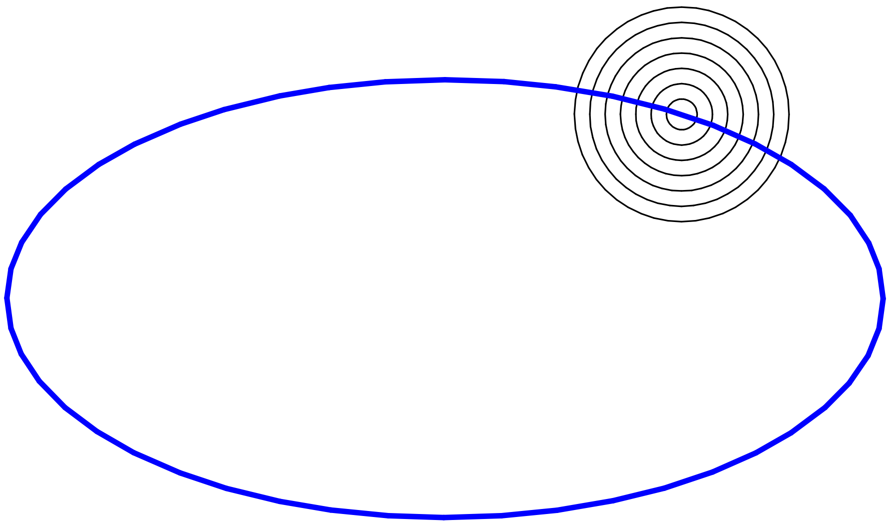
Wave fronts on random surfaces

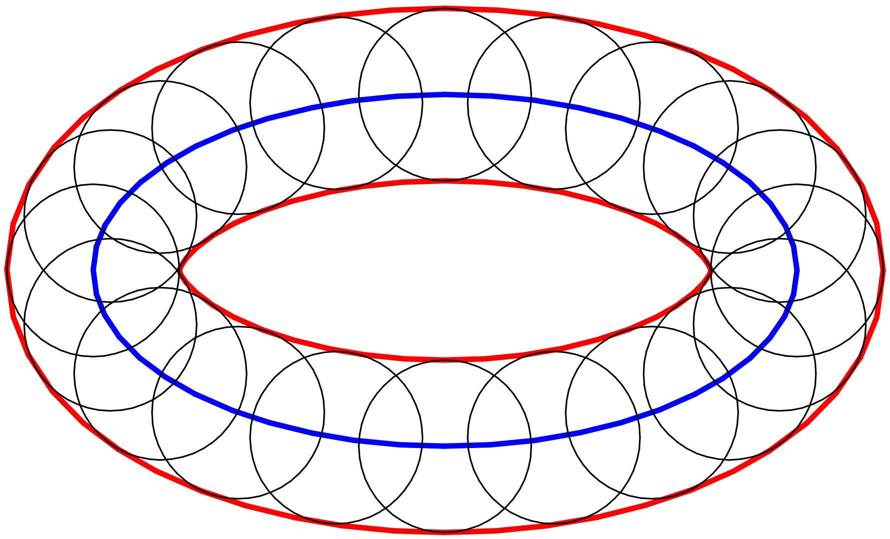
Planar wave fronts

Let every point of an ellipse emit a wave at $t = 0$



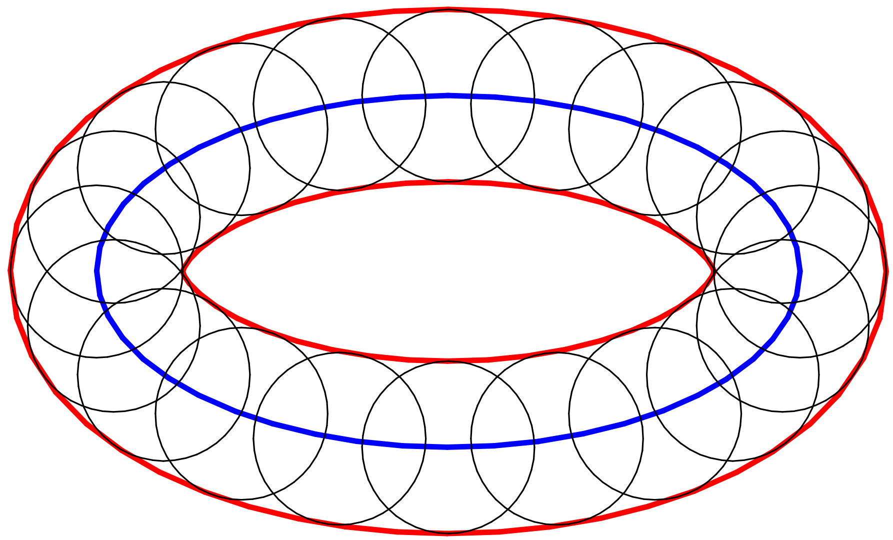
Points reached by that wave at some later time $t > 0$ form a circle.

The envelope of all these circles is called the wave front at time t.



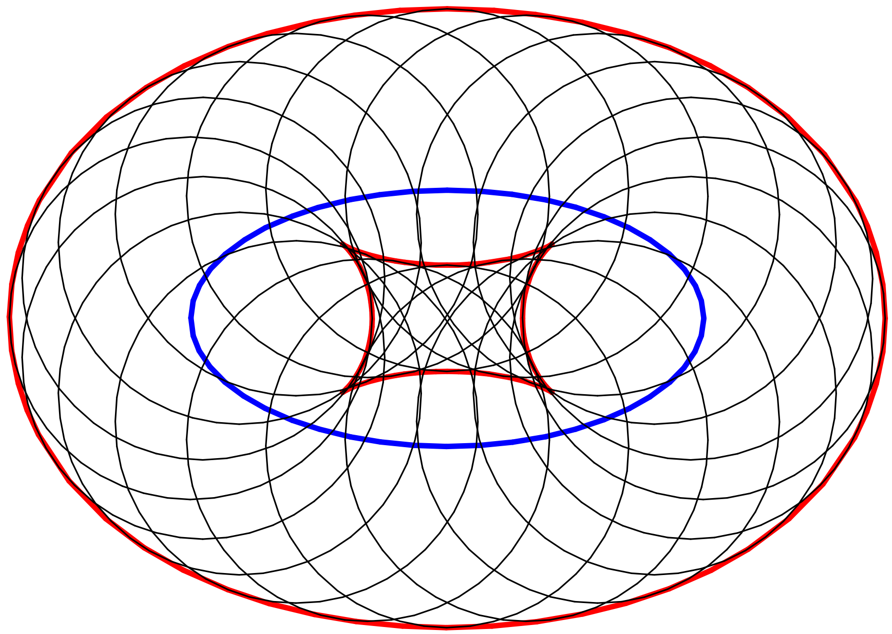
One can say that

$$\text{Curve} + \text{Circle} = \text{Wave front}$$



It is easy to see that the wave front of an ellipse is an algebraic curve which has degree 8 and genus 1, hence many (complex) singularities.

As the wave front evolves, some of these singularities become real.



Here we have 4 cusps corresponding to 4 points of the ellipse with curvature $= t^{-1}$.

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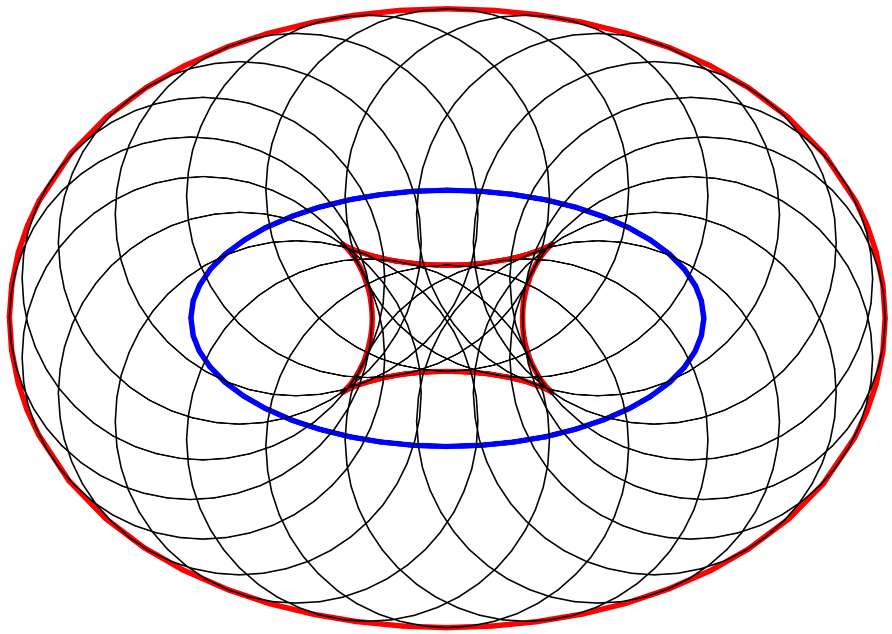

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<movie>

Geometry of wave fronts is, of course, much studied.

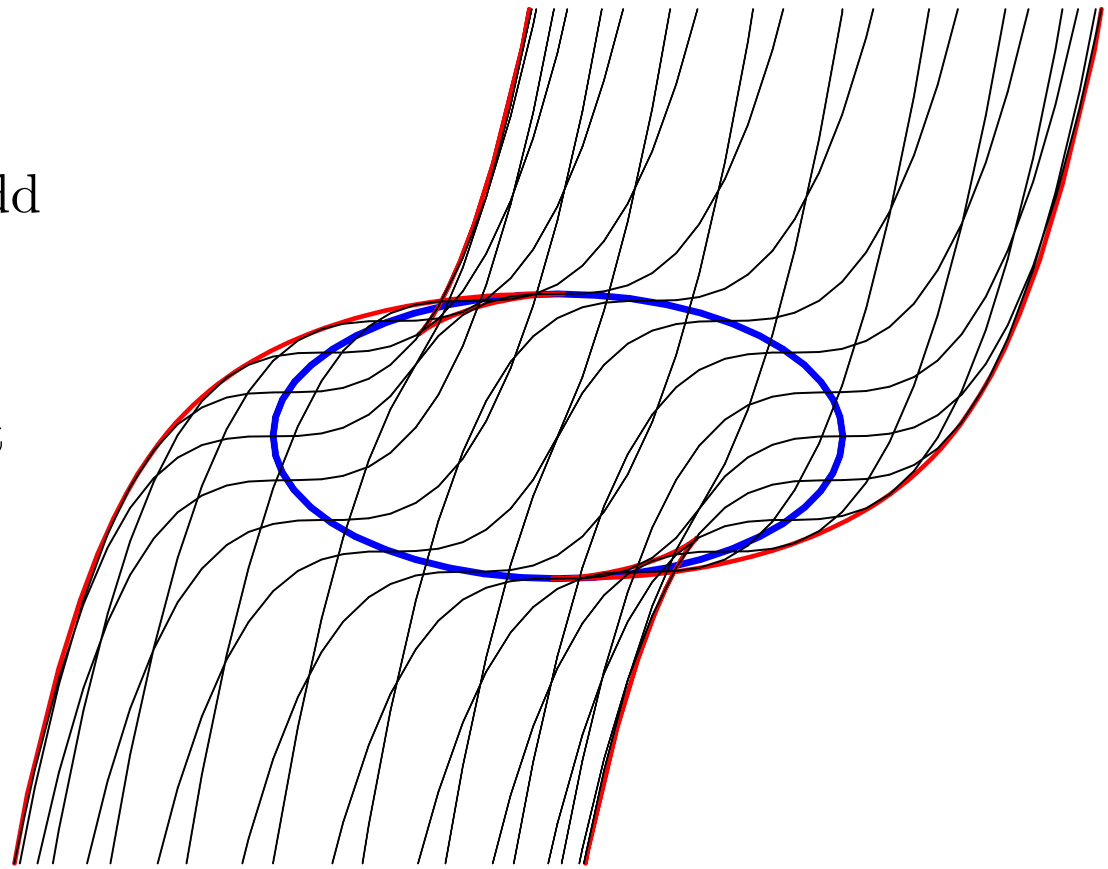
What if we replace the circle by some other algebraic curve, i.e.

$$\text{a curve } P + \text{another curve } Q = ?$$

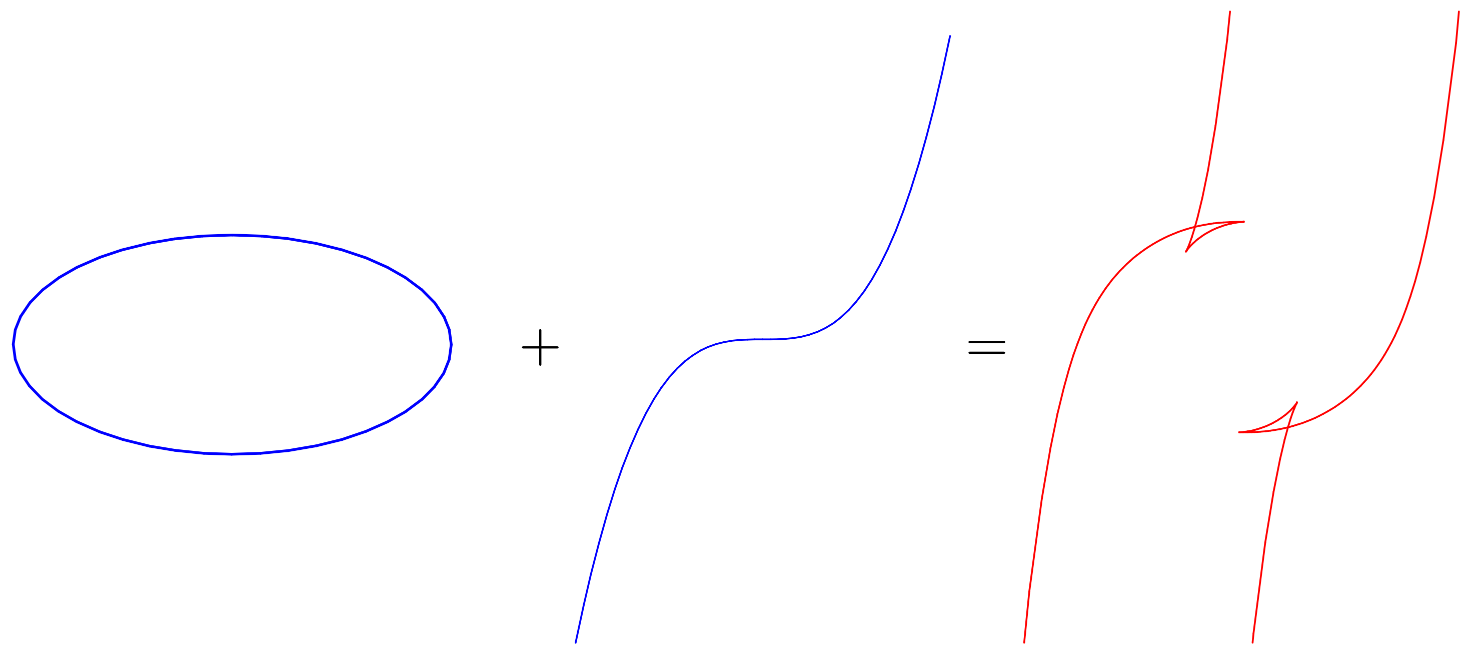
For example, let's add

$$y = x^3$$

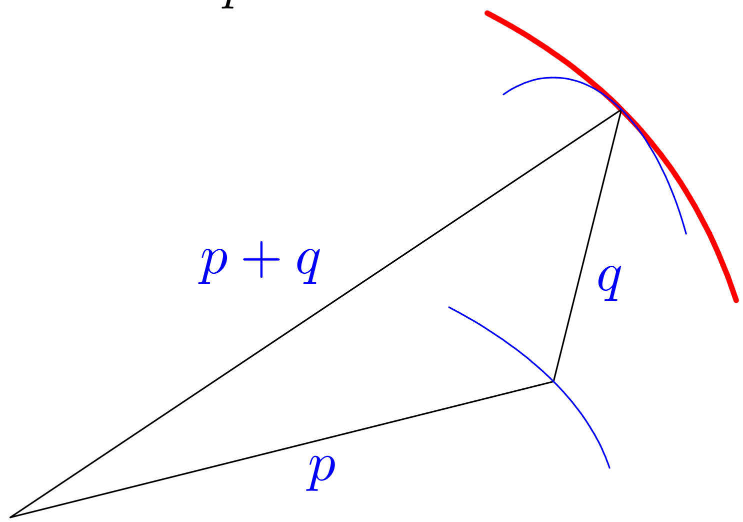
to an ellipse. We get



In other words,



It is easy to see that the envelope $P + Q$ is obtained by adding points $p \in P$ and $q \in Q$ with the same *slope of the tangent*, i.e. the same value of the *Gauss map*



Moreover,

$$\mathrm{Gauss}_{P+Q}(p+q) = \mathrm{Gauss}_{P}(p) = \mathrm{Gauss}_{Q}(q)\,.$$

Thus "+" is a commutative associative operation of plane curves.

It is not a group law, instead

$$P + Q - Q = (\deg \mathrm{Gauss}_Q) \cdot P + \dots$$

where dots stand for other irreducible components and the multiplicity of P is the degree of

$$\mathrm{Gauss}_Q : Q \to \mathbf{P}^1 \,.$$

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Note that from the previous slide that

$$\deg \mathrm{Gauss}_{P+Q} = \deg \mathrm{Gauss}_P \cdot \deg \mathrm{Gauss}_Q \ \ .$$

Addition may be replaced by multiplication. Exp and Log map one construction to another.

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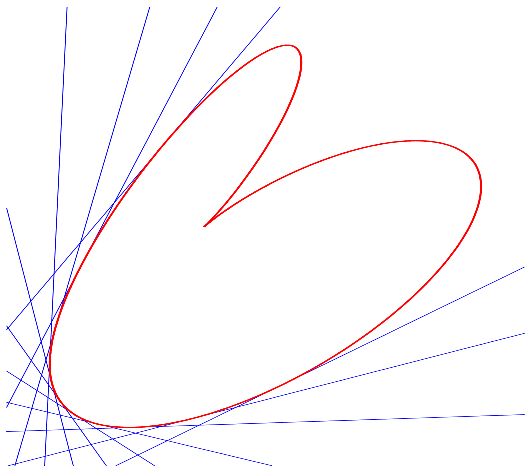
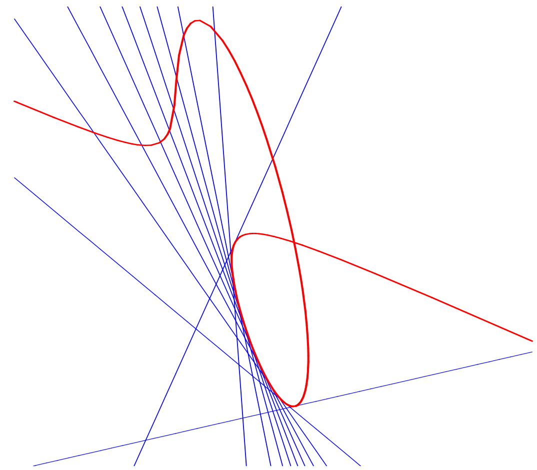
In the multiplicative case, one uses the *logarithmic Gauss map*.

For example

$$\textcolor{red}{\text{Line}}\Big/\textcolor{blue}{Q} = \text{dual curve}\,\textcolor{blue}{Q}^{\vee}\,,$$

i.e. the curve formed by all tangents $ax + by = 1$ to $Q(x, y) = 0$ in the (a, b)-plane





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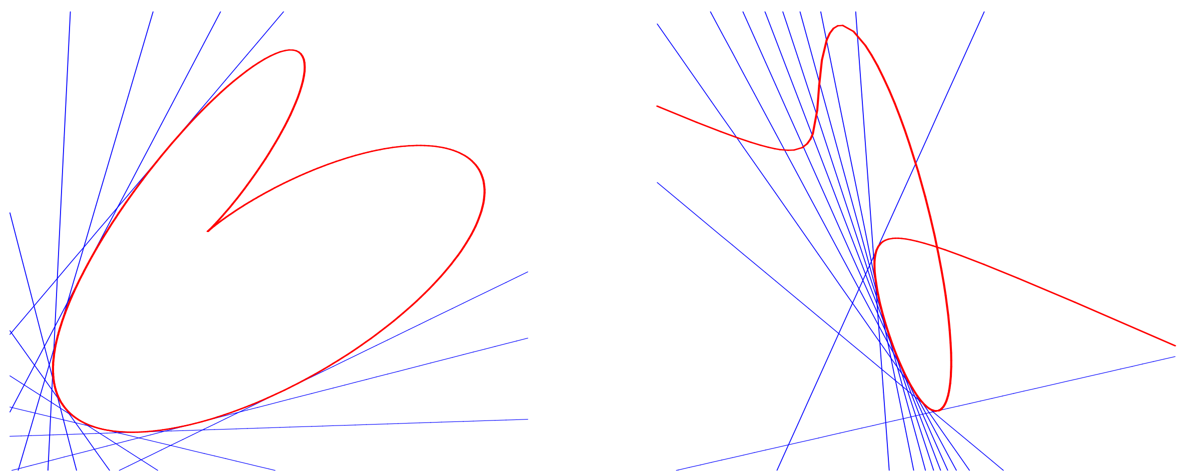

Note that the classical formula

$$Q^{\vee\vee} = Q$$

follows from $\quad \deg \, \text{LogGauss}_{\text{Line}} = 1\,.$

Classical *Plücker formulas* connect the geometry of $Q^\vee$ and Q, i.e. they relate the degrees, the numbers of cusps and nodes etc. of the two curves.

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It is possible to generalize Plücker formulas to all curves of the form P/Q, known as

Log-fronts

see `math.AG/0608018` (joint work with G. Mikhalkin).

Why is one interested in geometry of log-fronts ?

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Because they appear as

Frozen boundaries

in an interesting class of random surface models.

We are surrounded by random surfaces, in fact . . .

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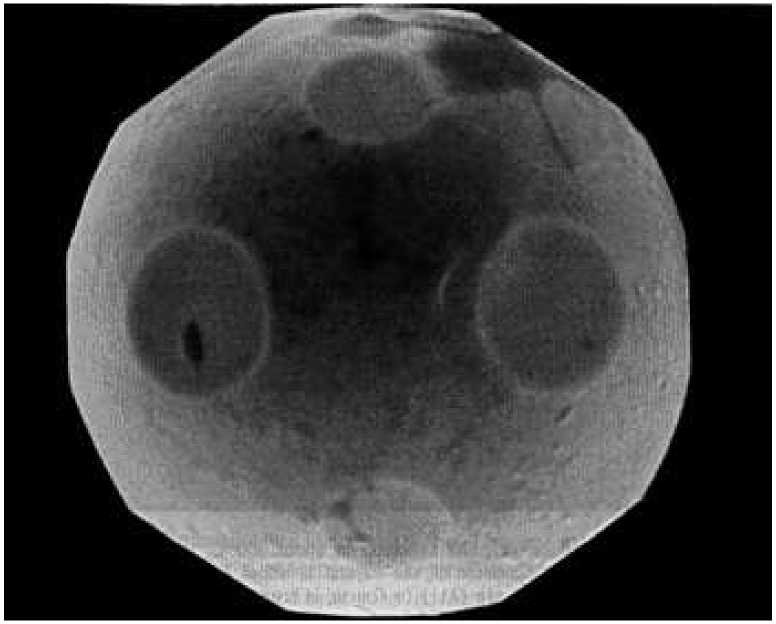
A standard challenge is to describe macroscopic equilibrium shapes from microscopic interactions, e.g.



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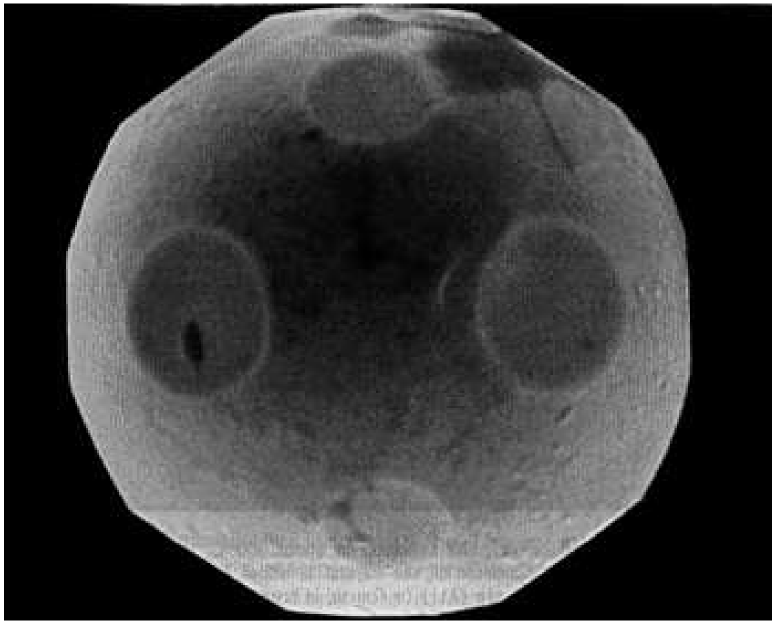
to determine the slopes and locations of crystalline facets.



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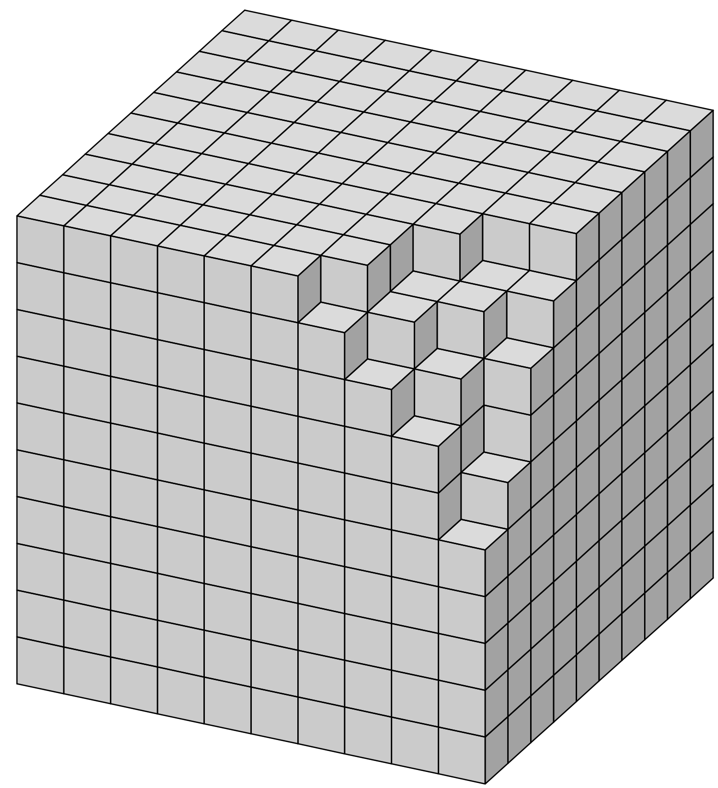
We will talk about

Stepped surfaces

Stepped surfaces are glued out of the sides of the unit cube without overhangs.



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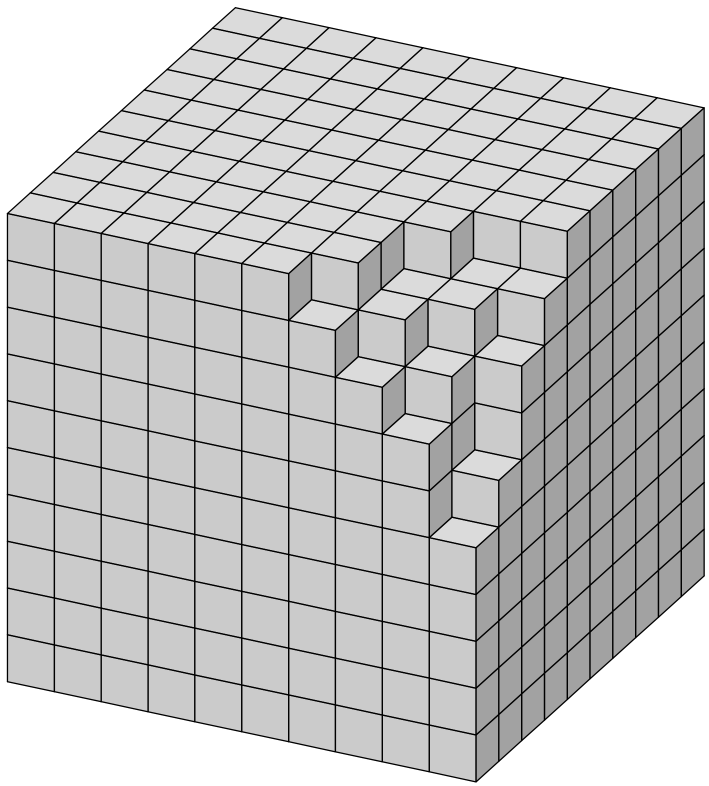
Note that for fixed boundary conditions they all have the same area.



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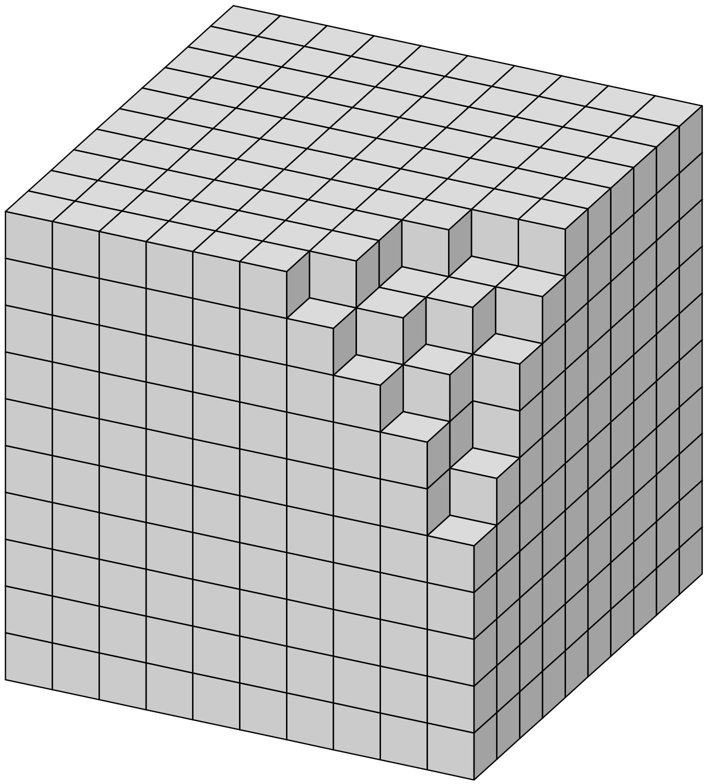
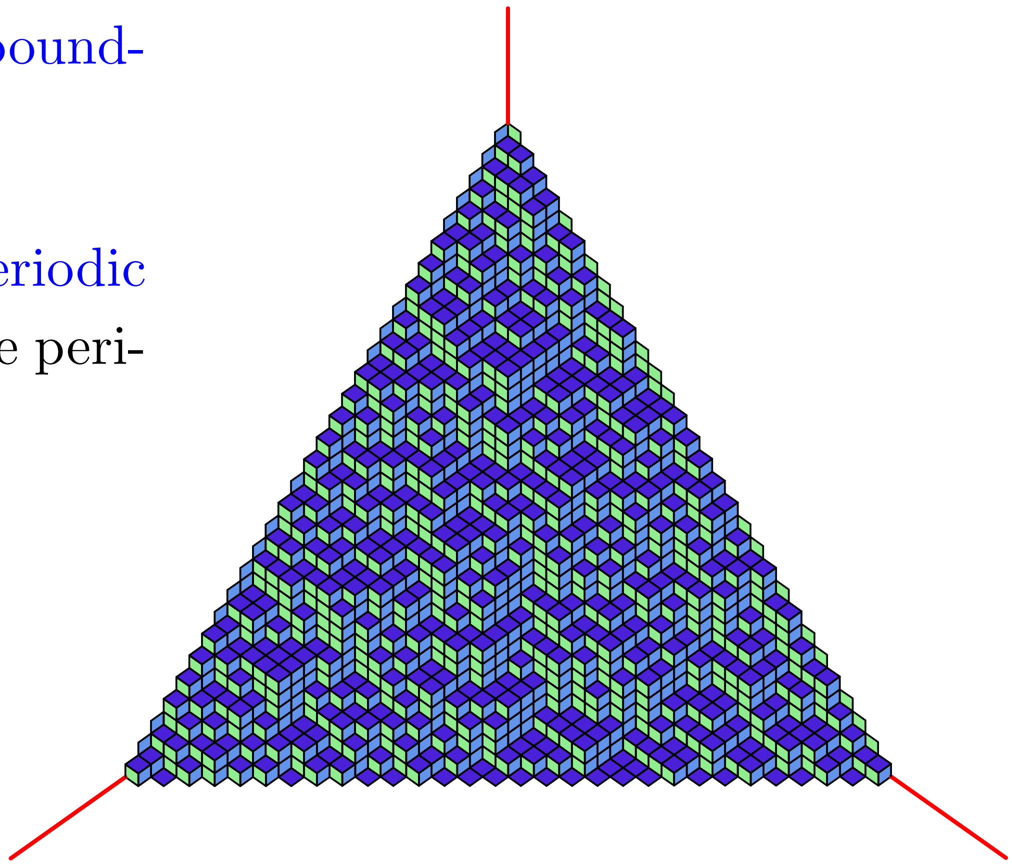
This is the 3D Ising interface at $T = 0$ and a 2D analog of a random walk.



For starters, we take all stepped surfaces spanning given boundary as equally probable.

Later, we will add periodic weights to better model the periodic nature of crystals.



The macroscopic shape forms through the

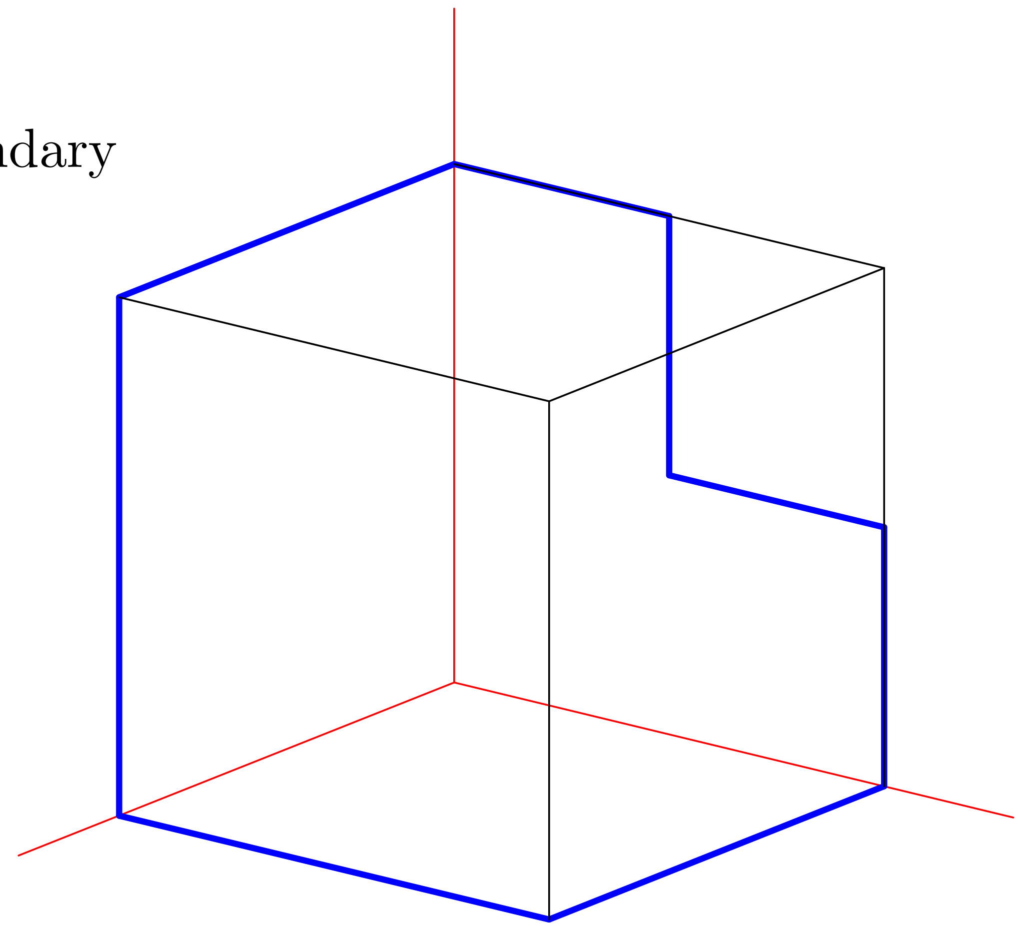
Law of large numbers

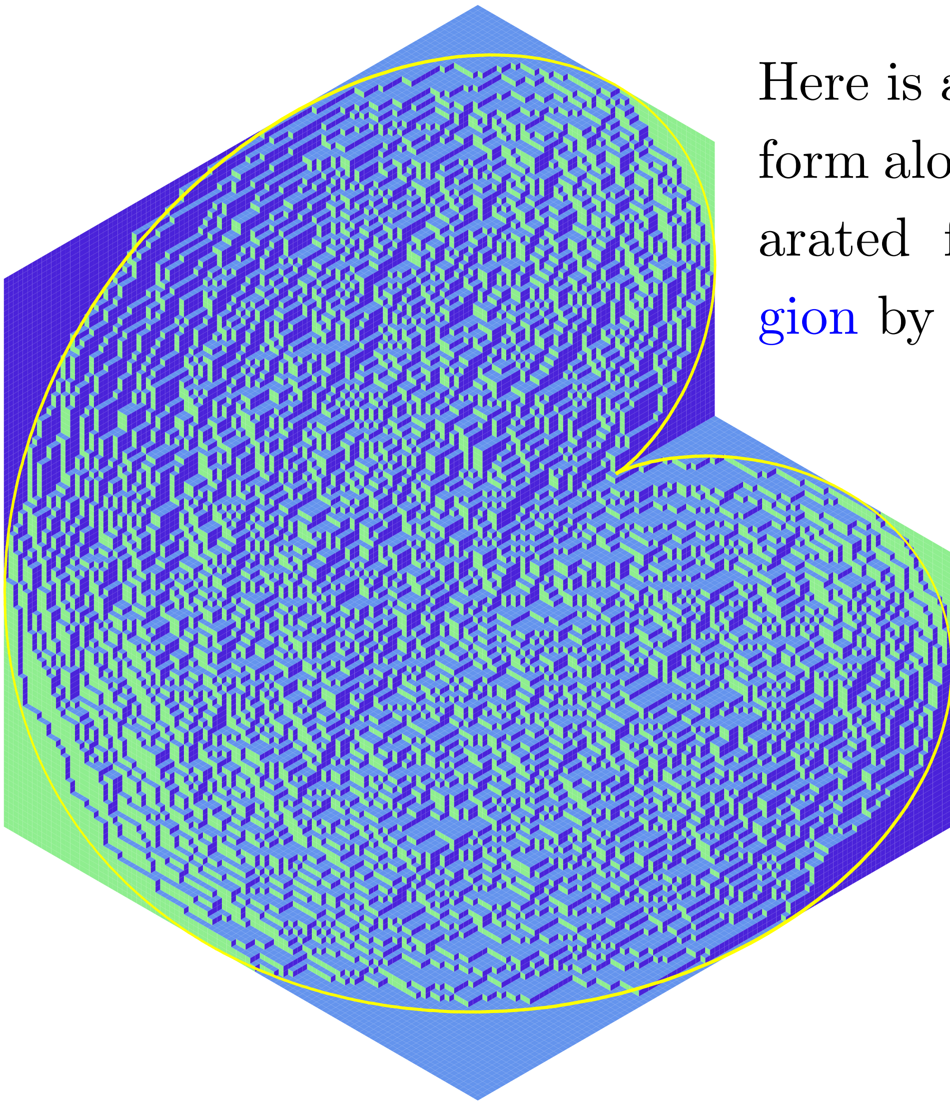
namely, as the mesh size $\to 0$, one finds that the random surface has a nonrandom limit.

A variational characterization of this limit shape was proven by Cohn, Kenyon, and Propp (2000).

How do these limit shape look ?

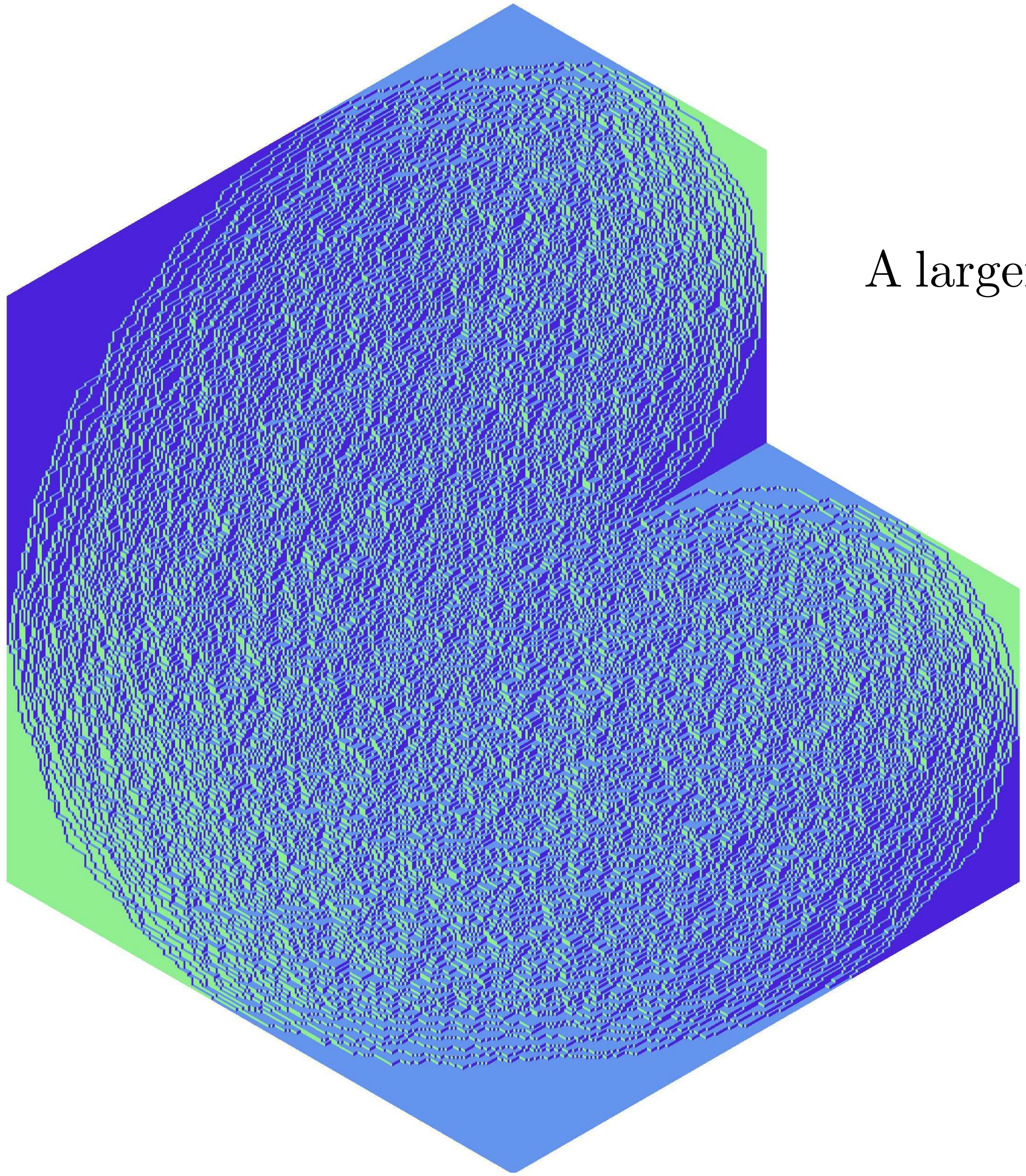
Take, for example, the boundary frame shown in blue:





Here is a simulation. Facets that form along the boundary are separated from the disordered region by the frozen boundary.

In fact, the frozen boundary is the yellow inscribed cardioid



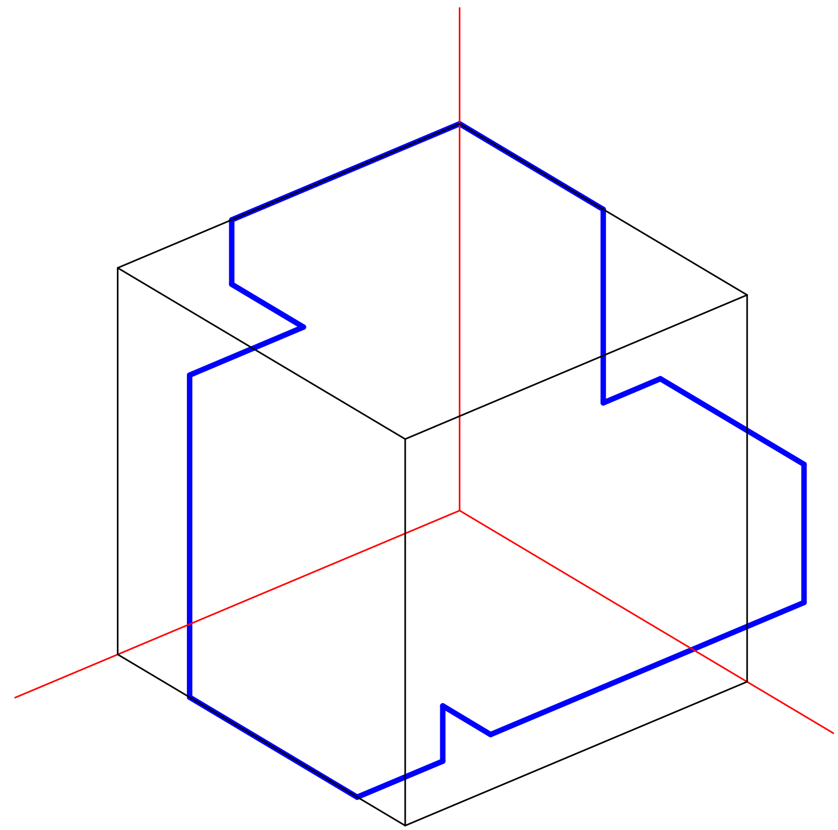
A larger simulation.

How general is such behavior ?

Every boundary contour can be approximated by polygons treated in the following

Theorem (R. Kenyon, A. O.)

Let the boundary Ω be formed by $3d$ segments in coordinate directions, then ...



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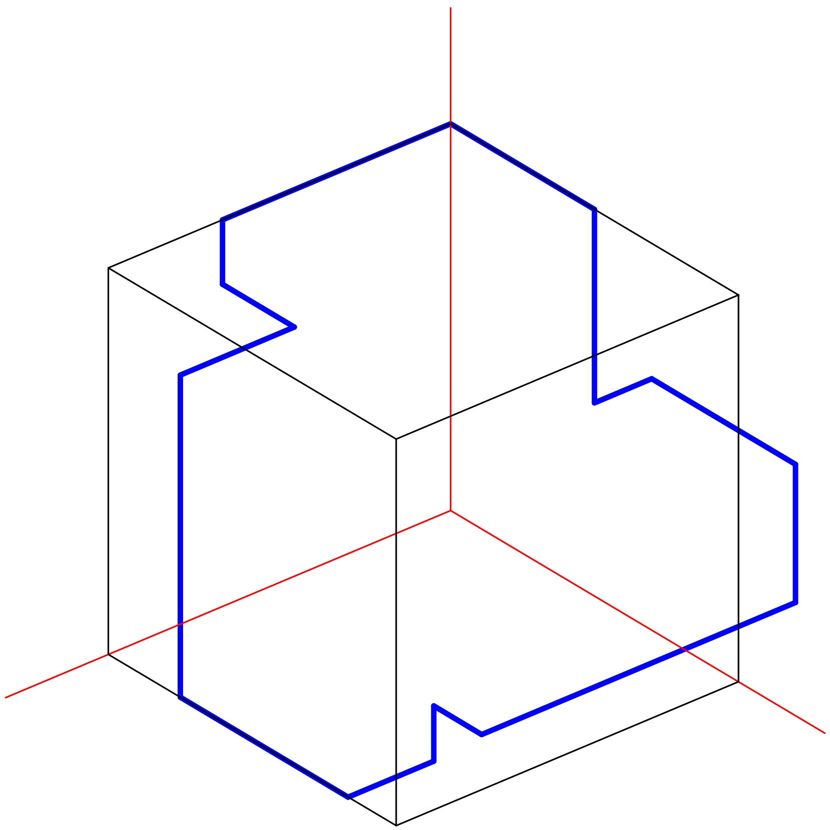

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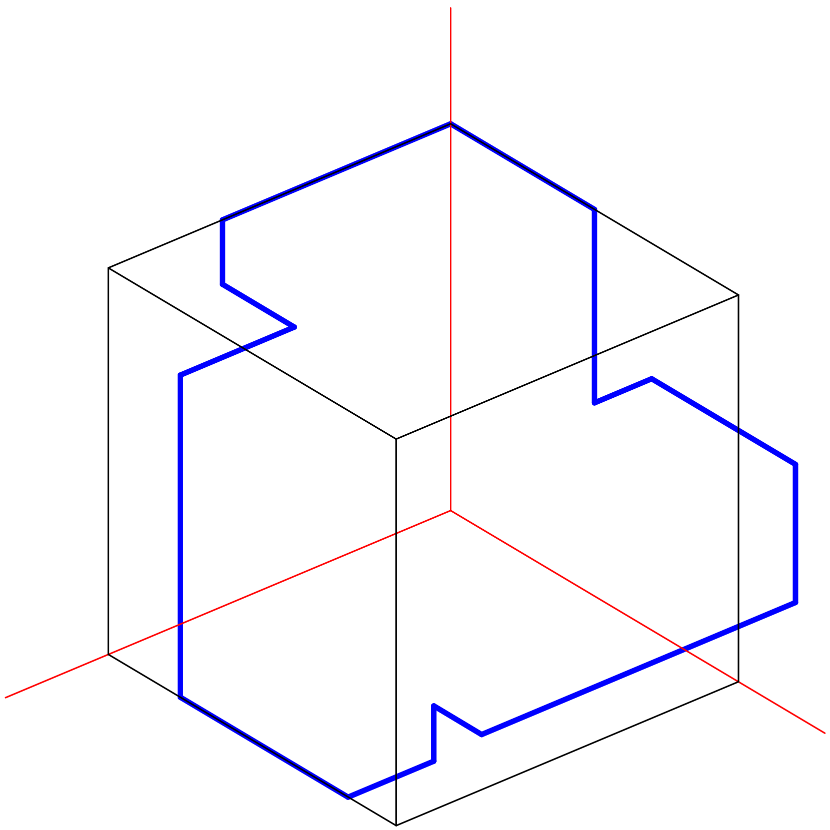

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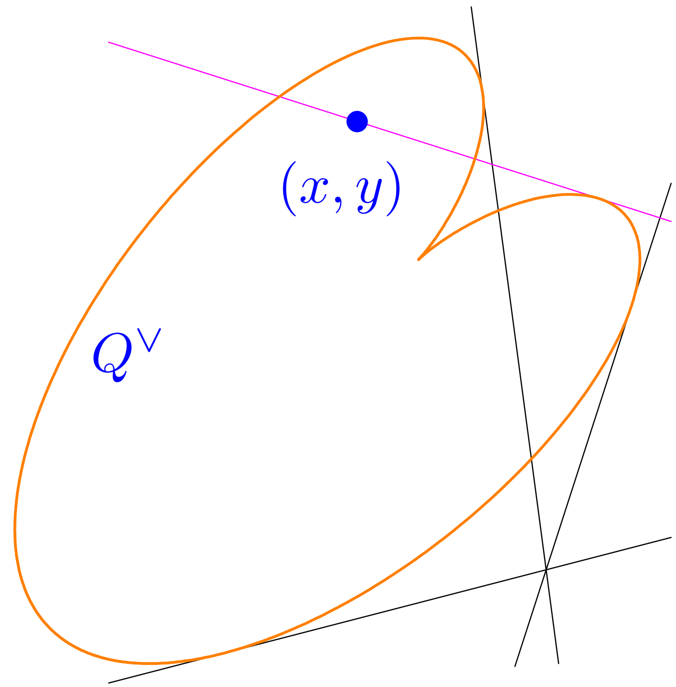
It determines the limit shape as follows ...

Theorem, continued

There is a unique (up to conjugation) complex tangent

$$ax + by + c = 0$$

to $Q^\vee$ through (x, y).



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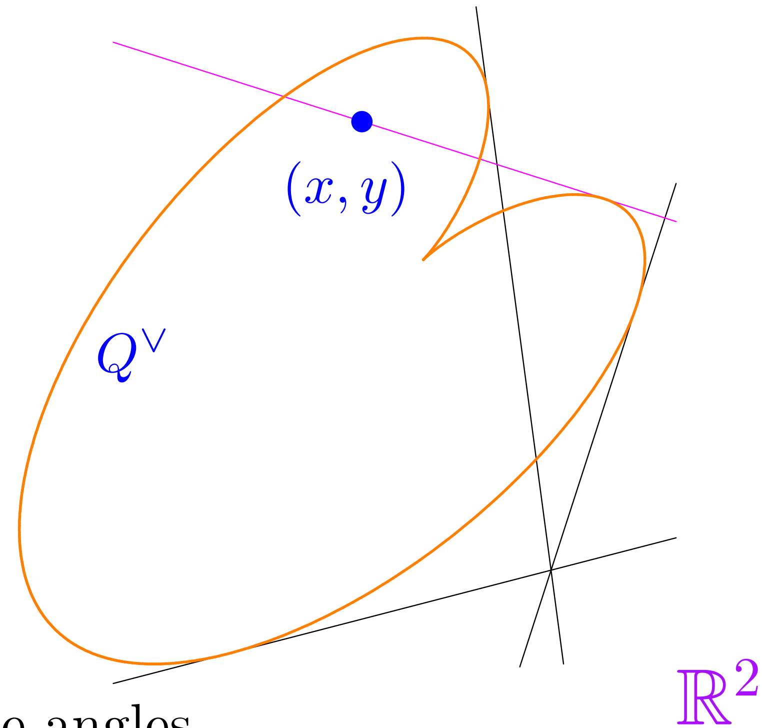
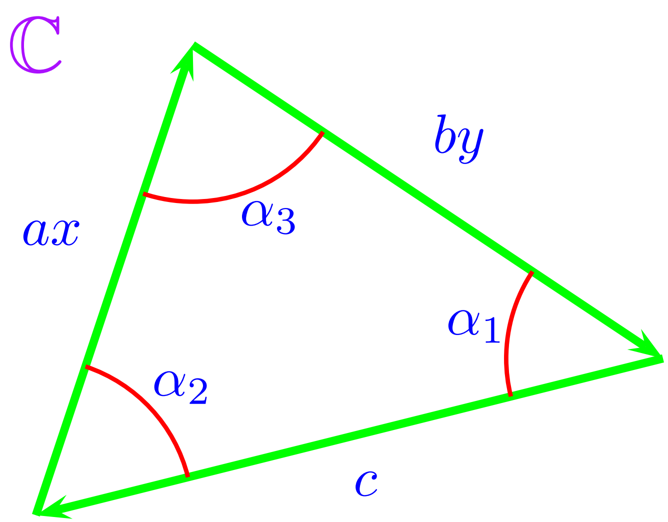




The angles

$$(\alpha_1, \alpha_2, \alpha_3)$$

give the normal to the limit shape at the point (x, y).

The map

$$(x, y) \mapsto \text{complex tangent} \in Q$$

gives the disordered region a natural complex structure, which plays a very important role.

We now add

Periodic weights

that is, we weight each box

$$\text{❒} = (i, j, k)$$

by a periodic function of $i - j$ and $j - k$ with period M.

How does this modify the limit shape ?

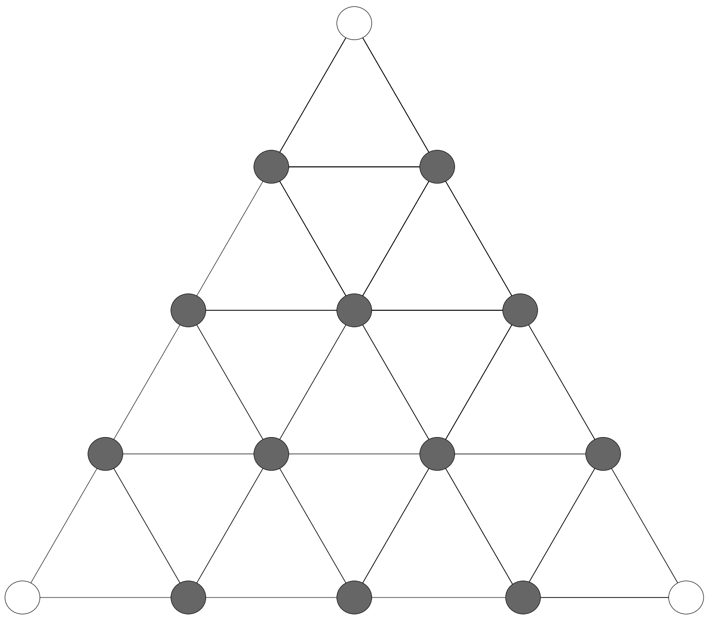
The effect of the periodic weights is to replace $Q^\vee$ as the frozen boundary by P/Q, where

- P a certain spectral curve associated to the periodic weights, while
- Q still needs to be found from the boundary conditions.

The spectral curve P is a high genus curve. Its handles and compact ovals correspond to resonant slopes.

All possible slopes form a triangle. The M-division points in it ($\bullet$), where M is the period (here $M = 4$), are resonant — they give rise to new facets.

The same points label monomials in the equation of P.



In the case of

Almost constant weights

that is, when periodic weights have the form $1 + o(\dots)$, the curve P has one large oval which approximates the curve

$$(z, w) = (t^M, (1-t)^M)$$

together with a tiny oval for each resonant slope.

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As a result, the log-front P/Q has the form $Q_0^\vee$, where Q_0 is the constant weight curve plus a tiny wavefront for each limit shape point s of resonant slope.

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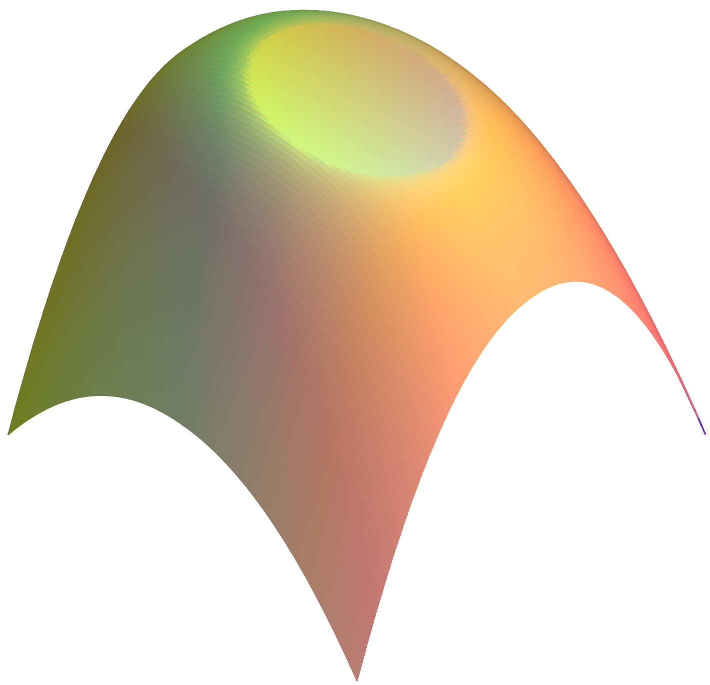
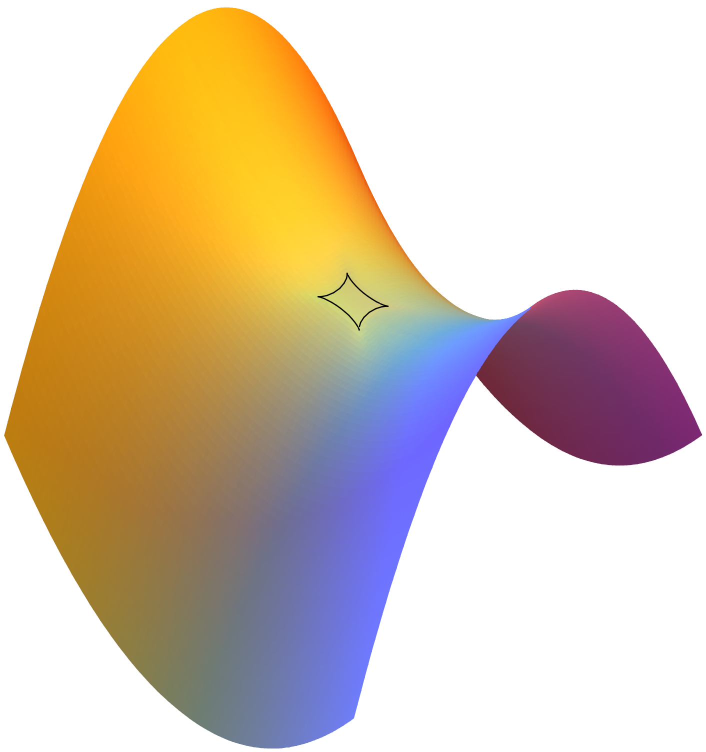
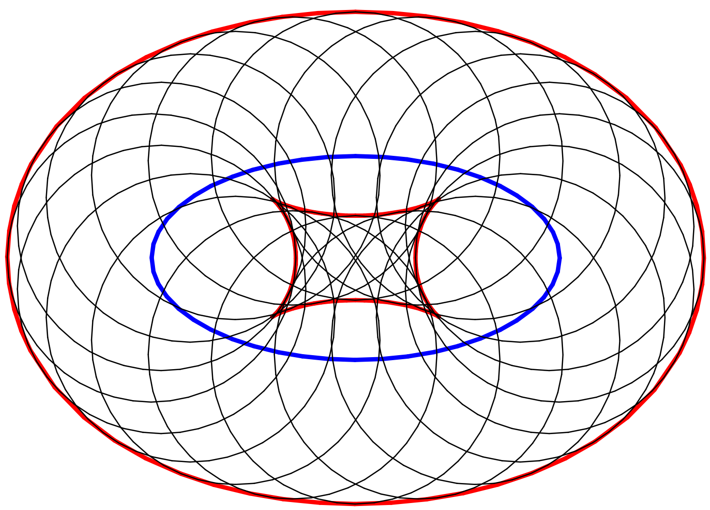
The shape of this small wavefront depends on the curvature of the limit shape at s.

If the curvature at s is nonzero, the ovals of Q are small ellipses and the wavefronts bounding the facets look like this:



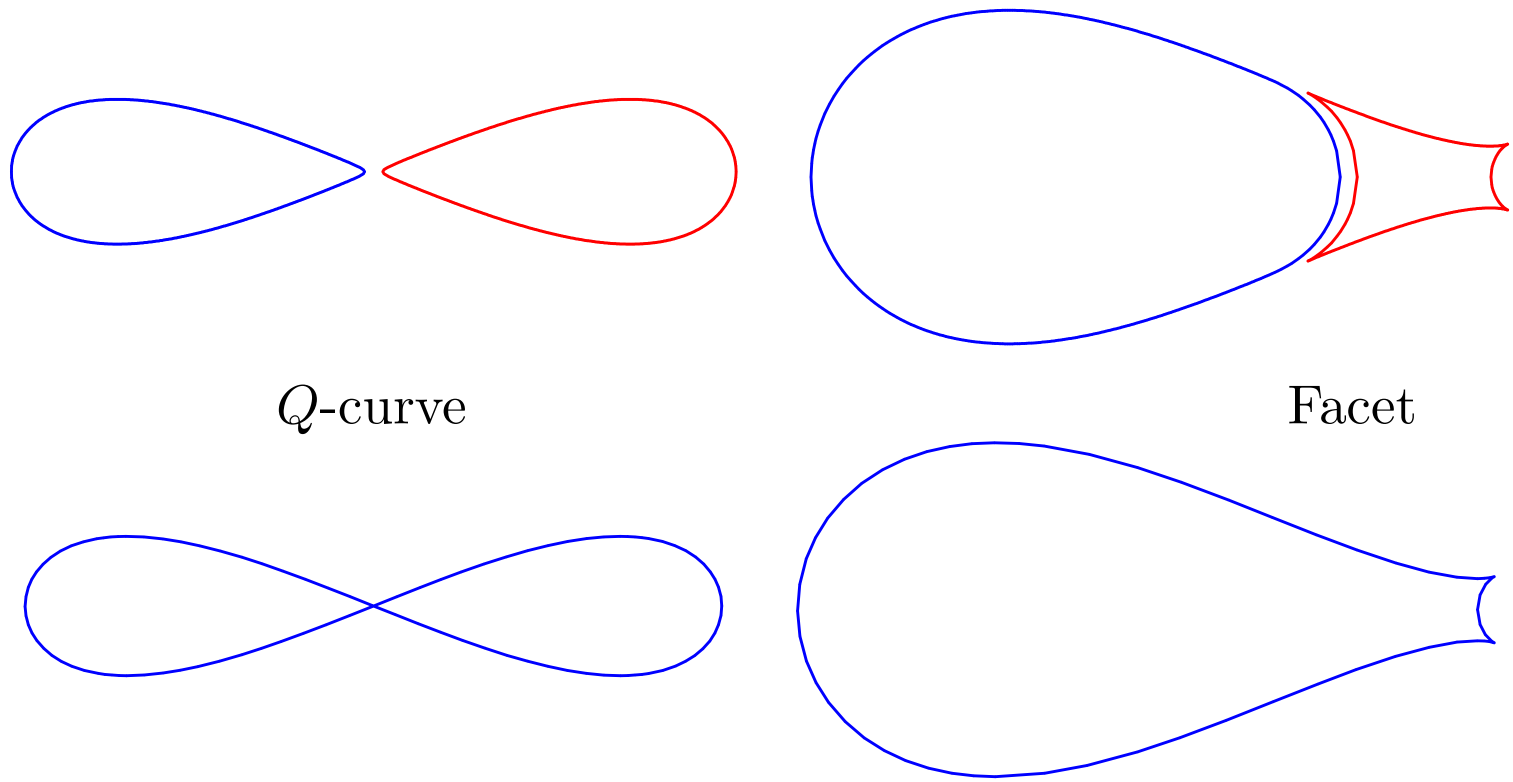




This is exactly the same picture as before.

The oriented areas of Q-ovals have to be multiples of the areas of P-ovals for the facets to be stable. Therefore, concave facets are smaller than the convex ones.

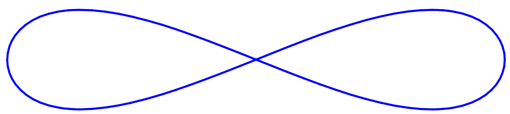
A point of zero curvature may be obtained as a coalescence of a saddle with a peak. For the Q-curve, this means formations of a node.



Note that facets merge along a portion of the P curve.

Several facets may be merging or splitting at the same time.

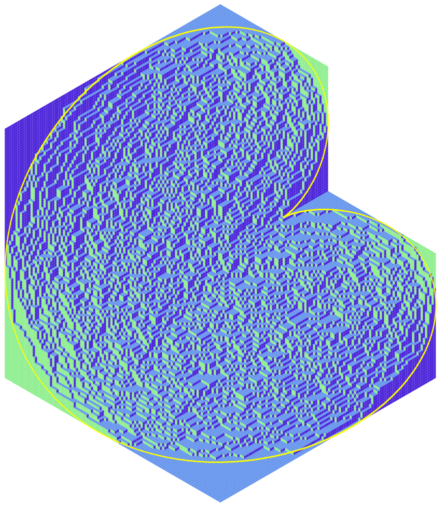
Ovals of Q with oriented zero area, such as a figure-eight oval



may disappear completely.

It remains a challenge to sort out all possible changes in the geometry of facets for general periodic weights.

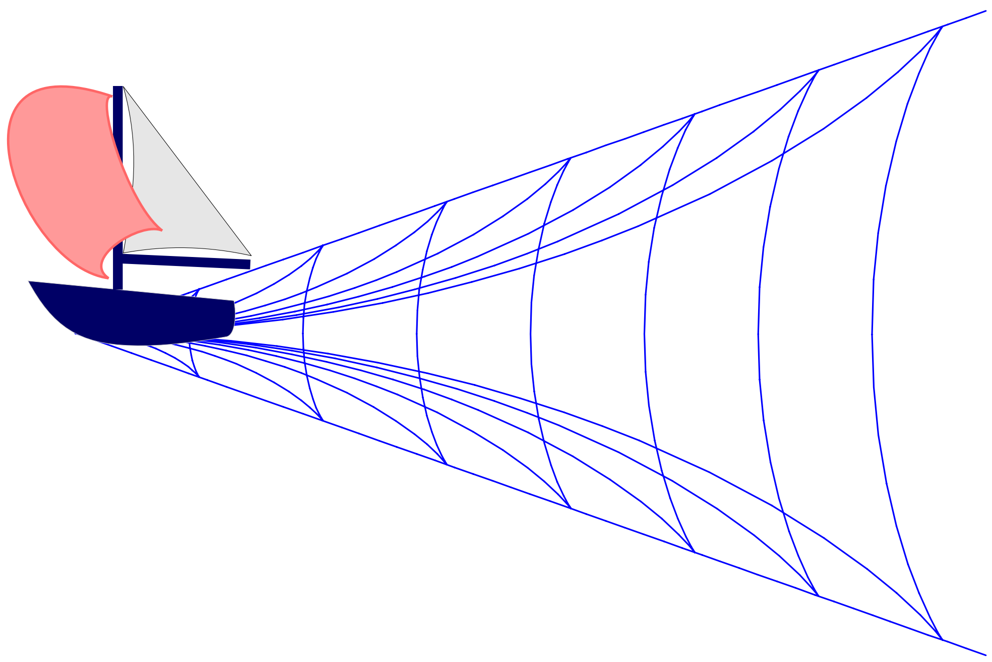
Physically, these wave patterns may be traced to the dispersion of periodic microscopic structures in the random surface (reflected by the oscillations of the correlation functions etc.)



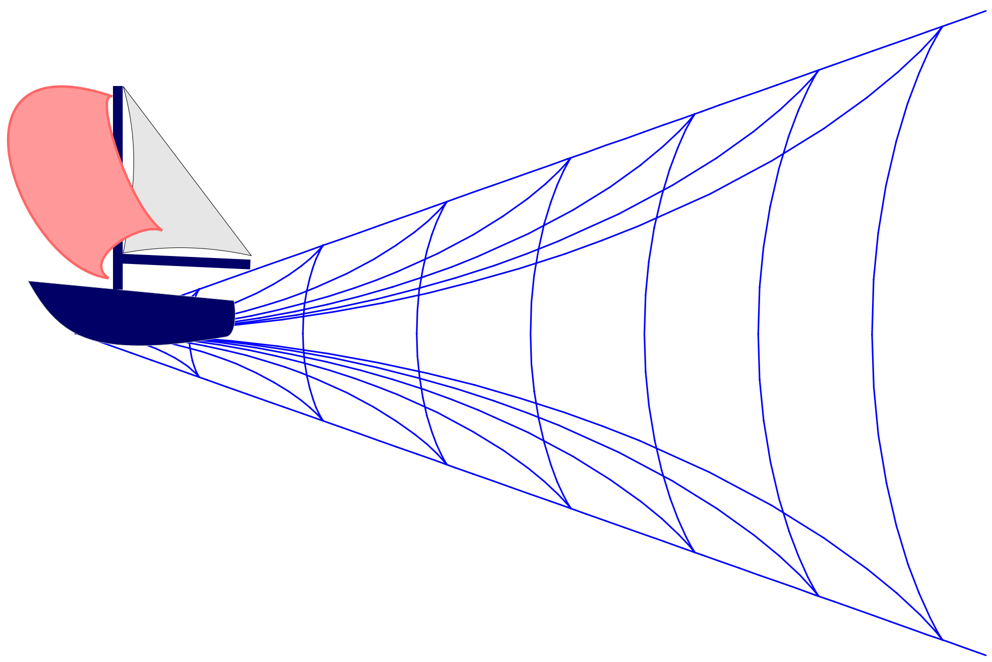
More mathematically, one can talk about *dispersion of Gibbs measures*.

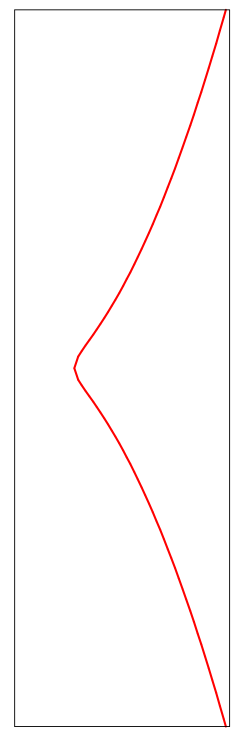
There are many parallels and formal connections to other kinds of dispersive phenomena such as water waves.

For example, the crests of ship waves



For example, the crests of ship waves are dilates of the dual curve of the relevant dispersion relation for the two wave numbers





Kelvin's angle of $\arcsin(1/3)$ is determined by the slope of the dispersion curve at its inflection point.