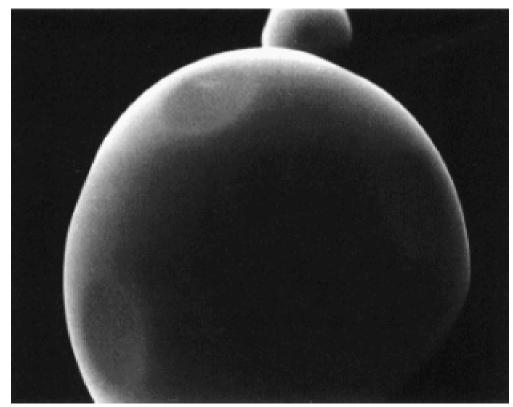
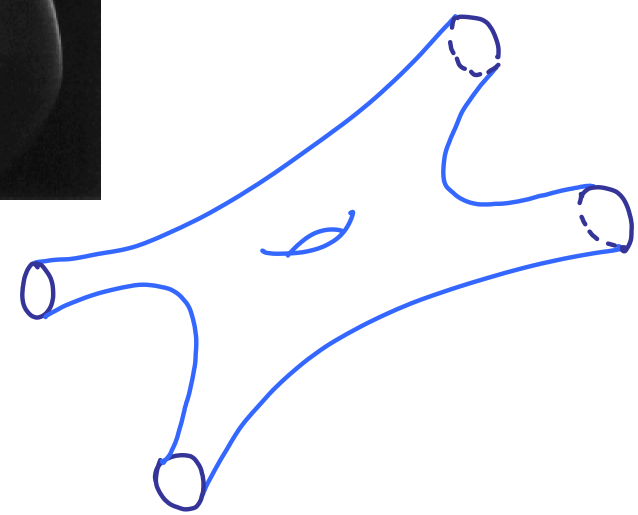
Algebra and geometry of random surfaces

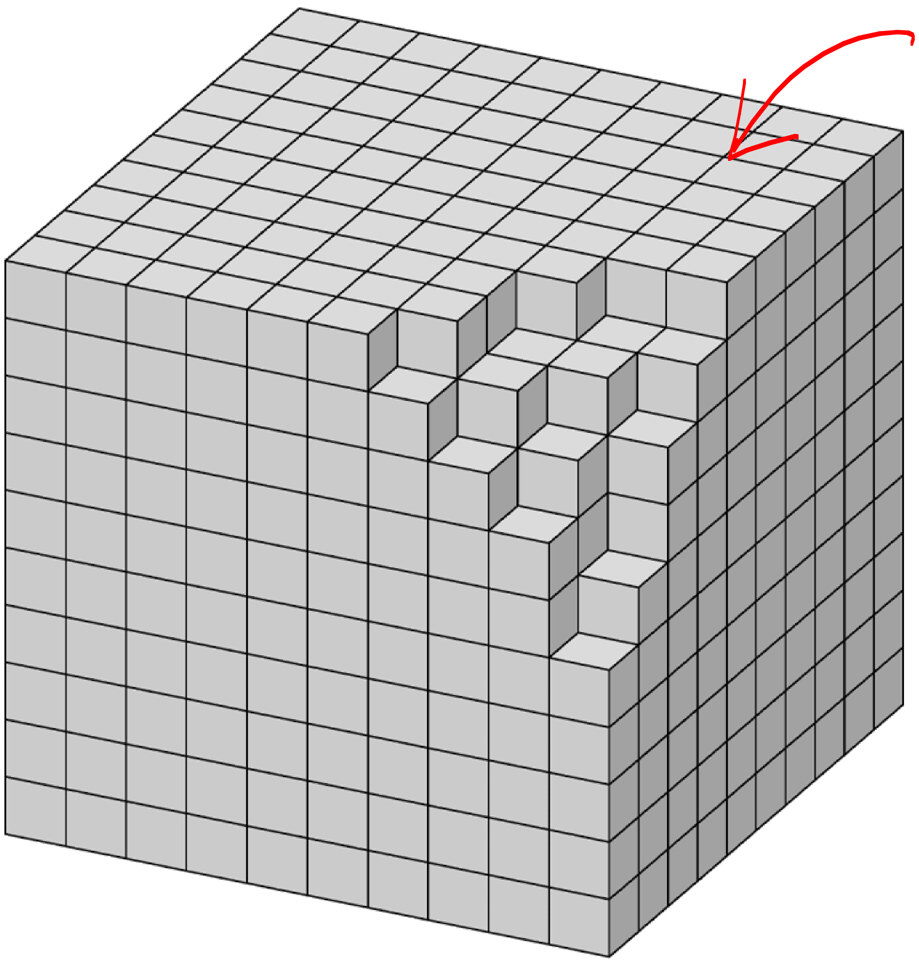
The subject of random surfaces
includes the study
of our whole world



and many
other worlds, too



Among the simplest random surface models are

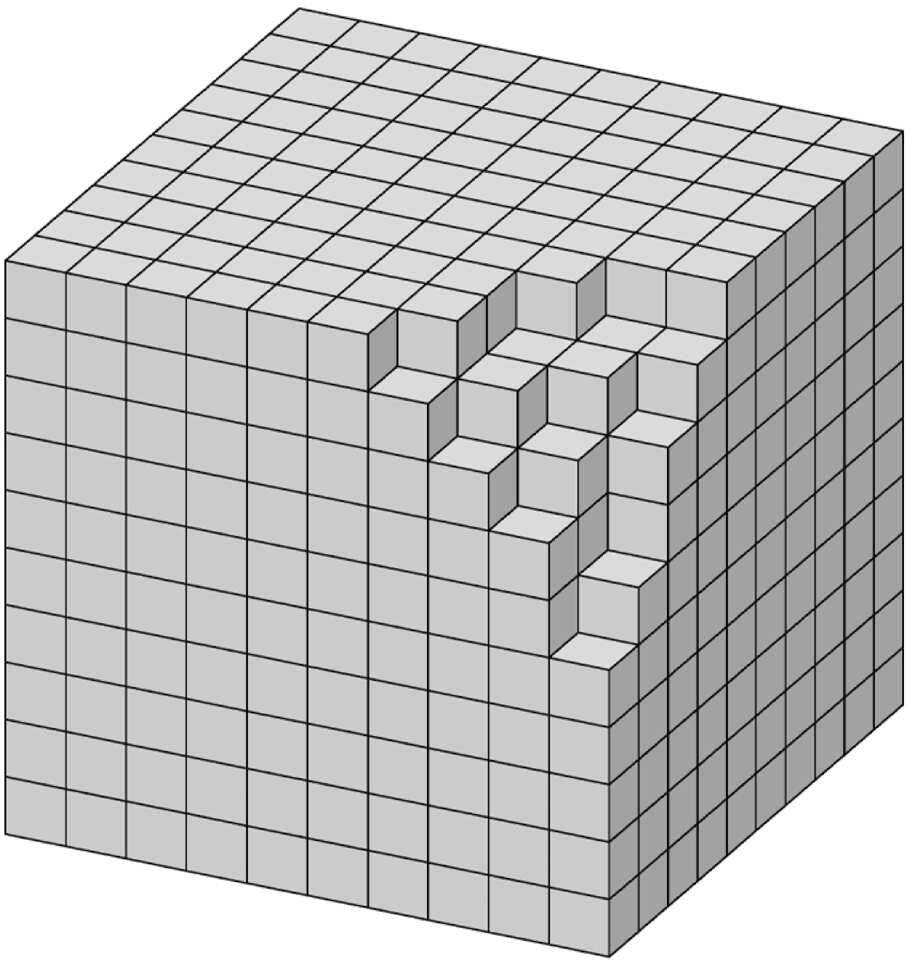


Stepped surfaces

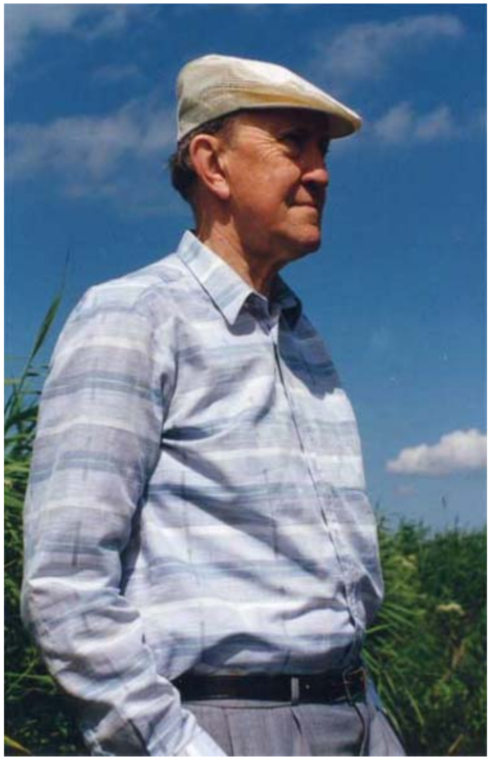
- glued out of 3 sides of a cube
- no overhangs

Discrete interfaces that minimize surface area

We study them because ...



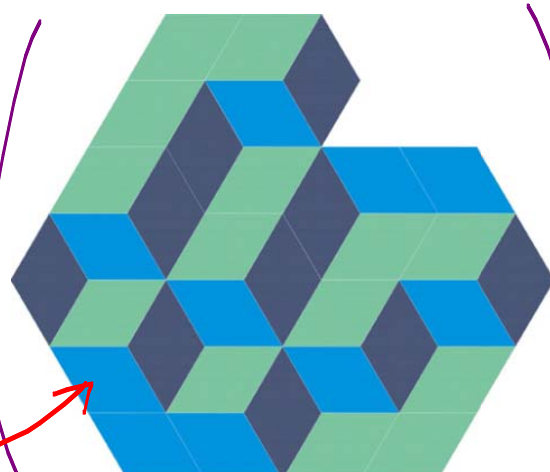
- Simple, but realistic models of crystalline interfaces
- come up in pretty advanced high-energy physics
- because we can!



Around 1960, P. Kasteleyn showed

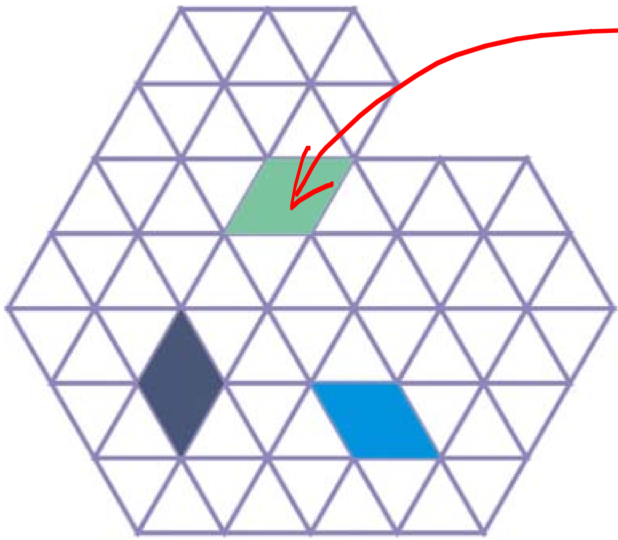
Stepped surfaces $\hookrightarrow$ Linear algebra

and namely

$\#$ () $= \pm \det \left(\begin{array}{c} \text{certain} \\ \text{matrix} \\ K \end{array} \right)$

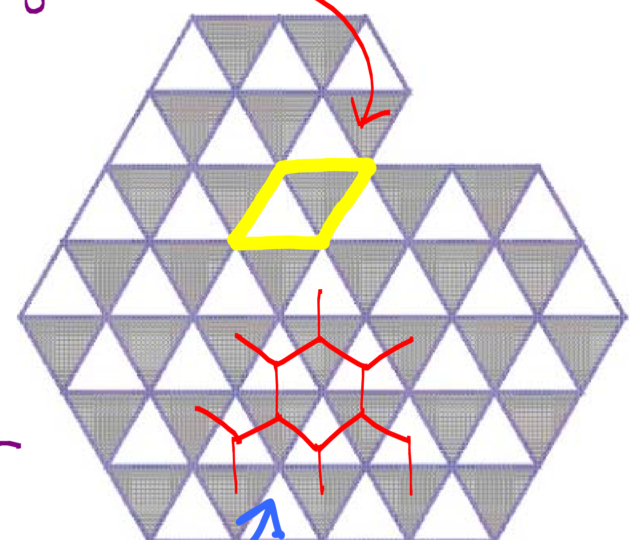
with given boundary

This Kasteleyn matrix K is defined as follows:



every rhombus in this picture is a union of 2 triangles

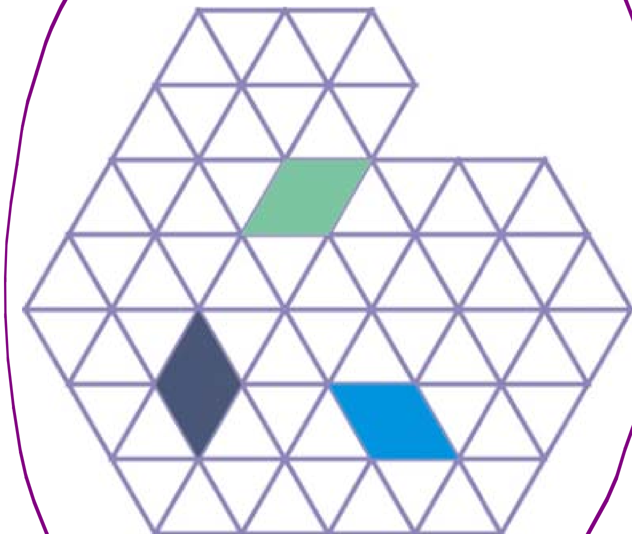
that is,



a stepped surface is a perfect matching / dimer cover of this graph

Def $K : \mathbb{R}^{\blacktriangledown} \longrightarrow \mathbb{R}^{\triangle}$ the adjacency matrix

Corollary

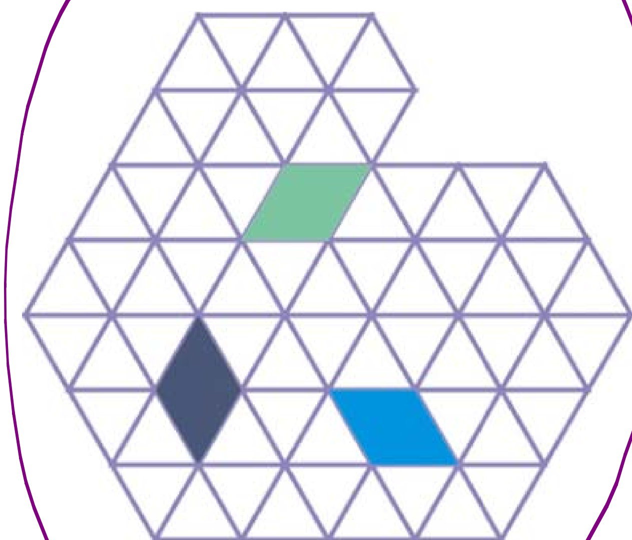
Prob  =

Correlation functions

find rhombi / tiles
some fixed places

Corollary

call this domain Ω

Prob ()

$$= \frac{\#\text{ tilings of } \Omega \setminus \text{fixed tiles}}{\#\text{ tilings of } \Omega} = \frac{\det \left(K \right)}{\det K}$$

= a minor of K^{-1}

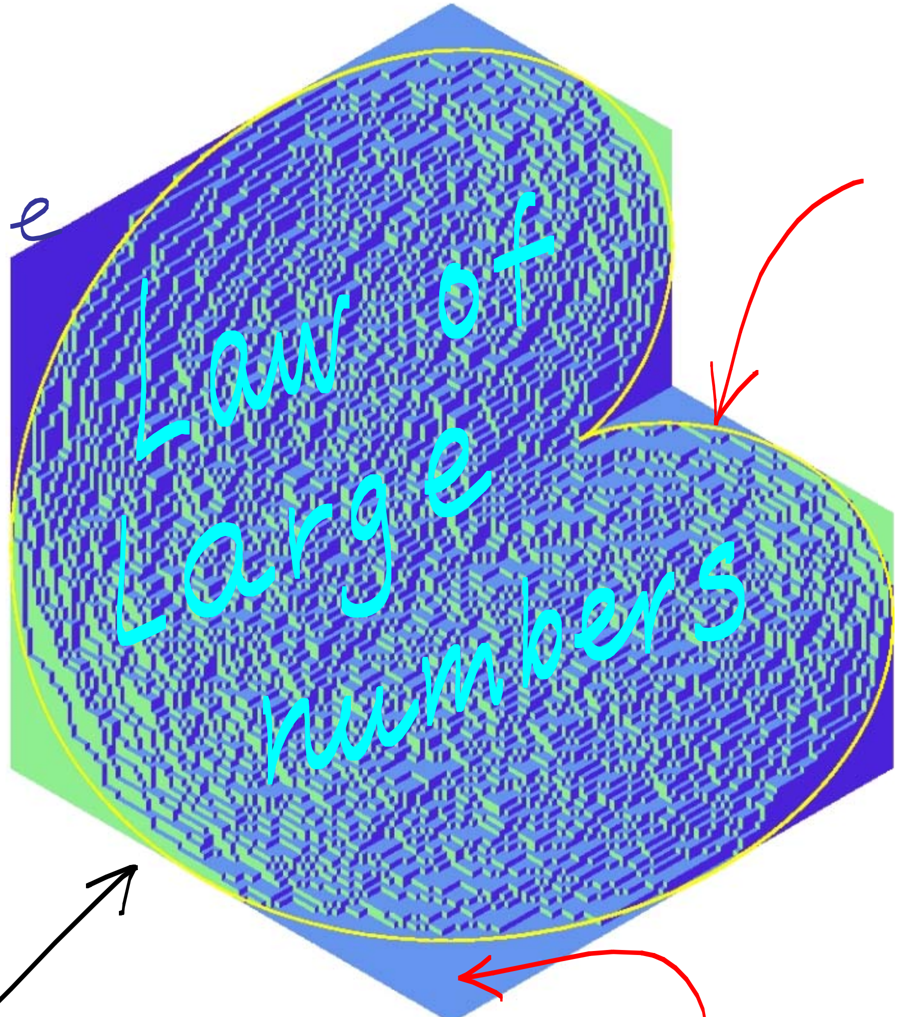

孔子問禮圖
Teacher, I want to learn about stepped surfaces
He who can invert the matrix knows all probabilities...
Rick Kenyon

Inverting K is not entirely trivial,

e.g. because stepped surfaces do

something interesting as they get large...

Here is a large stepped Surface with polygonal boundary



frozen boundary is here a cardioid (c.f. "arctic circle" of Cohn-Larsen-Propp)

boundaries in coordinate directions in $\mathbb{R}^3$

facet, ordered down to the atomic scale

In fact, some time ago, we proved

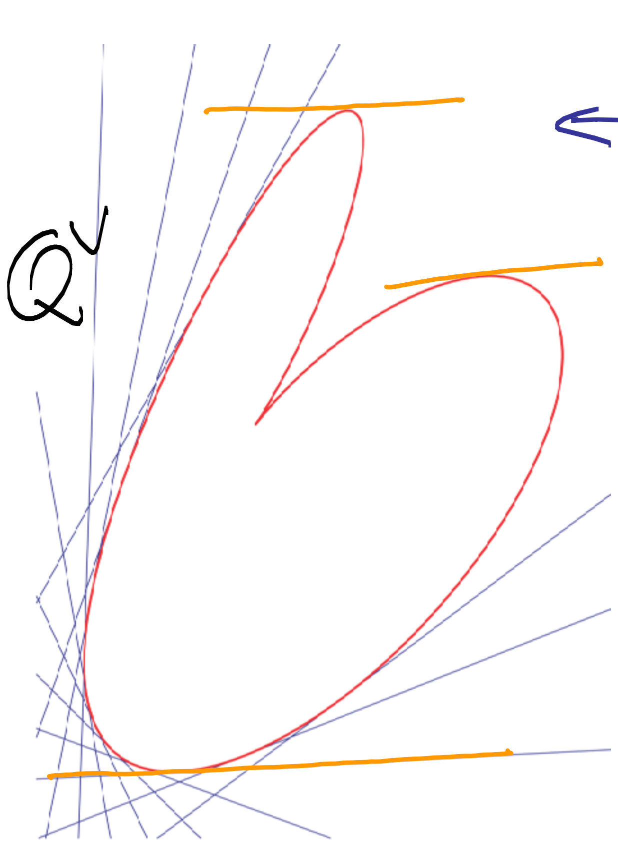
Theorem [Kenyon – O.] Take a random stepped surface bounded by $3d$ segments in coordinate directions (cyclicaly repeated). As mesh $\to 0$, these will have a limit shape with

$$\text{Frozen boundary} = \left(\begin{array}{c}\text{rational curve } Q \\ \text{of degree } d\end{array}\right)^{\vee}$$

dual

Further, Q determines the limit shape by ...

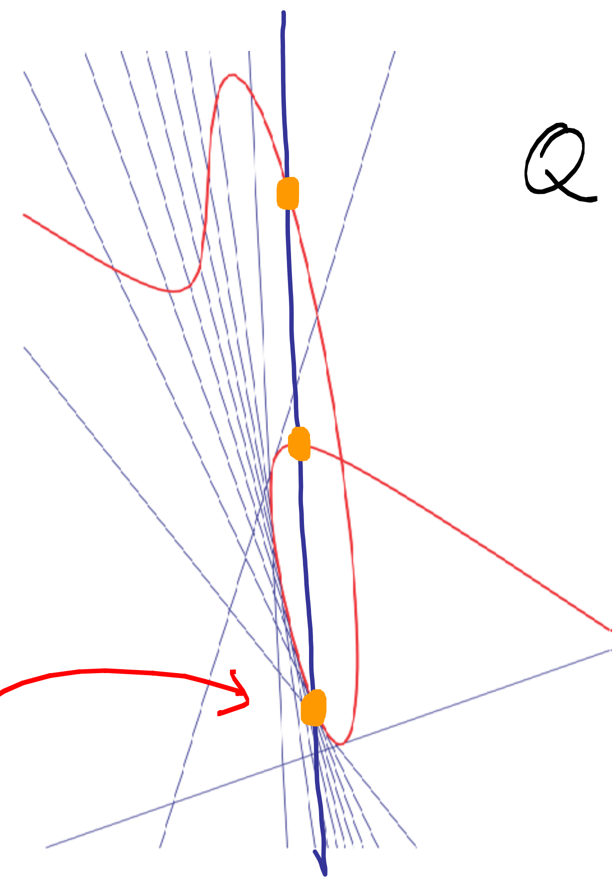
Q is found from boundary conditions as follows:



tangency to the boundary

means

incidence



So Q should be

- plane curve of degree d
- meeting $3d$ given points (1 automatic)
- rational

$\hookrightarrow$ large, but finite number of possibilities

only 1 will yield $Q^\vee$ that is inscribed

Finally, we come to the main point of this lecture:

For polygonal boundaries $K^{-1}(\cdot,\cdot)$ satisfies an additional difference equation in each argument

This extra equation is a quantization $\hat{Q}$ of Q

in every possible sense

Before we make this precise, here is the outlook

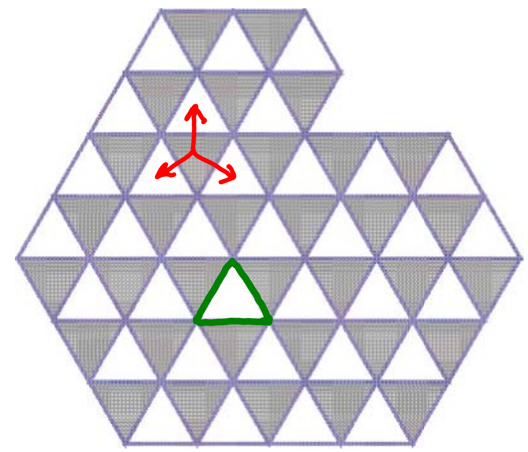
- In Lecture 2, we will discuss $\widehat{Q}$ as function of the boundary. We will see it satisfies a generalization of Painlevé equation, and find its asymptotics for large domains / small mesh. This provides an excellent control over K^{-1}.

Before we make this precise, here is the outlook

- In Lecture 3, we will talk about $\widehat{Q}$ as a successful example of higher genus mirror symmetry, and will try to draw some conclusions from it

Now back to math ....

Step 1 Recall $K : \mathbb{R}^{\blacktriangledown} \to \mathbb{R}^{\triangle}$



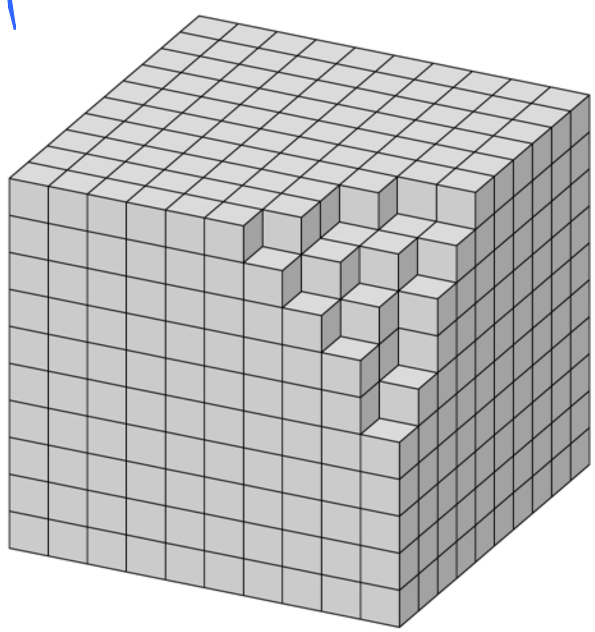
$$\mathbb{R}\, K^{-1}(\,\cdot\,, \triangle) = \operatorname{Ker} K$$

my domain minus $\triangle$

Therefore, we might as well study domains Ω such that $\dim \operatorname{Ker} K_{\Omega} = 1, 2, 3, \ldots$

"discrete holo. functions"

Step 2

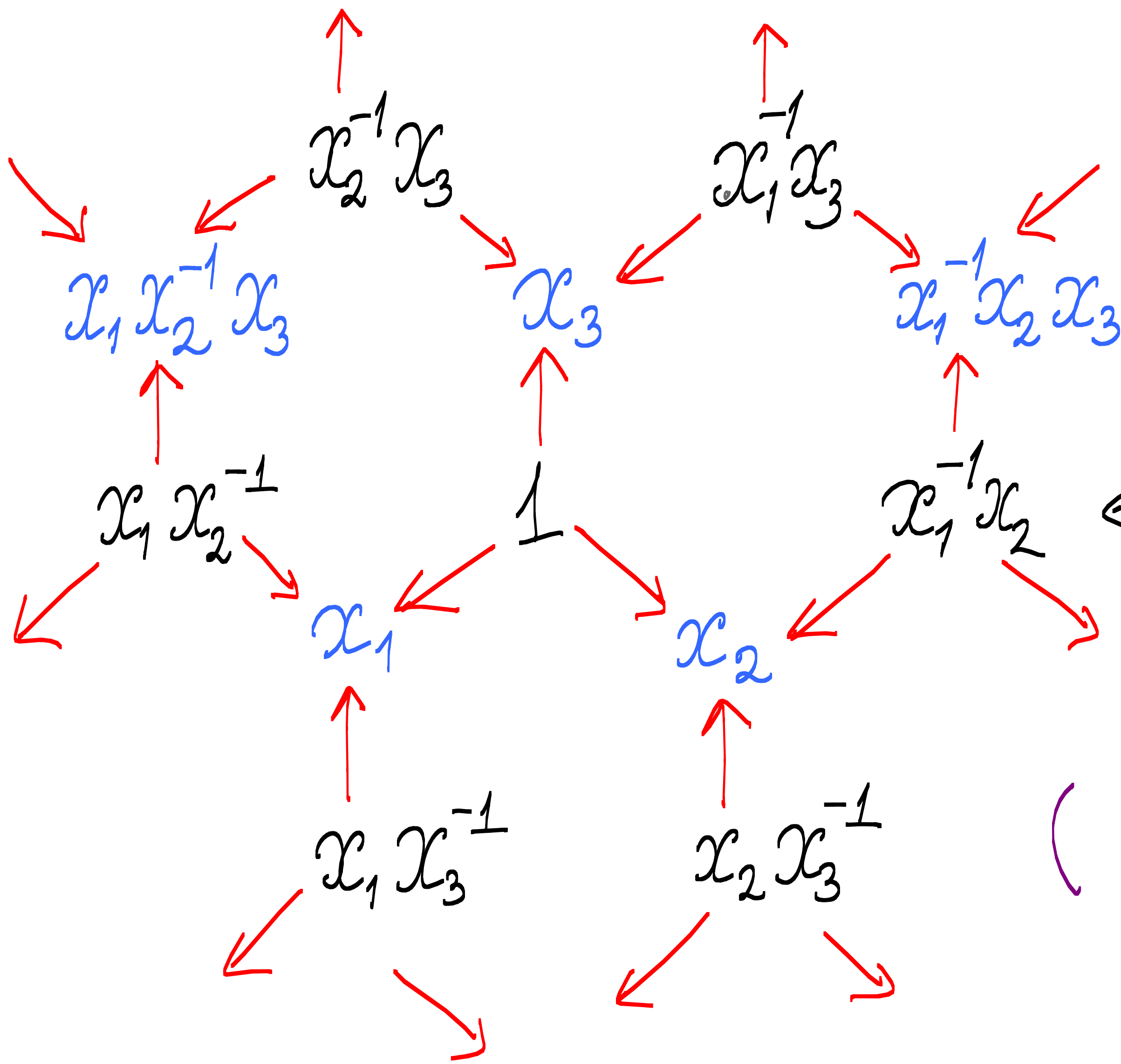


In this and other contexts, it is natural to weight surfaces by $q^{\text{volume enclosed}}$

parameter > 0

Kasteleyn's theorem continues to hold with certain weights best desribes as follows ...

The old K = multiplication by $x_1 + x_2 + x_3$



degree = 1

degree = 0

(Fourier transform, really)

The new, weighted K = right multiplication by $x_1 + x_2 + x_3$ in

$$A = \mathbb{C}\langle x_1, x_2, x_3 \rangle \Big/ \; x_j x_i = q_{ij} x_i x_j$$

where $q = q_{12} q_{23} q_{31}$

one of the simplest noncommutative $\mathbb{P}^2$'s

Left and right multiplication commute

Anonymous

and act by difference operators, in our case

Theorem The left A-module $\widehat{Q}$ generated by $\operatorname{Ker} K$ in a polygonal domain Ω is torsion (means difference equations)

Moreover it is a

"line bundle $\mathcal{L}$ on a degree $d = \deg \Omega$ curve"

with $\chi(\mathcal{L}) = \dim \operatorname{Ker} K$, meaning say for $\chi = 1$ that it has a presentation of the form

this map tells us the equations

$$0 \to A(-2)^{d-1} \to A(-1)^{d-2} \oplus A \to \hat{Q} \to 0$$

relations $\qquad$ generators

Same as the free resolution for a generic $\mathcal{L}$ on $\mathbb{P}^2$ with $\operatorname{rk} \mathcal{L} = 0$, $c_1(\mathcal{L}) = d$, $\chi(\mathcal{L}) = 1$

how do we find out what $\widehat{Q}$ is ?

- we know its degree
- we know $3d$ points that it meets
- what would "rational" mean ???

More tomorrow, ...