Problem set 1

Due Wednesday, February 1

The goal of this homework is prepare ourselves for the classifications of discrete groups \( \Gamma \subset \text{Iso}(\mathbb{R}^n) \) generated by reflections of a Euclidean space \( \mathbb{R}^n \).

Let \( \{r_\alpha\} \subset \Gamma \) be the set of reflections and let \( \{H_\alpha\} \subset \mathbb{R}^n \) be the corresponding reflecting hyperplanes. We denote by

\[ \Delta \subset \mathbb{R}^n \]

the closure of a connected components of \( \mathbb{R}^n \setminus \{H_\alpha\} \). Let \( \{H_i\}, i = 1, \ldots, k \), be the hyperplanes that bound \( \Delta \) and \( e_i \in \mathbb{R}^n \) the corresponding unit outer normal vectors.

1. Show that \( \Gamma \) is generated by the reflections \( r_1, \ldots, r_k \) in the hyperplanes bounding \( \Delta \).

2. Show that

\[ (e_i, e_j) = -\cos(\pi/m_{ij}), \quad i \neq j, \quad \text{(1)} \]

where \( m_{ij} = 2, 3, \ldots, \infty \).

We collect these inner products \( a_{ij} = (e_i, e_j) \) into a matrix \( A = (a_{ij}) \). It is convenient to assume in what follows that this matrix is indecomposable, that is

\[ A \neq A' \oplus A'' \]

for two proper submatrices \( A' \) and \( A'' \) of size \( n' + n'' = n \).

3. Show that if \( A \) is decomposable then \( \Gamma \) is a product

\[ \Gamma = \Gamma' \times \Gamma'' \subset \text{Iso}(\mathbb{R}^{n'}) \times \text{Iso}(\mathbb{R}^{n''}) \]

of two groups generated by reflections and \( \Delta = \Delta' \times \Delta'' \).

4. If \( A \) is indecomposable then either \( k = n + 1 \) and \( \Delta \) is a simplex or \( k = n \) and \( \Delta \) is a cone over a simplex. These cases are called parabolic and elliptic, respectively\(^1\).

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\(^1\)Elliptic groups generated by reflections fix a point in \( \mathbb{R}^n \) and correspond to reflection groups of isometries of the sphere \( S^{n-1} \).
5. Show that in the elliptic case $A$ determines $\Delta$ up to an isometry of $\mathbb{R}^n$, while in the parabolic case it determines it up to an isometry and scaling.

6. An indecomposable matrix $A$ of the form (1) corresponds to an elliptic reflection group if and only if it is positive definite. It corresponds to a parabolic reflection group if it has corank 1 and all of its principal minors are positive definite.

7. If $A$ is not positive definite, then there is a vector $v$ all of which coordinates are positive such that
\[
(Av, v) \leq 0.
\]