Problem set 1

Due Wednesday, February 1

The goal of this homework is prepare ourselves for the classifications of discrete groups $\Gamma \subset \mathrm{Iso}(\mathbb{R}^n)$ generated by reflections of a Euclidean space $\mathbb{R}^n$.

Let $\{r_\alpha\} \subset \Gamma$ be the set of reflections and let $\{H_\alpha\} \subset \mathbb{R}^n$ be the corresponding reflecting hyperplanes. We denote by

$$\Delta \subset \mathbb{R}^n$$

the closure of a connected components of $\mathbb{R}^n \setminus \{H_\alpha\}$. Let $\{H_i\}$, $i = 1, \ldots, k$, be the hyperplanes that bound Δ and $e_i \in \mathbb{R}^n$ the corresponding unit outer normal vectors.

1. Show that Γ is generated by the reflections $r_1, \ldots, r_k$ in the hyperplanes bounding Δ.

2. Show that

$$(e_i, e_j) = -\cos(\pi/m_{ij}), \quad i \neq j, \tag{1}$$

where $m_{ij} = 2, 3, \ldots, \infty$.

We collect these inner products $a_{ij} = (e_i, e_j)$ into a matrix $A = (a_{ij})$. It is convenient to assume in what follows that this matrix is *indecomposable*, that is

$$A \neq A' \oplus A''$$

for two proper submatrices A' and A'' of size $n' + n'' = n$.

3. Show that if A is decomposable then Γ is a product

$$\Gamma = \Gamma' \times \Gamma'' \subset \mathrm{Iso}(\mathbb{R}^{n'}) \times \mathrm{Iso}(\mathbb{R}^{n''})$$

of two groups generated by reflections and $\Delta = \Delta' \times \Delta''$.

4. If A is indecomposable then either $k = n+1$ and Δ is a simplex or $k = n$ and Δ is a cone over a simplex. These cases are called *parabolic* and *elliptic*, respectively[1].

[1] Elliptic groups generated by reflections fix a point in $\mathbb{R}^n$ and correspond to reflection groups of isometries of the sphere S^{n-1}.

5. Show that in the elliptic case A determines Δ up to an isometry of $\mathbb{R}^n$, while in the parabolic case it determines it up to an isometry and scaling.

6. An indecomposable matrix A of the form (1) corresponds to an elliptic reflection group if and only if it is positive definite. It corresponds to a parabolic reflection group if it has corank 1 and all of its principal minors are positive definite.

7. If A is not positive definite, then there is a vector v all of which coordinates are positive such that

$$(Av, v) \leq 0\,.$$