
y \cos(xy) & x \cos(xy)
\end{bmatrix}
\]

(b) (5 points) Let \( u = g(s, t) = s^2 - t^3 + 3 \). If \( s(x, y) = 2xy \) and \( t(x, y) = \sin(xy) \), find \( \frac{\partial u}{\partial x} \) and \( \frac{\partial u}{\partial y} \).

Solution:

From the chain rule, we know that \( D(g \circ f) = Dg \cdot Df \), so

\[
\begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial u}{\partial s} & \frac{\partial u}{\partial t}
\end{bmatrix}
\cdot
\begin{bmatrix}
\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\
\frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}
\end{bmatrix}
\]
so
\[
\begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}
\end{bmatrix}
= \begin{bmatrix}
2s & -3t^2
\end{bmatrix}
\begin{bmatrix}
2x & 2y
y \cos(xy) & x \cos(xy)
\end{bmatrix}
= \begin{bmatrix}
2s \cdot 2y - 3t^2 \cdot y \cos(xy) & 2s \cdot 2x - 3t^2 \cdot x \cos(xy)
\end{bmatrix}
\]

So
\[
\frac{\partial u}{\partial x} = 2s \cdot 2y - 3t^2 \cdot y \cos(xy)
= 2(2xy) \cdot 2y - 3 \sin^2(xy) \cdot y \cos(xy)
\]

and
\[
\frac{\partial y}{\partial y} = 2s \cdot 2x - 3t^2 \cdot x \cos(xy)
= 2(2xy) \cdot 2x - 3 \sin^2(xy) \cdot x \cos(xy)
\]

8. (12 points) Consider the function \( f(x, y) = (1 + xy)(x + y) \). Find the critical points, and determine whether each is a minimum, a maximum, or a saddle point.

**Solution:** Firstly, we will find the critical points of \( f \). For this we need:
\[
\frac{\partial f}{\partial x} = y(x + y) + (1 + xy) = 1 + 2xy + y^2
\]
\[
\frac{\partial f}{\partial y} = x(x + y) + (1 + xy) = 1 + 2xy + x^2
\]
Now these must both be zero, so we must solve
\[
1 + 2xy + y^2 = 0
\]
\[
1 + 2xy + x^2 = 0
\]
We can subtract the equations to get the condition \( y^2 - x^2 = 0 \), which means that \( y = \pm x \). Taking this and using it with the first equation, that becomes either \( 1 + 2x^2 + x^2 = 0 \) or \( 1 - 2x^2 + x^2 = 0 \), depending on the sign. But this first equation is \( 3x^2 = -1 \), which is impossible, so we must have \( y = -x \) and the equation \( 1 - x^2 = 0 \). The possible solutions for \( x \) are now \( \pm 1 \), and the critical points are \((-1, 1)\) and \((1, -1)\).
Next we need to find the second order partial derivatives.

\[
\frac{\partial^2 f}{\partial x^2} = 2y \\
\frac{\partial^2 f}{\partial x \partial y} = 2x + 2y \\
\frac{\partial^2 f}{\partial y^2} = 2x
\]

This means that the determinant of the Hessian matrix is

\[
\begin{vmatrix}
2y & 2x + 2y \\
2x + 2y & 2x
\end{vmatrix} = 4xy - (2x + 2y)^2 = -4x^2 - 4xy - 4y^2
\]

At our first critical point, \((-1, 1)\), we get the value of \(-4\), which means that we have a saddle point. At our second critical point, \((1, -1)\), we also get the value of \(-4\), so both critical points are saddle points, and there are neither relative minima nor maxima for this function.

9. (12 points) Find the minimum value of the function \(f(x, y) = x^2 + y^2 + z^2\) subject to the constraint that \(x + y + z = 12\).

**Solution:**

The method of Lagrange Multipliers tells us that the minimum will occur when the derivatives of \(f\) and \(g(x, y, z) = x + y + z\) are scalar multiples. This means that we can solve the system of equations

\[
\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \\
\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \\
\frac{\partial f}{\partial z} = \lambda \frac{\partial g}{\partial z}
\]

\(x + y + z = 12\)

We can compute the partial derivatives to get

\[
2x = \lambda \\
2y = \lambda \\
2z = \lambda \\
x + y + z = 12
\]
We see that $\lambda \neq 0$, or else we get $x$, $y$, and $z$ are all 0, violating the final condition. Let’s first subtract the first two equations, giving

$$2x - 2y = 0$$

from which we see that $y = x$. Subtracting the first and third equations similarly gives $z = x$, so we know $x = y = z$. So the last equation becomes

$$3x = 12$$

so we get $x = y = z = 4$ is the unique minimum point. We can now find the minimum value that our function takes by evaluating:

$$4^2 + 4^2 + 4^2 = 48$$