While these are only practice problems, and therefore do not need to be turned in, I suggest that everyone try to do these. There is no problem set this week as we wrap up the course, which means that the only way to get any familiarity with Taylor Series is to do these problems.

Something to keep in mind: since the class has such a ridiculous schedule, much of the material here won’t be covered until Monday or Tuesday. So don’t panic if you haven’t seen all of this yet—just try to do as the problem asks, and you’ll get more familiar with the concepts as the week progresses.

Problems:

Power Series and Analytic Functions:

1. Give the radii of convergence for the following power series. At which endpoints of the interval of convergence, if any, does the series converge?

   (a). \[ \sum_{n=1}^{\infty} \frac{x^n}{n3^n} \]

   We compute

   \[ \alpha = \lim_{n \to \infty} \left| \frac{1}{(n+1)3^{n+1}} \frac{n3^n}{1} \right| = \frac{1}{3} \lim_{n \to \infty} \left| \frac{n}{n+1} \right| = \frac{1}{3} \]

   So \( R = 1/\alpha = 3 \). So the power series converges in \((-3, 3)\). At the endpoint \(-3\) we get the alternating harmonic series, which converges; at the endpoint 3 we get the harmonic series which diverges. So the interval of convergence is \([-3, 3)\).

   (b). \[ \sum_{n=1}^{\infty} (2n-1) \]

   We compute

   \[ \alpha = \lim_{n \to \infty} \left| \frac{1}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1) \cdot (2n+1)} \right| = 0 \]

   so \( R = 1/\alpha = \infty \) and the series converges on all of \( \mathbb{R} \).

   (c). \[ \sum_{n=2}^{\infty} \frac{1}{\ln n} (x-3)^n \]

   We compute

   \[ \alpha = \lim_{n \to \infty} \left| \frac{1}{\ln n + 1} \frac{\ln n}{1} \right| = \lim_{x \to \infty} \left| \frac{\ln x}{\ln (x+1)} \right| = 1 \]

   so \( R = 1/\alpha = 1 \), so the series converges for all \( x \in (2, 4) \). At the endpoints, the series for \( x = 2 \) converges by the alternating series test; it diverges by the comparison test for \( x = 4 \), since for \( t \geq 2, t \geq \ln t \), so \( \frac{1}{\ln n} \geq \frac{1}{n} \) for all \( n \geq 2 \), and the comparison test gives divergence.
2. If the radius of convergence of a power series \( \sum_{n=0}^{\infty} c_n (x - a)^n \) is \( R > 0 \), then what is the radius of convergence of \( \sum_{n=0}^{\infty} nc_n (x - a)^n \)? Why?

The radius of convergence is also \( R \), since within the radius of convergence \( |x - a| < R \), \( \sum_{n=0}^{\infty} c_n (x - a)^n \) defines an analytic function, whose derivative is also analytic, and if we write \( f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n \), then
\[
\sum_{n=0}^{\infty} nc_n (x - a)^n = (x - a) f'(x)
\]
is also analytic in the same region. So it will have the same radius of convergence (this is a little non-trivial, but you should know this fact even if you don’t know the proof).

3.
(a). What is the radius of convergence of the power series
\[
f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}
\]
The radius of convergence is computed by
\[
\alpha = \lim_{n \to \infty} \left| \frac{n!}{(n+1)!} \right| = 0
\]
so \( R = \frac{1}{\alpha} = \infty \). So the radius of convergence is \( \infty \), i.e. the series converges everywhere.

(b). By a., \( f(x) \) is an analytic function defined everywhere. Expand out
\[
\left(1 + \frac{x}{n}\right)^n
\]
for \( n = 1, 2 \).

This gives \( \left(1 + \frac{1}{n}\right) \) and \( 1 + 2 \left(\frac{x}{n}\right) + \left(\frac{x}{n}\right)^2 \) for \( n = 1, 2 \) respectively.

Write out the expansion for arbitrary \( n \) in terms of binomial coefficients, then use that \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \) to see that
\[
\left(1 + \frac{x}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{n}\right)^k
\]
We expand
\[
\left(1 + \frac{x}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{n}\right)^k
= \sum_{k=0}^{n} \frac{n!}{k! (n-k)!} \frac{1}{n^k} x^k
= \sum_{k=0}^{n} \frac{n!}{(n-k)! n^k} \frac{1}{k!} x^k
= \sum_{k=0}^{n} \frac{n (n-1) (n-k+1) 1}{n^k} \frac{1}{k!} x^k
= \sum_{k=0}^{n} \frac{n n-1 \cdots n-k+1 x^k}{n \cdots k!}
\]
Note, sort of loosely because there are quite a few technical details we’ve ignored, that as \( n \) gets larger and larger these coefficients \( \frac{n-1}{n} \cdots \frac{n-k+1}{n} \) simply tend to 1, for each \( k \). So it’s quite plausible that \( (1 + \frac{x}{n})^n \to \sum_{k=0}^{\infty} \frac{x^k}{k!} \) for all \( x \in \mathbb{R} \).
This shows (modulo some unpleasant details) that for each $x \in \mathbb{R}$, $e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = \sum_{k=0}^{\infty} \frac{x^k}{k!} = f(x)$.

4. (a) Starting with the equality $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, valid for $-1 < x < 1$, find a similar equality for

$$\sum_{n=1}^{\infty} n x^{n-1}$$

We use term by term differentiation and integration to get

$$\sum_{n=1}^{\infty} n x^{n-1} = \frac{d}{dx} \left( \frac{1}{1-x} \right) = \frac{1}{(1-x)^2} \sum_{n=1}^{\infty} \frac{x^n}{n}$$

**Hint:** For this second one, you need to check the value at $x = 0$ to confirm your guess, since it could be off by a constant.

Similarly, we have $\sum_{n=1}^{\infty} \frac{x^n}{n} = \int \frac{1}{1-x} dx = -\ln (1-x) + C$. We check $C = 0$ by plugging in $x = 0$. So

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln (1-x)$$

(b) Use these to evaluate the values of $\sum_{n=1}^{\infty} n x^{n-1}$ for $|x| < 1$

We’ve already done this: $\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$.

and

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

We see $\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2} \sum_{n=1}^{\infty} n x^{n-1}$ for $x = \frac{1}{2}$. So we get

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2} \left( \frac{1}{1-\frac{1}{2}} \right)^2 = \frac{1}{2} \cdot 4 = 2$$

Notice this is legitimate since $\frac{1}{2} \in (-1, 1)$ so lies in the domain of convergence of the power series.

(c) Write down a power series centered at 0 for

$$\ln (1 + x)$$

based on your work in (a).

We have

$$\ln (1 + x) = -(- \ln (1 - t))$$

where $t = -x$. We find

$$\ln (1 + x) = -\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$
We have to find the domain of convergence here: the radius of convergence is \((-1, 1)\) since that was the radius of convergence of the geometric series (we can also check it by hand!). At the endpoint \(x = 1\) the series converges; at \(x = -1\) it diverges.

Plug in \(x = 1\) to confirm that the sum of the alternating harmonic series is \(\ln 2\).

We get

\[
\ln 2 = \ln (1 + 1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}
\]

5. Write down a power series solution to

\[
\int \frac{1}{1+x^4} \, dx
\]

We have

\[
\frac{1}{1-t} = 1 + t + t^2 + \ldots = \sum_{k=0}^{\infty} t^k
\]

so taking \(t = -x^4\), we have

\[
\frac{1}{1+x^4} = 1 - x^4 + x^8 - \ldots = \sum_{k=0}^{\infty} (-1)^k t^{4k}
\]

and integrating termwise we find

\[
\int \frac{1}{1+x^4} \, dx = C + \sum_{k=0}^{\infty} (-1)^k \frac{1}{4k+1} t^{4k+1}
\]

How would you evaluate this integral if you weren’t using power series? Don’t actually do it unless you’re feeling masochistic, just tell me. (The polynomial factors as \(x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)\))

Would use partial fractions, completing the square, substitution and possibly trig substitution to evaluate the integral.

**Taylor Series:**

1. Suppose a function \(f(x)\) has a Taylor series \(\sum_{n=0}^{\infty} a_n (x-a)^n\) centered at \(a\) with radius of convergence \(R > 0\). How can you tell that \(f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n\) in \((a-R, a+R)\)? What I’m really asking: what inequality do you have to use?

We have to use Taylor’s inequality together with a bound on \(f(x)\) and all of its derivatives, valid for all \(x \in (a-R, a+R)\).

2. Find the Maclaurin series of \(\cos(x^4)\).

We know the Maclaurin series of \(\cos t\), it is

\[
\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!}
\]

so taking \(t = x^4\), we find

\[
\cos x^4 = 1 - \frac{x^8}{2!} + \frac{x^14}{4!} - \frac{x^{24}}{6!} + \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{8k}}{(2k)!}
\]
(b).

\[
f(x) = \begin{cases} 
  e^{-1/x^4} & \text{if } x \neq 0 \\
  0 & \text{if } x = 0
\end{cases}
\]

Here we can’t be sneaky and use Maclaurin series we already know to get the answer. So instead we have to compute derivatives. Note that \( f(x) \) has continuous \( n \)-th derivative for all \( n \), since

\[
\lim_{x \to 0} \frac{d^n}{dx^n} \left( e^{-1/x^4} \right) = 0
\]

for all \( x \). For example, the first derivative

\[
\frac{d}{dx} \left( e^{-1/x^4} \right) = \frac{4}{x^5} e^{-1/x^4}
\]

and the limit of this, as \( x \to 0 \), is 0 (Check this!). Similarly

\[
\frac{d^2}{dx^2} \left( e^{-1/x^4} \right) = -\frac{20}{x^6} x^{-1/x^4} + \left( \frac{4}{x^5} \right)^2 e^{-1/x^4}
\]

which also tends to 0 as \( x \to 0 \). We can see, by repeated application of the product rule, that

\[
\frac{d^n}{dx^n} \left( e^{-1/x^4} \right) = e^{-1/x^4} \left( \sum_{l=1}^{N} c_l \frac{x^l}{x^l} \right)
\]

where \( N \) depends on \( n \) and \( c_l \) are the coefficients we get by repeated differentiation. The limit of anything of this form, as \( x \to 0 \), is 0, so we can conclude that \( f(x) \) is \( n \)-differentiable at \( x \) for all \( x \), with continuous \( n \)-th derivative for all \( n \), and with \( f^{(n)}(0) = 0 \) for all \( n \). So the Maclaurin series is just

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0
\]

Note that the Maclaurin series of this function is not equal to the function!

\textit{Where are these convergent?}

Both of these series are convergent for all values of \( x \). The first, because the series for \( \cos x \) is; the second, because the series which is just 0 for all terms is always convergent.

3. \textit{Use series to approximate}

\[
\int_0^{1/2} x^3 \tan^{-1} x \, dx
\]

to within four decimal places. (Use Taylor’s inequality.)

4. \textit{Evaluate}

\[
\lim_{x \to 0} \frac{1 - \cos x}{1 + x - e^x}
\]

using series.

Using series, we have

\[
1 - \cos x = 1 - \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \right)
\]

\[
= \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots
\]
while
\[ 1 + x - e^x = 1 + x - \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \right) \]
\[ = -\frac{x^2}{2!} - \frac{x^3}{3!} - \ldots \]
so
\[ \lim_{x \to 0} \frac{1 - \cos x}{1 + x - e^x} = \lim_{x \to 0} \frac{x^2 - \frac{x^4}{3!} + O(x^6) \ldots}{-x^2 - \frac{x^3}{3!} - \ldots} = -1 \]

since the lowest order terms dominate. [To be perfectly rigorous in this argument, it’s better to cite Taylor’s Theorem to say that near 0,]
\[ 1 - \cos x = \frac{x^2}{2!} + O(x^3) \]
and
\[ 1 + x - e^x = -\frac{x^2}{2!} + O(x^3) \]
so their limit is $-1$.

Evaluate it again using Euler’s identity to write $\cos x = e^{ix} + e^{-ix}$ and use L’Hopital’s rule.

This is simple too: L’Hopital’s rule give
\[ \lim_{x \to 0} \frac{1 - \cos x}{1 + x - e^x} = \lim_{x \to 0} \frac{\sin x}{1 - e^x} \]
\[ = \lim_{x \to 0} \frac{\cos x}{-e^x} \]
\[ = -1 \]

You never actually have to use Euler’s Identity. Whoops!

**Sequences and Series:**

1. What does it mean for a sequence $\{x_n\}_{n=1}^{\infty}$ to converge? What does it mean for a function $f(x)$ to be continuous?

   A sequence $\{x_n\}_{n=1}^{\infty}$ converges to a limit $x$, if, for any $\varepsilon > 0$, there exists an $N$ so that for all $n \geq N$,
   \[ |x - x_n| < \varepsilon \]
   That is, given any tolerance $\varepsilon$, you can find a point after which the whole sequence is inside that tolerance.

   A function $f$ is continuous at $x$ if, for any sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \to x$,
   \[ f(x_n) = f(x) \]

   If $f$ is continuous at $x$ for all $x$, then we say $f$ is continuous.

2. In the integral test, we required that the $f$ for which $a_n = f(n)$ be continuous, positive, and non-increasing. Why do we need each of these requirements? Hint: the condition that $f$ is continuous is actually stronger than necessary. What would be a good enough condition to conclude? Think of the conditions we needed at the beginning of the class, when we proved the fundamental theorem of calculus.

   We need continuous to ensure integrable, so that we can even talk about the integral of $f(x)$.

   We need positive to ensure that the $a_n \geq 0$, so that the partial sums $s_n = \sum_{k=0}^{n} a_n$ are a non-decreasing sequence.

   We need non-increasing to make sure that the boxes we draw lie, respectively, below the curve and above the curve.

3. Test the following for convergence or divergence. Justify your conclusions.
(a). \[
\sum_{n=1}^{\infty} \frac{n^2 + 1}{n^3 + 1}
\]
This diverges by the integral test:
\[f(x) = \frac{x^2 + 1}{x^3 + 1}\]
is decreasing, positive, and continuous, and
\[\int_{1}^{\infty} f(x) \, dx = \infty\]
by comparing \(\frac{x^2 + 1}{x^3 + 1} \geq \frac{x^2 + 1}{2x^3} = \frac{1}{2x} + \frac{1}{2x^3}\) for \(x \geq 1\) and noting that
\[\int_{1}^{\infty} \frac{1}{2x} + \frac{1}{2x^3} \, dx = \frac{1}{2} \int_{1}^{\infty} \frac{dx}{x} + \frac{1}{2} \int_{1}^{\infty} \frac{dx}{x^3}\]
The second integral converges, but the first integral diverges.

(b).
\[\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}\]
Use the limit comparison test with \(\frac{1}{n}\). We find
\[\lim_{n \to \infty} \frac{1/n}{1/n^{1+1/n}} = \lim_{n \to \infty} n^{1/n}\]
Let’s compute this limit:
\[\lim_{x \to \infty} x^{1/x} = \lim_{x \to \infty} e^{\ln(x^{1/x})} = e^{\lim_{x \to \infty} \ln x/x} = e^{\lim_{x \to \infty} \frac{1}{x} \cdot x} = e^{\lim_{x \to \infty} \frac{1}{x} \cdot x} = 1\]
using L’Hopital’s rule. So \(\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}\) converges if and only if \(\sum_{n=1}^{\infty} \frac{1}{n}\) converges. Since this latter sum diverges, the first sum diverges as well.

(c).
\[\sum_{n=1}^{\infty} \tan \left( \frac{1}{n} \right)\]
We again use the limit comparison test with \(\frac{1}{n}\) to see, since
\[\lim_{n \to \infty} \frac{1/n}{\tan (1/n)} = 1\]
that the series diverges.

(d).
\[\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}\]
This converges by the alternating series test. We have to show
\[\frac{\ln n}{\sqrt{n}}\]
is non-increasing. We do so by computing the derivative of \( \frac{\ln x}{x^{1/2}} \), which is

\[
\frac{\sqrt{x^{-1}} - \ln x \cdot \frac{1}{2\sqrt{x}}}{x} = 1 - \frac{1}{2} \ln x
\]

this is less than 0 for \( x \geq e^2 \), so we can apply the alternating series test, since it is also obvious that \( \ln n/\sqrt{n} \geq 0 \) for all \( n \geq 1 \) and \( \lim_{n \to \infty} \frac{\ln n}{\sqrt{n}} = 0 \).

**Other Questions:**

1. Evaluate the integral

\[
\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} dx
\]

Here, \( \xi \) is a constant and \( i \) is the imaginary number \( i = \sqrt{-1} \). Remember, when it comes to manipulations, treat it no differently than you would a real number.

We complete the square in the exponent to see

\[
\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} dx = \int_{-\infty}^{\infty} e^{-\pi (x^2 - 2ix\xi - \xi^2)} e^{-\pi \xi^2} dx
\]

\[
= e^{-\pi \xi^2} \int_{-\infty}^{\infty} e^{-\pi (x + i\xi)^2} dx
\]

now, doing the change of variables \( u = x + i\xi \), we see this latter integral becomes

\[
\int_{-\infty}^{\infty} e^{-\pi (x + i\xi)^2} dx = \int_{-\infty}^{\infty} e^{-\pi u^2} du
\]

Now, this latter integral is 1 since \( e^{-\pi u^2} \) is the density function of the normal probability distribution with deviation \( \sigma = \frac{1}{\sqrt{2\pi}} \) and mean \( \mu = 0 \). So

\[
\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} dx = e^{-\pi \xi^2}
\]

*Use the above to evaluate

\[
\int_{-\infty}^{\infty} e^{-\pi x^2} \cos (2\pi x \xi) dx
\]

We have, since \( e^{-2\pi i x \xi} = \cos 2\pi x \xi + i \sin 2\pi x \xi \) by Euler’s identity, that

\[
\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} dx = \int_{-\infty}^{\infty} e^{-\pi x^2} \cos 2\pi x \xi dx + i \int_{-\infty}^{\infty} e^{-\pi x^2} \sin 2\pi x \xi dx
\]

But

\[
\int_{-\infty}^{\infty} e^{-\pi x^2} \sin 2\pi x \xi dx = 0
\]

since the function \( e^{-\pi x^2} \sin 2\pi x \xi \) is odd, and the integral

\[
\int_{-\infty}^{\infty} e^{-\pi x^2} \sin 2\pi x \xi dx
\]

converges since \( \int_{0}^{\infty} e^{-\pi x^2} dx \) converges and \( \sin 2\pi x \xi \leq 1 \) for all \( x \). So

\[
e^{-\pi \xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} dx = \int_{-\infty}^{\infty} e^{-\pi x^2} \cos 2\pi x \xi dx
\]
2. Evaluate
\[ \int \frac{1}{\sinh x} \, dx \]

We compute
\[
\int \frac{1}{\sinh x} \, dx = \int \frac{2}{e^x - e^{-x}} \, dx \\
= \int \frac{2e^x}{e^{2x} - 1} \, dx
\]

taking \( u = e^x, \, du = e^x \, dx \), we have
\[
\int \frac{1}{\sinh x} \, dx = \int \frac{2du}{u^2 - 1} \\
= \int \frac{1}{u - 1} - \frac{1}{u + 1} \, du \\
= \ln (u - 1) - \ln (u + 1) + C \\
= \ln (e^x - 1) - \ln (e^x + 1) + C
\]