Additional Problems:

1. I promised you a formula for \( \int p(t) e^t \, dt \)

where \( p(t) \) is a polynomial. Here it is. Let \( n \) be the degree of the polynomial \( p(t) = a_n t^n + a_{n-1} t^{n-1} + ... + a_0 \), that is, the highest power of \( t \) in the polynomial (here \( a_n \neq 0 \)). Then

\[
\int p(t) e^t \, dt = \left( \sum_{k=0}^{n} (-1)^k p^{(k)}(t) \right) e^t + C
\]

We’re going to prove this.

(a). Consider the formula of 7.1.52,

\[
\int t^n e^t \, dt = t^n e^t - n \int t^{n-1} e^t \, dt
\]

Use this to show

\[
\int t^n e^t \, dt = \left( \sum_{k=0}^{n} (-1)^k \frac{d^k}{dt^k} (t^n) \right) e^t + C
\]

While I would prefer a completely rigorous proof, I am ok with you just writing in words how to deduce this formula from the one in 7.1.52. If rigor is your thing, use an induction argument.

Let’s do a formal proof by induction. Remember that this proceeds as follows: first we show that the statement we want to show holds in a base case \( n = 0 \); then show that if it holds for \( n - 1 \), it holds for \( n \).

The base case: we are trying to consider

\[
\int t^0 e^t \, dt = \int e^t \, dt = e^t + C = \left( \sum_{k=0}^{0} (-1)^k \frac{d^k}{dt^k} (t^0) \right) e^t + C
\]

so we have verified equality in this case.

The inductive step: We assume the statement holds for \( n - 1 \), and consider the statement for \( n \). This means we consider \( \int t^n e^t \, dt \). But we can use the formula 7.1.52

\[
\int t^n e^t \, dt = t^n e^t - n \int t^{n-1} e^t \, dt
\]

But, by the assumption, the formula holds here, i.e.

\[
\int t^{n-1} e^t \, dt = \left( \sum_{k=0}^{n-1} (-1)^k \frac{d^k}{dt^k} (t^{n-1}) \right) e^t + C
\]
So
\[
\int t^n e^t \, dt = t^n e^t - n \int t^{n-1} e^t \, dt
\]
\[
= t^n e^t - \left( \sum_{k=0}^{n-1} (-1)^k \frac{d^k}{dt^k} (t^{n-1}) \right) e^t + C
\]
\[
= t^n e^t - \left( \sum_{k=0}^{n-1} (-1)^k \frac{d^k}{dt^k} (t^n) \right) e^t + C
\]
\[
= t^n e^t + \left( \sum_{k=0}^{n-1} (-1)^{k+1} \frac{d^{k+1}}{dt^{k+1}} (t^n) \right) e^t + C
\]
\[
= \left( \sum_{k=0}^{n-1} (-1)^k \frac{d^k}{dt^k} (t^{n-1}) \right) e^t + C
\]

This is a formal proof.
If you don’t like proofs by induction, I also would have accepted something like this:
To integrate \( \int t^n e^t \, dt \), we need to integrate by parts \( n \) times. Each time we differentiate the \( t^n \) part in the \( \int udv \) part of the integration by parts formula \( \int udv = uv - \int vdu \), and this comes with a minus sign.
So each time we pick up another successive derivative of \( t^n \) with alternating sign. The \( e^t \) term remains there always.

(b). Why does this prove the formula for general polynomials? (hint: How do derivatives work with sums and multiplication by constants?)
Since polynomials \( p(t) \) are of the form \( p(t) = a_n t^n + \ldots + a_1 t + a_0 \), and everything in the formula commutes with taking sums and multiplication by constants, the formula holds for general polynomials.

(c). Write down and prove a similar formula for
\[
\int p(t) e^{at} \, dt
\]
where \( a \) is a constant.
The formula we need is
\[
\int t^n e^{at} \, dt = \frac{1}{a} t^n e^{at} - \frac{1}{a} \int nt^{n-1} e^{at} \, dt
\]
and the full formula is
\[
\int p(t) e^{at} \, dt = \left( \sum_{k=0}^{n} (-1)^k \frac{d^k}{dt^k} p(t) \frac{1}{a^{k+1}} \right) e^{at}
\]
The proof is nearly identical, we simply carry a \( \frac{1}{a} \) throughout.

2. On the first day, I computed some Riemann sums to approximate the definite integral
\[
\int_{0}^{1} x^2 \, dx
\]
(a). What is this value?
\[
\int_{0}^{1} x^2 \, dx = \left( \frac{x^3}{3} \right)_{0}^{1} = \frac{1}{3}
\]
Now that we know the answer, let’s prove this, but directly, without using the Fundamental Theorem of Calculus. Take the following partition \( P_n \) of the interval \([0, 1] : 0 < \frac{1}{n} < \frac{2}{n} < \ldots < \frac{n-1}{n} < 1 \). That is, partition
the unit interval into smaller intervals of size \( \frac{1}{n} \). That means when we write \( 0 = x_0 < x_1 < \ldots < x_n = 1 \), that \( x_k = \frac{k}{n} \).

(b). In each interval \([x_{k-1}, x_k]\), for what value of \( x_k^{\text{min}} \in [x_{k-1}, x_k] \) is \( f(x_k^{\text{min}}) = \min_{x \in [x_{k-1}, x_k]} f(x) \). That is, at what point in the interval \([x_{k-1}, x_k]\) does \( f(x) \) hit its minimum value. What about its maximum value? What are these maximum and minimum values in each interval \([x_{k-1}, x_k]\) = \([\frac{k-1}{n}, \frac{k}{n}]\)? What is \( x_k^{\text{min}} = \frac{k-1}{n} \) and \( x_k^{\text{max}} = \frac{k}{n} \) since \( x^2 \) is an increasing function in the interval \([0, 1]\). Also, clearly

\[
\begin{align*}
    f(x_k^{\text{min}}) &= \left(\frac{k-1}{n}\right)^2 \\
    f(x_k^{\text{max}}) &= \frac{k^2}{n^2}
\end{align*}
\]

Finally

\[
\Delta x_k = \frac{k}{n} - \frac{k-1}{n} = \frac{1}{n}
\]

(c). Write down the Riemann sums that I called \( \overline{S} \) and \( \underline{S} \) in class, i.e.

\[
\begin{align*}
    \overline{S} &= \underline{S}(P_n) = \sum_{k=1}^{n} f(x_k^{\text{max}}) \Delta x_k \\
    \underline{S} &= \overline{S}(P_n) = \sum_{k=1}^{n} f(x_k^{\text{min}}) \Delta x_k
\end{align*}
\]

for the values of \( f(x_k^{\text{max}}), f(x_k^{\text{min}}), \) and \( \Delta x_k \) you computed in (b). Note that we are using the notation \( \overline{S}(P_n) \) and \( \underline{S}(P_n) \) to emphasize the dependence the quantities I in class called \( \overline{S} \) and \( \underline{S} \) have on the partition \( P_n \) we took.

\[
\begin{align*}
    \overline{S}(P_n) &= \sum_{k=1}^{n} \frac{k^2}{n^2} \frac{1}{n} \\
    &= \frac{1}{n^3} \sum_{k=1}^{n} k^2
\end{align*}
\]

and

\[
\begin{align*}
    \underline{S}(P_n) &= \sum_{k=1}^{n} \frac{(k-1)^2}{n^2} \frac{1}{n} \\
    &= \frac{1}{n^3} \left( \left( \sum_{k=1}^{n} k^2 \right) - n^2 \right)
\end{align*}
\]

(d). Why do we have the following inequalities? (hint: it’s a stupidly easy reason)

\[
\overline{S}(P_n) \leq \int_{0}^{1} x^2 \, dx
\]
\[ \int_0^1 x^2 \, dx \leq \overline{S}(P_n) \]

We defined \( \int_0^1 x^2 \, dx = \max_{P_n} (S(P_n)) \geq \overline{S}(P_n) \) and \( \int_0^1 x^2 \, dx = \min_{P_n} (\overline{S}(P_n)) \leq \overline{S}(P_n) \).

(e). Use the formula

\[
\sum_{k=1}^{n} k^2 = \frac{1}{3} n(n+1) \left( n + \frac{1}{2} \right)
\]

to write \( \overline{S}(P_n) \) and \( \overline{S}(P_n) \) as functions of \( n \).

\[
\overline{S}(P_n) = \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \frac{n(n+1)(n+\frac{1}{2})}{3 n^3}
\]

and

\[
\overline{S}(P_n) = \frac{1}{n^3} \left( \left( \sum_{k=1}^{n} k^2 \right) - n^2 \right) = \frac{1}{3} \frac{n(n+1)(n+\frac{1}{2})}{n^3} - \frac{1}{n}
\]

(f). Take the limit as \( n \to \infty \) of both, and note that they both tend towards the value in (a). This means, by the Squeeze Theorem that \( \int_0^1 x^2 \, dx = \int_0^1 x^2 \, dx \). So we have shown from scratch that \( x^2 \) is integrable on \([0, 1]\) and that its integral is what the Fundamental Theorem says it is.

Both of these clearly have limit \( \frac{1}{3} \).

Extra Credit: Prove the formula \( \sum_{k=1}^{n} k^2 = \frac{1}{3} n(n+1)(n+\frac{1}{2}) \).

Induction works. You can also be creative: