1 Introduction to Power Series

We have already seen an example of a power series. There is the prototypical example

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

which we showed is valid for all values of $x$ with $|x| < 1$. What does this mean, in terms of the partial sums? Well, let’s plot these partial sums together with the function in question, $\frac{1}{1-x}$. [Draw on board.]

So $s_0 = 1$, $s_1 = 1 + x$, $s_2 = 1 + x + x^2$, $s_3 = 1 + x + x^2 + x^3$, ...

As is visible from the picture, for each $x \in (-1,1)$, the partial sums $s_0, s_1, ...$ progressively better approximate $\frac{1}{1-x}$ as long as we remain in the interval $(-1,1)$.

So what is this kind of phenomenon saying? Well $\frac{1}{1-x}$ is not a polynomial, but we are saying that, in the interval $(-1,1)$, we can approximate it quite well by polynomials.

Well, it takes a bit of thought to realize that in this canonical example, there are really two things going on:

1. The function $\frac{1}{1-x}$ can literally be realized as an infinite sum for $|x| < 1$.
2. The function $\frac{1}{1-x}$ is, for $x$ close to zero, well-approximated by the polynomials $s_0, s_1, ...$

These two phenomenon, which in this example seem to blend together, should really be thought of as distinct. The first is summarized by saying that $f(x) = \frac{1}{1-x}$ is an analytic function. The second is summarized by saying that $f(x) = \frac{1}{1-x}$ is well-approximated by its Taylor polynomials.

But before we dig into these interesting ideas, let’s make a preliminary definition.

**Definition 1** A power series centered around a point $a \in \mathbb{R}$ is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

The sum of the series defines a function $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$, in the domain where it is defined, i.e. in the domain of all $x$ for which the series converges.

Note that the example above was $a = 0$ and $c_n = 1$ for all $n$.

In some sense, a power series is completely analogous to a polynomial, only with (possibly) infinite degree.

So how can we find the domain of definition of a power series? We should use the Ratio test (or Root test, if you prefer). We see, letting $a_n = c_n (x-a)^n$

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{c_{n+1} (x-a)^{n+1}}{c_n (x-a)^n} \right| = \lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right| |x-a|$$

so if $\lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right| = \alpha$, then we have that the series converges absolutely if

$$\alpha |x-a| < 1$$

$$|x-a| < \frac{1}{\alpha}$$
and will diverge if $|x - a| > \frac{1}{\alpha}$. Call $R = \frac{1}{\alpha} = \lim_{n \to \infty} \left\lfloor \frac{c_n}{c_{n+1}} \right\rfloor$. As usual, we adopt the convention that if $\lim_{n \to \infty} \left\lfloor \frac{c_{n+1}}{c_n} \right\rfloor = \infty$, then we write $\alpha = \infty$ and $R = \frac{1}{\alpha} = 0$. Similarly, if $\alpha = 0$, then $R = \infty$. We have effectively shown the following (modulo some important technical details)

**Theorem 1** For power series $\sum_{n=0}^{\infty} c_n (x - a)^n$, there are only three possibilities:

(i). The series converges only when $x = a$. $[\alpha = \infty, R = 0]$

(ii). The series converges for all $x$. $[\alpha = 0, R = \infty]$

(iii). The series converges for $|x - a| < R$ and diverges for $|x - a| > R$. $[0 < R < \infty]$

At the endpoints, we have to check for convergence, usually using the alternating series test.

**Example 1** For what values of $x$ does

$$\sum_{k=1}^{\infty} \frac{(x - 3)^k}{k}$$

converge? Use ratio test.

$$\alpha = \lim_{n \to \infty} \left| \frac{n + 1}{n} \right| = 1$$

so $R = 1/\alpha = 1$. If $x = 4$, we get $\sum_{k=1}^{\infty} \frac{1}{k}$ which is divergent; if $x = 2$, we get the alternating harmonic series, which is convergent. So the domain is $[2, 4]$.

Let’s do another example:

**Example 2** Find the domain of the Bessel function

$$J_0 (x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^k (k!)^2}$$

We use the ratio test. Note that here, since we’re not given the power series in a nice form where the power of $x$ is $k$, we have to go back to the ratio test on the full thing. This gives

$$\lim_{k \to \infty} \left| \frac{(-1)^{k+1} x^{2k+2}}{2^{2k+2} ((k + 1)!)^2} \right| = \left| \frac{x^2}{4 (k + 1)^2} \right| = 0$$

for all $x \in \mathbb{R}$. So by the Ratio test, the series converges for all $x \in \mathbb{R}$. So the domain is $\mathbb{R}$.

One more, for good measure:

**Example 3** Find the radius of convergence for the power series

$$\sum_{k=0}^{\infty} \left( 1 + \frac{1}{k} \right)^k (x + 3)^k$$

Well, we use the Root Test here. We find

$$\lim_{k \to \infty} \left| \left( 1 + \frac{1}{k} \right)^k \right|^{1/k} = \lim_{k \to \infty} \left| 1 + \frac{1}{k} \right| = 1$$

so $\alpha = 1$ and $R = 1$. Note that here it is very hard to say anything about convergence at the endpoints!

Good, so time for a definition:
**Definition 2** A function \( f(x) \) is said to be analytic around a point \( a \) if \( f(x) \) can be expanded in a power series around \( a \)

\[
f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n
\]

which has a positive radius of convergence, i.e. \( \lim_{n \to \infty} \left| \frac{c_n}{c_{n+1}} \right| \neq 0 \).

So, almost tautologically, a convergent power series defines an analytic function. Here are some properties of analytic functions. We won’t prove these, because the proofs are a little messy and not terribly enlightening if we insist on trying to do them by hand. [An aside: the proofs are quite easy if we knew some complex analysis, but that would take us too far afield.]

**Sums, Products of Analytic Functions** Suppose that \( f(x) = \sum c_n (x-a)^n \) and \( g(x) = \sum d_n (x-a)^n \) are both analytic in \( (a-R,a+R) \). Then \( f(x) + g(x) \) and \( f(x)g(x) \) are both analytic as well, and in the same region \( (a-R,a+R) \). In fact, the power series found are

\[
f(x) + g(x) = \sum (c_n + d_n) (x-a)^n
\]

\[
f(x)g(x) = \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} (c_n d_k) (x-a)^n
\]

\[
= c_0 d_0 + (c_1 d_0 + c_0 d_1) (x-a) + (c_2 d_0 + c_1 d_1 + c_0 d_2) (x-a)^2 + ...
\]

**Quotients by Non-Vanishing Analytic Functions** Suppose that \( f(x) = \sum c_n (x-a)^n \) and \( g(x) = \sum d_n (x-a)^n \) are both analytic in \( (a-R,a+R) \). Suppose further than \( g(x) \neq 0 \) for all \( x \in (a-R,a+R) \). Then \( f(x)/g(x) \) is also analytic in \( (a-R,a+R) \).

**Derivatives and Antiderivatives** If \( f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \) is an analytic function in \( (a-R,a+R) \), then so is \( f'(x) \), and it is given, within the radius of convergence of the series \( \sum c_n (x-a)^n \), by

\[
f'(x) = \sum_{n=0}^{\infty} n c_n (x-a)^{n-1}
\]

Similarly, any antiderivative of \( f(x) \) is also analytic in \( (a-R,a+R) \). They are given by

\[
F(x) = C + \sum_{n=1}^{\infty} \frac{c_{n-1}}{n} (x-a)^n
\]

These are called term-by-term differentiation and integration—we simply differentiate and integrate termwise.

### 1.1 Some Exercises on Basic Power Series

Here are some nice exercises from the book that we’re going to do together. I think they will help clarify some things.

**pp 746**

37. Suppose \( f \) is defined by

\[
f(x) = 1 + 2x + x^2 + 2x^3 + x^4 + ...
\]

That is, its coefficients are \( c_{2n} = 1 \) and \( c_{2n+1} = 2 \) for all \( n \geq 0 \). What is the interval of convergence for this series and what is a formula for \( f(x) \)?
Well,
\[
\frac{1}{1 - x} = 1 + x + x^2 + x^3 + ...
\]
and
\[
\frac{1}{1 - x^2} = 1 + x^2 + x^4 + x^5 + ...
\]
so
\[
\frac{x}{1 - x^2} = x + x^3 + x^5 + ...
\]
and so
\[
\frac{1 + 2x}{1 - x^2} = \frac{1}{1 - x} + \frac{x}{1 - x^2} = 1 + 2x + x^3 + 2x^4 + ...
\]

since the intervals of convergence of both of these series is \((-1,1)\), we conclude that this formula is true for all \(x\) in \((-1,1)\).

41. Suppose the series \(\sum c_n x^n\) has a radius of convergence 2 and the series \(\sum d_n x^n\) has a radius of convergence 3. Then the radius of convergence of
\[
\sum (c_n + d_n) x^n
\]
is 2. This is because for \(|x| < 2\), the series \(\sum c_n x^n\) and \(\sum d_n x^n\) converge absolutely; on the other hand, for \(|x| > 2\), \(\sum c_n x^n\) diverges, so \(\sum (c_n + d_n) x^n\) should as well.

42. Suppose that the radius of convergence of the power series \(\sum c_n x^n\) is \(R\). What is the radius of convergence of the power series
\[
\sum c_n x^{2n}
\]
Well, we compute: we want
\[
\lim_{n \to \infty} \left| \frac{c_{n+1} x^{2n+2}}{c_n x^{2n}} \right| = \lim_{n \to \infty} |x|^2 \left| \frac{c_{n+1}}{c_n} \right| = |x|^2 \frac{1}{R} < 1
\]
so we want \(|x| < \sqrt{R}\).

**Examples:**

**Ex1:** Find a power series expansion for \(f(x) = \ln (1 + x)\). We have
\[
\frac{1}{1 - t} = \sum_{k=0}^{\infty} t^k
\]
and substituting \(t = -x\), we find
\[
\frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n x^n
\]
and integrating termwise to get \(\ln (1 + x)\) gives
\[
\ln (1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}
\]

**Ex2:** Find a power series expansion for \(f(x) = \tan^{-1} x\). Well:
\[
\frac{d}{dx} \tan^{-1} (x) = \frac{1}{1 + x^2}
\]

4
and we can write a power series for $\frac{1}{1+x^2}$ using the power series

$$\frac{1}{1-t} = \sum_{k=0}^{\infty} t^k$$

and substituting in $t = -x^2$ to find

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

so

$$\tan^{-1}(x) = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

evaluating at $x = 0$, we find $C = 0$.

**Ex3:** Evaluate $\int \frac{1}{1+x^2} \, dx$ as a power series. Well:

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

so integrating, we find

$$\int \frac{1}{1+x^2} \, dx = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

## 2 Analytic Functions and Taylor Series

The main point here is the following: suppose that $f(x)$ is an analytic function, i.e. for $|x-a| < R$,

$$f(x) = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots$$

We know:

$$f(a) = c_0$$

can we determine the other coefficients $c_1, c_2, \ldots$? Well,

$$f'(x) = c_1 + 2c_2 (x-a) + 3c_3 (x-a)^2 + \ldots$$

so

$$f'(a) = c_1$$

similarly

$$f''(x) = 2c_2 + (3)(2)c_3 (x-a) + (4)(3)c_4 (x-a)^2 + \ldots$$

so

$$f''(a) = 2c_2$$

and

$$f'''(x) = (3)(2)c_3 + (4)(3)(2)(x-a) + (5)(4)(3)(x-a)^2 + \ldots$$

so

$$f'''(a) = 3!c_3$$

We can clearly see the pattern, and can prove by induction that

$$f^{(n)}(a) = n!c_n$$

for all $n$. This gives us the following:
Theorem 2 Let \( f(x) \) be an analytic function around \( a \), i.e. let

\[
f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n
\]

for \( |x - a| < R \). Then the coefficients \( c_n \) are given by the formula

\[
c_n = \frac{f^{(n)}(a)}{n!}
\]

so we have

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n
\]

This is called the Taylor Series expansion of the analytic function \( f \).

If we only look at the case \( a = 0 \), i.e. power series centered at the origin, we have:

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n
\]

2.1 Important Analytic Functions

Let’s see what happens if we apply this to \( e^x \). This gives, since

\[
\frac{d^n}{dx^n} (e^x) \bigg|_{x=0} = 1
\]

for all \( n \), so we write the power series as

\[
\sum_{n=0}^{\infty} \frac{x^n}{n!}
\]

So if \( e^x \) is analytic at 0, then it is given by the series \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \). Moreover, we can check that the radius of convergence for this power series is \( \infty \), so if it analytic at 0, then for all \( x \in \mathbb{R} \),

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}
\]

We’ll show tomorrow, when we cover Taylor’s inequality, that in fact \( e^x \) is analytic at 0.

Then we can do a similar argument to see, if \( f(x) = \sin x \), that

\[
f^{(n)}(0) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n = 1, 5, 9, 13, \ldots \\ -1 & \text{if } n = 3, 7, 11, 15, \ldots \end{cases}
\]

so if \( \sin x \) is analytic at 0, then (check radius of convergence using \( n \)-th root test) it is equal to:

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots
\]

We can do a similar computation for \( g(x) = \cos x \) to see

\[
g^{(n)}(0) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n = 0, 4, 8, 12, \ldots \\ -1 & \text{if } n = 2, 6, 12, 16, \ldots \end{cases}
\]

then

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots
\]