The M-theory index

joint work with Nikita Nekrasov

While my talk will be purely mathematical, it may be useful to say a few words about the physical ideas that motivate our work ...

Part I The dream of M-theory

The great Einstein equation

Einstein tensor = const * Energy-momentum tensor

still has a numer of weak spots, some of which were pointed out by Einstein himself, who said

Sie (equation) *gleicht aber einem Gebäude, dessen einer Flügel aus vorzüglichem Marmor, dessen anderer Flügel aus minderwertigem Holze gebaut ist.*

Einstein probably knew Aristotle's ὕλη meant wood in other words, the LHS of

$$R_{ab} - \frac{1}{2} R\, g_{ab} - \Lambda\, g_{ab} = - \frac{8\pi G}{c^4} T_{ab}$$

is tightly constrained by symmetry and geometry, while in the RHS one can put matter of any kind and shape

Matter and forces enter the scene through two separate doors ...

A lot of work in theoretical physics has been put into constructing a theory in which all fields and all interactions between them follow from a single geometric principle

Representation theory determines one candidate related to supergravity in 10+1 dimensions.

The fields of this supergravity are the graviton, i.e. the metric, its superpartner gravitino, and a 3-form analogous to the 4-vector potential in Maxwell theory. They form a single representation of the supersymmetry algebra.

The unique Lagrangian for the 10+1 dimensional supergravity was written down by Cremmer, Julia, and Scherk in 1978. It is nonrenormalizable, meaning that the standard QFT techniques fail to produce a quantum theory from it.

It is believed that a consistent quantum theory includes extended objects, namely M2 branes and M5 branes, analogous to the worldlines of electrically and magnetically charged particles (0-branes) in Maxwell theory. This theory-in-progress is called the M-theory.

M-theory is very difficult because it is unique.

In particular, it does not have any parameters in which one could expand it into a perturbation series.

One learns about M-theory by putting it on various manifolds with symmetries. Most importantly, for manifolds of the form $X^{10} \times S^1$ one expects an equivalence to superstrings on X with the string coupling constant determined by the size of S^1.

Despite the difficulties, many important insights into M-theory were made by C. Hull, P. Townsend, C. Vafa, E. Witten, and many others. Of course, I won't be able to survey them in this talk.

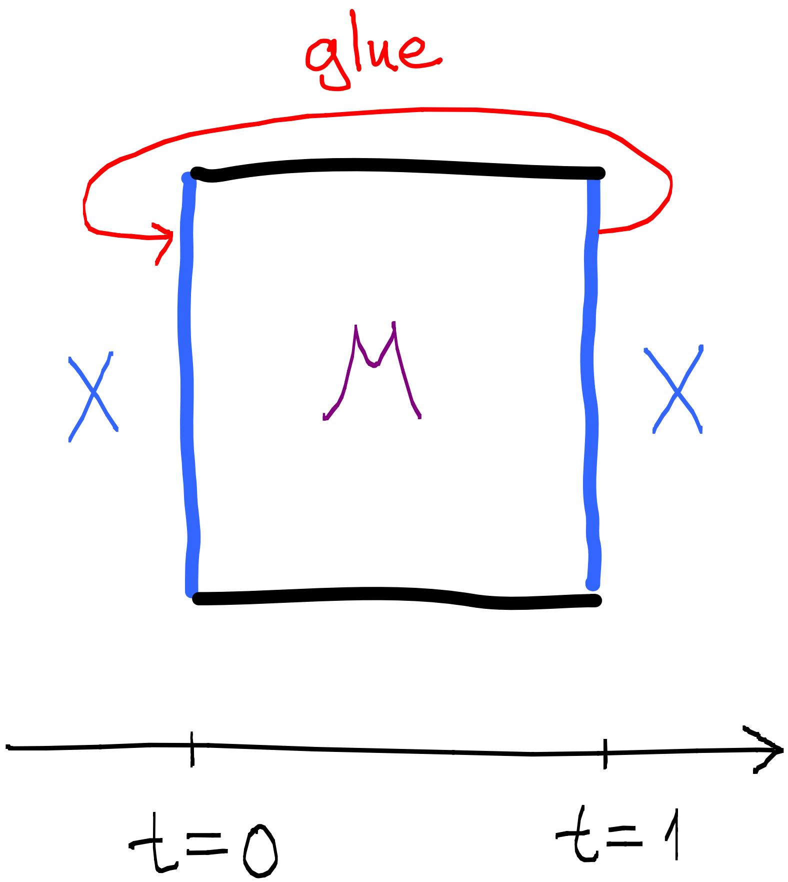
Part II Our project

Our goal is to give a mathematical definition of the M-theory index for manifolds that fiber over a circle like this:

Calabi-Yau complex 5-fold

$$M^{11} = \underset{\text{time}}{\mathbb{R}^1} \times X \Big/ (1, \text{isometry } g \text{ of } X)$$

$$S^1 = \mathbb{R}/\mathbb{Z}$$



From general principles, the partition function Z on such manifold equals

$$\mathrm{tr}_{\mathcal{H}(X)}\ \text{Evolution} \cdot \text{Gluing}$$

a vector space associated to X

operators in it

Fermions contribute to the trace with a sign

$$(\pm 1)^{\#\text{ of fermions}}$$

depending on how one glues them.

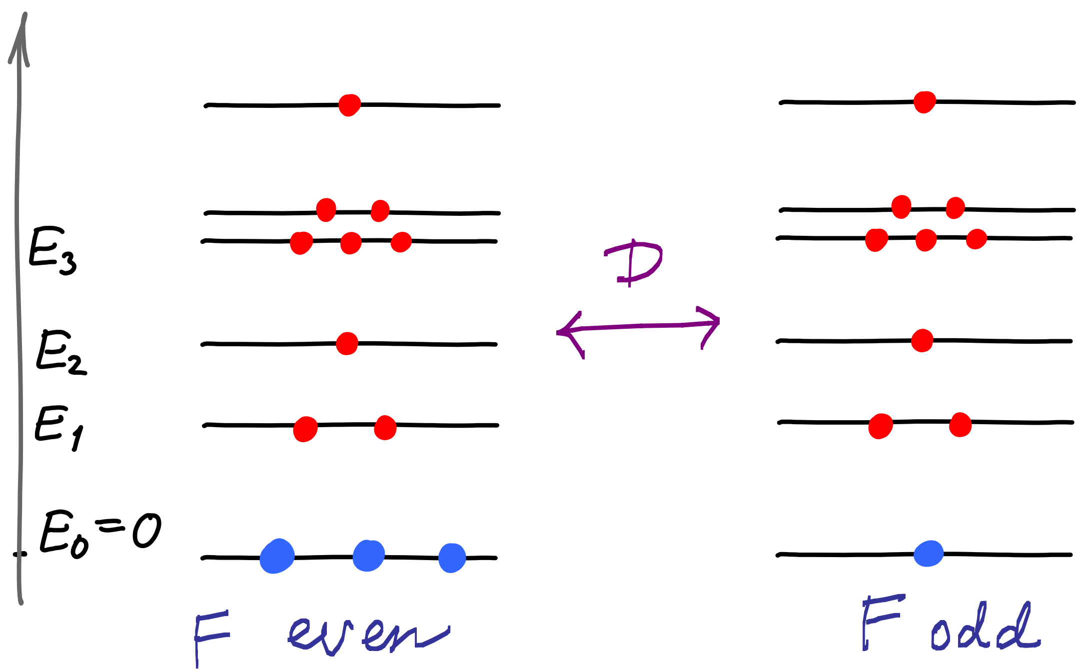
We choose (-1) and then supersymmetry gives an almost perfect cancellation of bosonic and ferminionic terms in the trace. Essentially, we are computing the index of a Dirac operator D such that

$$D^2 = \text{time translation}$$

except our D acts on a very infinite-dimensional manifold.

Dirac operator D pairs nonzero eigenvalues of D^2 of opposite parity. As a a result

$$\mathrm{tr}_{\mathcal{H}}(-1)^F e^{\beta D^2} g = \mathrm{tr}_{\mathrm{Ker}\, D}(-1)^F g$$



Here we have supersymmetric quantum mechanics on the space of 2d objects in 10d, so its Hilbert space $\mathcal{H}(X)$ is really badly infinite-dimensional

Nonetheless, the index of D is, formally, the holomorphic Euler characteristic of a coherent sheaf $\mathcal{E}$ on a certain countable union M2(X) of algebraic varieties, so, potentially, a perfectly well-defined mathematical object.

This M2(X) should be a certain compactification of the moduli space of immersed holomorphic curves in X.
It is an algebraic variety for a fixed degree of a curve, but the union over all degrees is countable. We will call it the moduli space of stable M2 branes in X.

This is a very singular space, but in a certain very technical sense, the sheaf $\mathcal{E}$, the Euler of which gives the index, is the square root $\mathcal{K}_{M2}^{1/2}$ of its canonical bundle.

We have a proposal for M2(x) as the moduli space of maps

$$f: C \longrightarrow X$$

from 1-dimensional schemes to X that are slope-stable for

$$\text{Slope}(f) = \chi(\mathcal{O}_C) \big/ \text{degree } f(C)$$

Slope-stability bounds the Euler characteristic of C in terms of degree and this is crucial for defining the M-theory index, since in M-theory there is no field to couple to the Euler chracteristic.

This proposal comes from many experiments with another conjecture...

Part III

Curves in 5-folds & 3-folds

We conjecture something special to happen for those CY 5-folds X that admit a $\mathbb{C}^*$-action of a certain kind.

This $\mathbb{C}^*$ is required to

(1) preserve the holo 5-form, and

(2) have a purely 3-dimensional set of fixed points Y.

While this class of 5-folds X is very special, it includes the usual suspects for the M-theory vacua, that is, products of a CY-3 fold with a flat space.

We then conjecture that

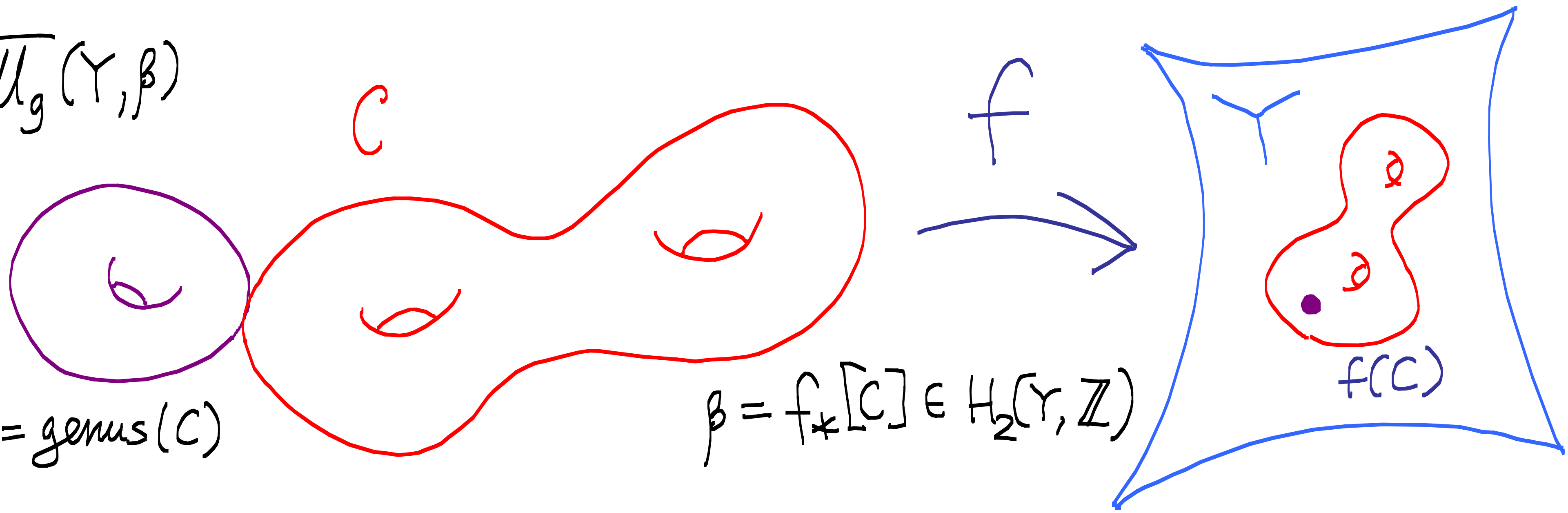
M-theory index of X = a certain specific K-theoretic Donaldson-Thomas invariant of Y

DT is one of the enumerative theories of curves in 3-folds

Counting curves in 3-folds

There are several seemingly different but deeply related ways to count curves in 3-folds. One of them is the Gromov-Witten theory, which does intersection theory on the moduli spaces of stable holomorphic maps to Y

$\overline{\mathcal{M}}_g(Y,\beta)$

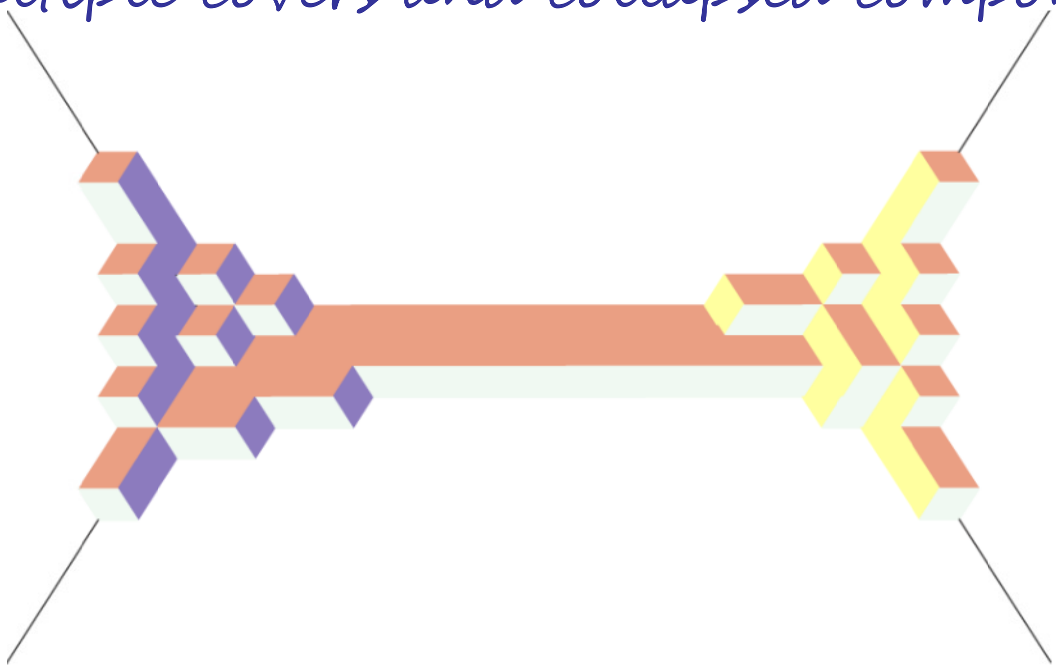


$g = \text{genus}(C)$

$\beta = f_*[C] \in H_2(Y,\mathbb{Z})$

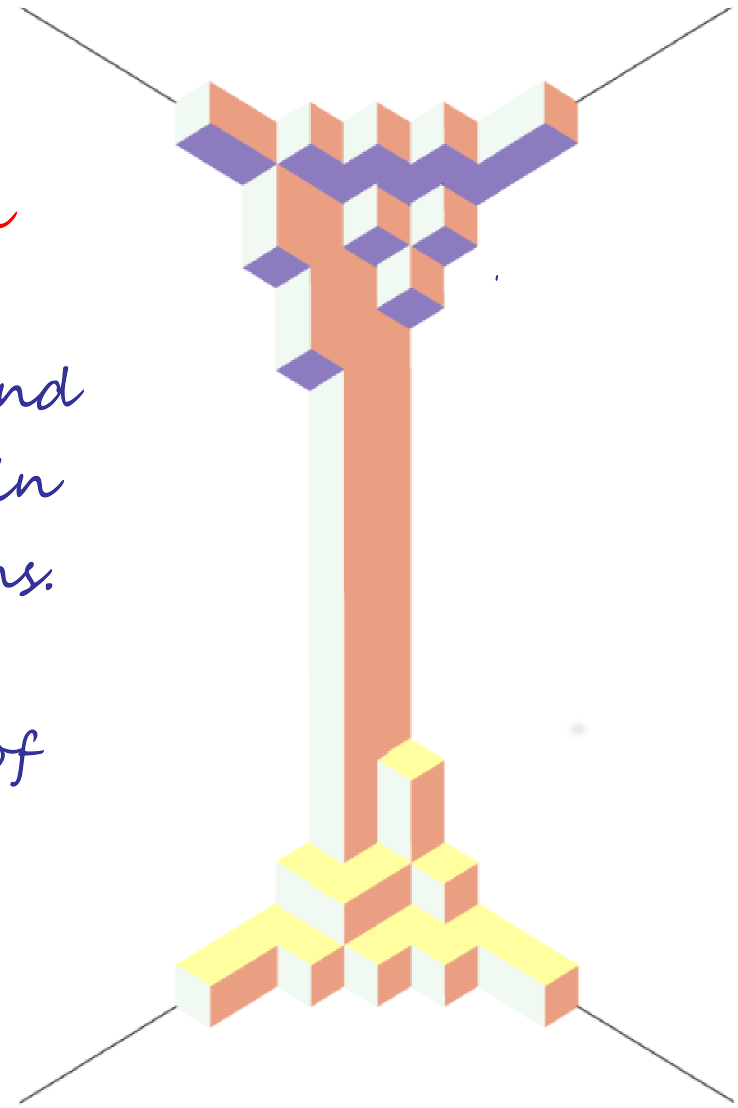
one can have multiple covers and collapsed components

Donaldson-Thomas theory is an alternative way to count curves. What it really counts is 1-dimensional coherent sheaves on Y, like all functions on Y modulo those that vanish on a given curve. Nonreduced structures take the place of multiple covers and collapsed componets.



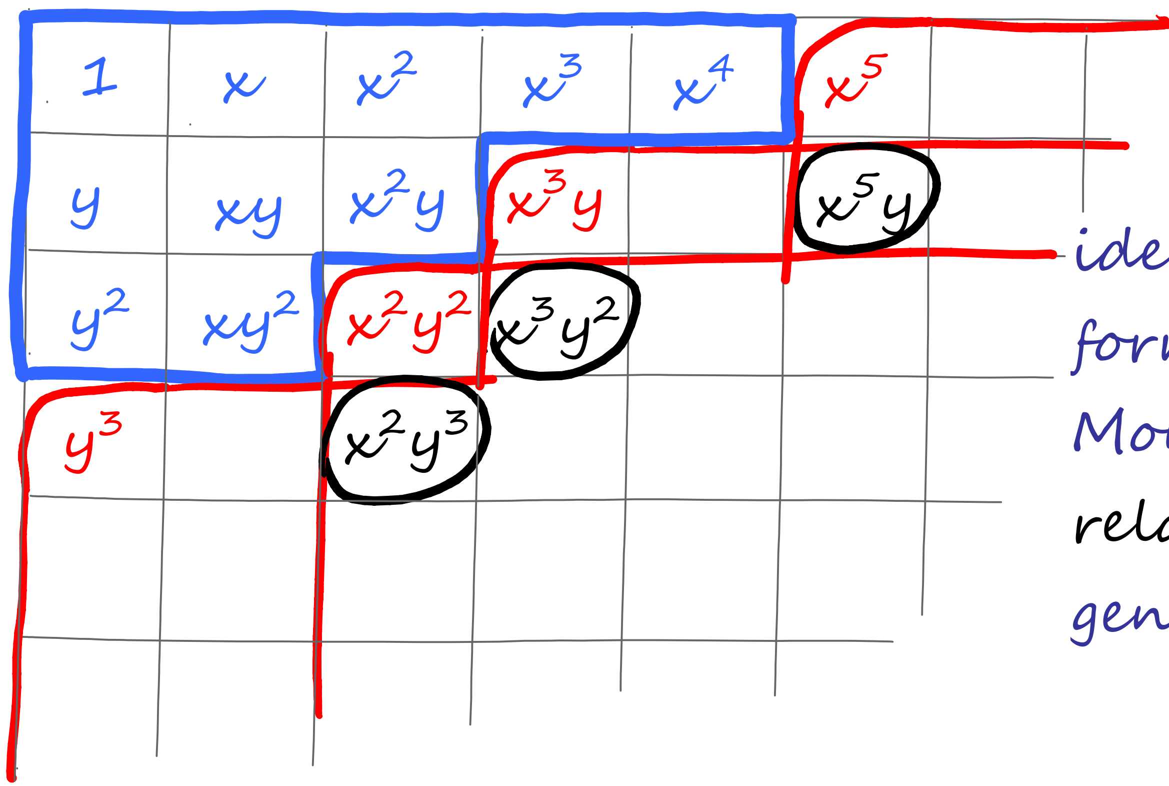
Equivalent to GW, but in a very subtle way

Pictures like this represent monomial curves in toric varieties. For toric varieties, DT invariants, both usual and K-theoretic may be computed as certain weighted sums over such configurations. In K-theory, the weigth of the configuration resembles a 3d version of the Macdonald measure on partitions times q^{#of boxes}



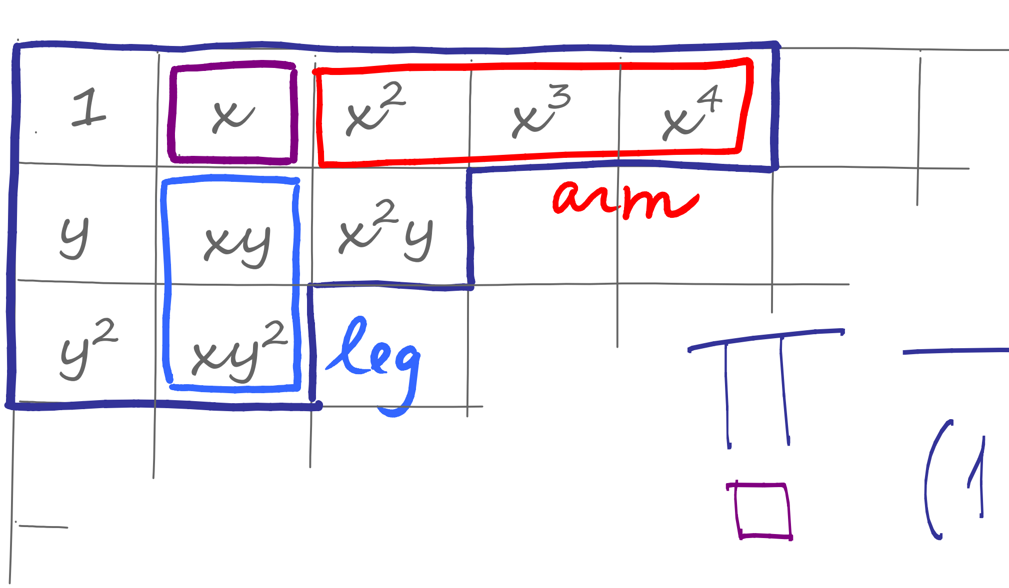
in lieu of the full 3d story, let's go over the "baby" 2d case:

- 2d partitions represent monomial ideals in $\mathbb{C}[x,y]$



Monomials in red are the generators of an ideal I. Monomials in blue form a basis of $\mathbb{C}[x,y]/I$. Monomials in black are the relations among the generators.

From the analysis of the generators and relations, one derives the following Macdonald weight

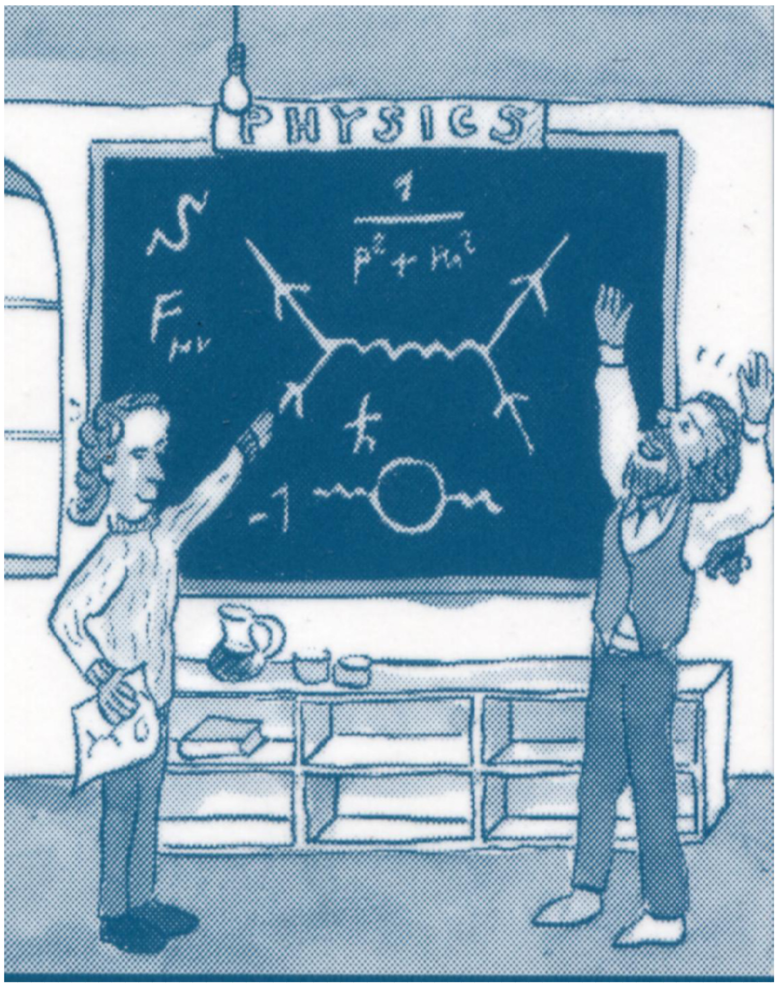


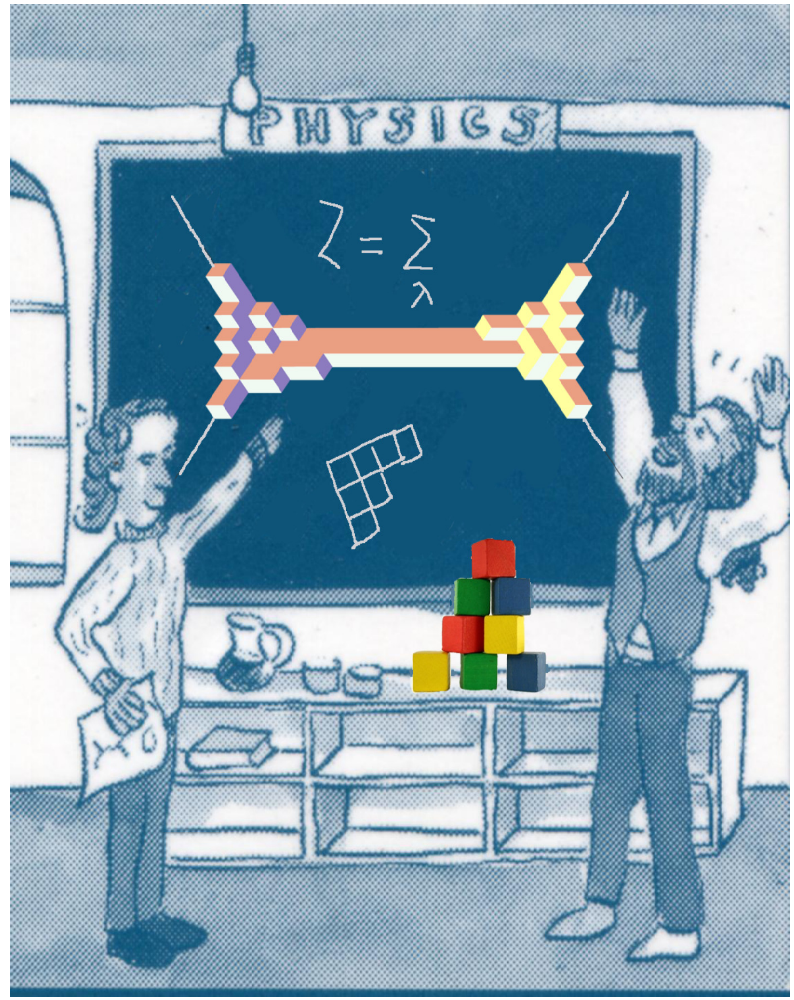
recognize Vandermode ?

$$\prod_{\square} \frac{1}{\left(1-t_1^{a+1} t_2^{-\ell}\right)\left(1-t_1^{-a} t_2^{\ell+1}\right)}$$

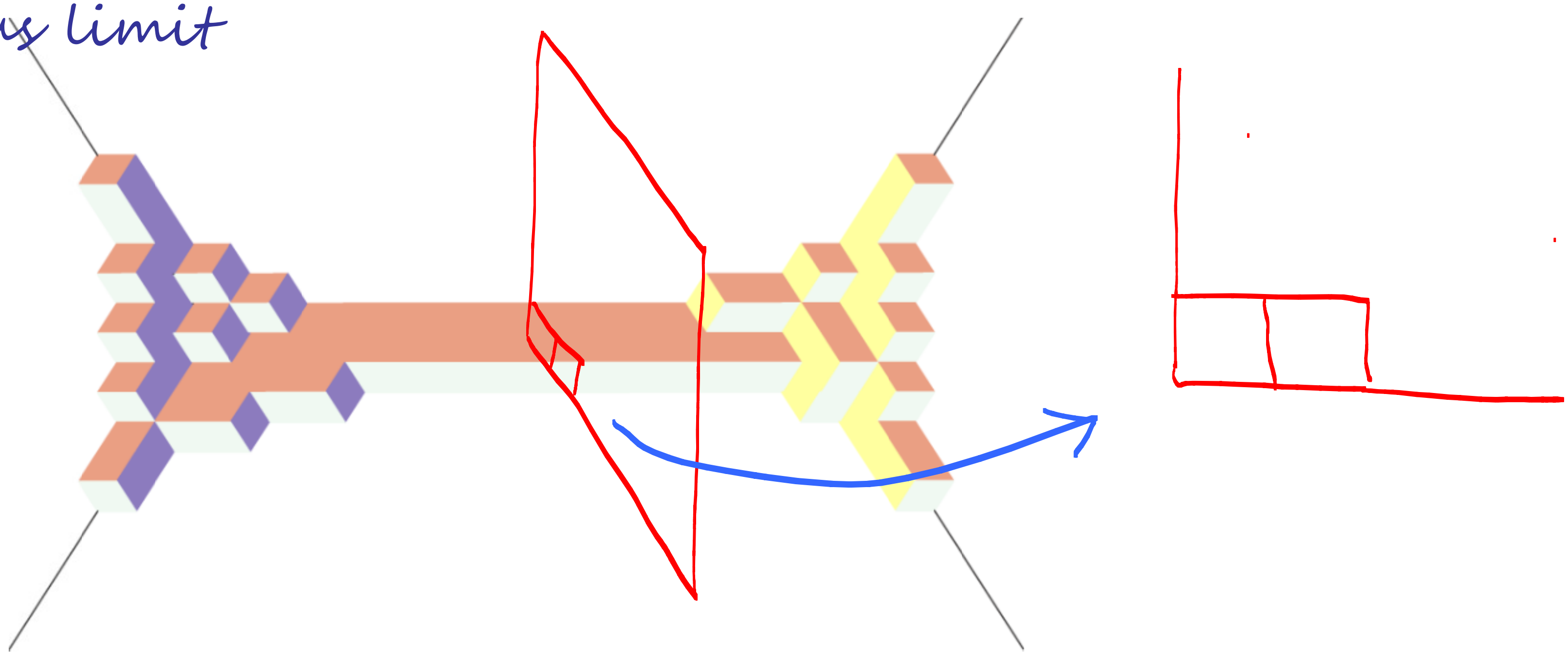
The 3d weight of a 3d partirion is similar in spirit but much more involved .....

Our ability to compute or analyze such boxcounting sums is a very important source of both theoretical and experimental information





Many such sums may be converted to sums over ordinary partitions that are the cross-sections of the edges. These generalize discrete integrals with Macdonald measure and degenerate to all possible random matrix integrals in a continuous limit



In fact, these evaluations are best seen from 5d

So, what we conjecture, is an equality between such boxcounting functions for Y and a somewhat similar in spirit, but technically very different membrane counting function for X.

In fact, we conjecture the equality not the whole sums, that is, of the entire Euler characteristics, but of sheaves themselves, once they are pushed forward to the "greatest common denominator" of the two moduli spaces.

The Chow variety of cycles in Y serves as such "gcd", all moduli spaces of curves map to it.

we consider

$$\begin{tikzcd}
M2(X) & M2(X)^{\mathbb{C}^*} \arrow[l, hook'] \arrow[dr, "\pi_{M2}"'] & & \text{DT-moduli}\,(Y) \arrow[dl, "\pi_{DT}"', "\text{forgets \# of boxes}"] \\
& & \text{Chow variety of cycles in } Y &
\end{tikzcd}$$

Both maps to the Chow variety forget everything except the multiplicities of irreducible components. E.g. in boxcounting sums we only remember the thickness of edges, but not the actual partitions along the edges.

Tautological

Conjecture

$$S^{\bullet}_{\mathrm{Chow}}\, \pi_{M2*}\, \iota_*^{-1}\, \mathcal{K}_{M2}^{1/2} = \pi_{DT*}\, (-q)^{\chi}\, \mathcal{K}_{DT}^{1/2} \otimes \begin{array}{c}\text{Extrinsic}\\ \text{term}\end{array}$$

where $q \in \mathbb{C}^*$ is an equivariant parameter on the left and a box counting parameter on the right.

The tautological extrinsic term in DT theory depends on how Y sits inside X. It is absent if $X = Y \times \mathbb{C}^2$

Translates into a myriad of boxcounting identities

Obviously implies rationality of DT counts a function of q.
This rationality is a very important feature of GW/DT correspondence, in which GW invariants arise as the coefficient in the expansion as $q \to 1$, via $q=\exp(iu)$

Also predicts many highly nontrivial relations for DT counts by choosing different $\mathbb{C}^*$-actions for the same X. E.g. take X to be the total space of 4 lines bundles over a curve, then any two can be Y. Extends and generalizes many dualities of geometric representation theory.