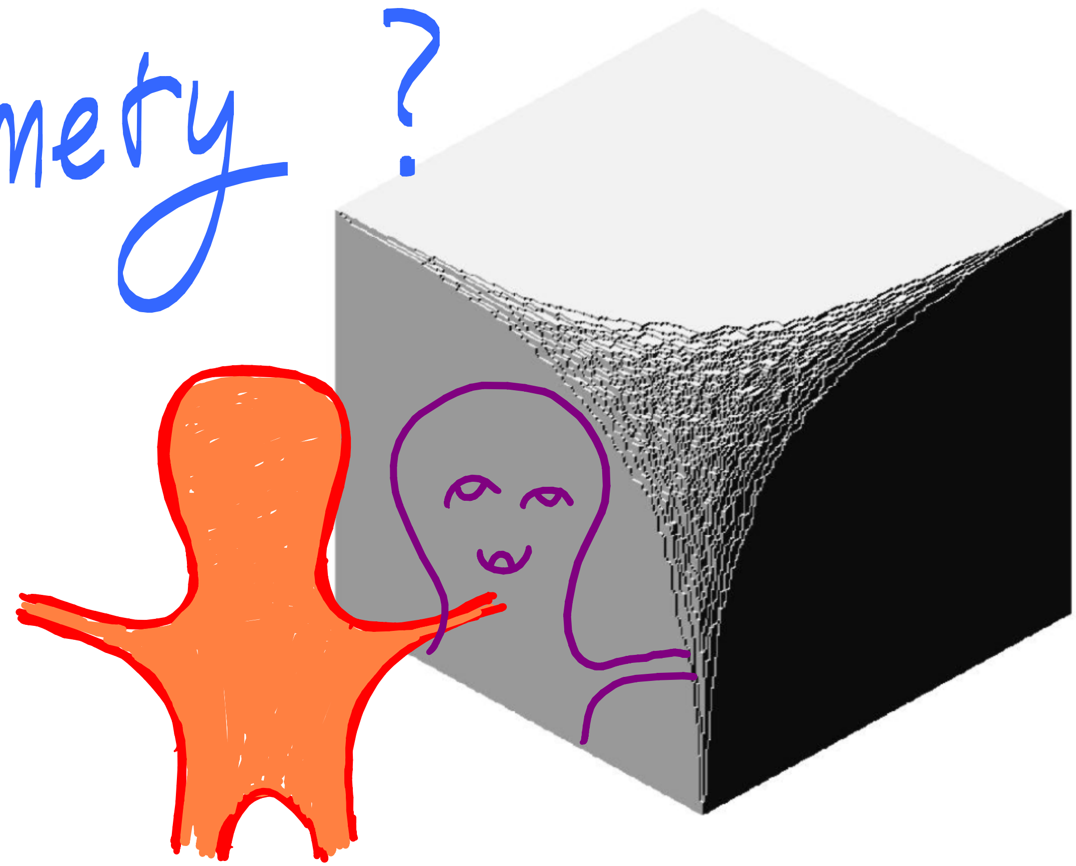
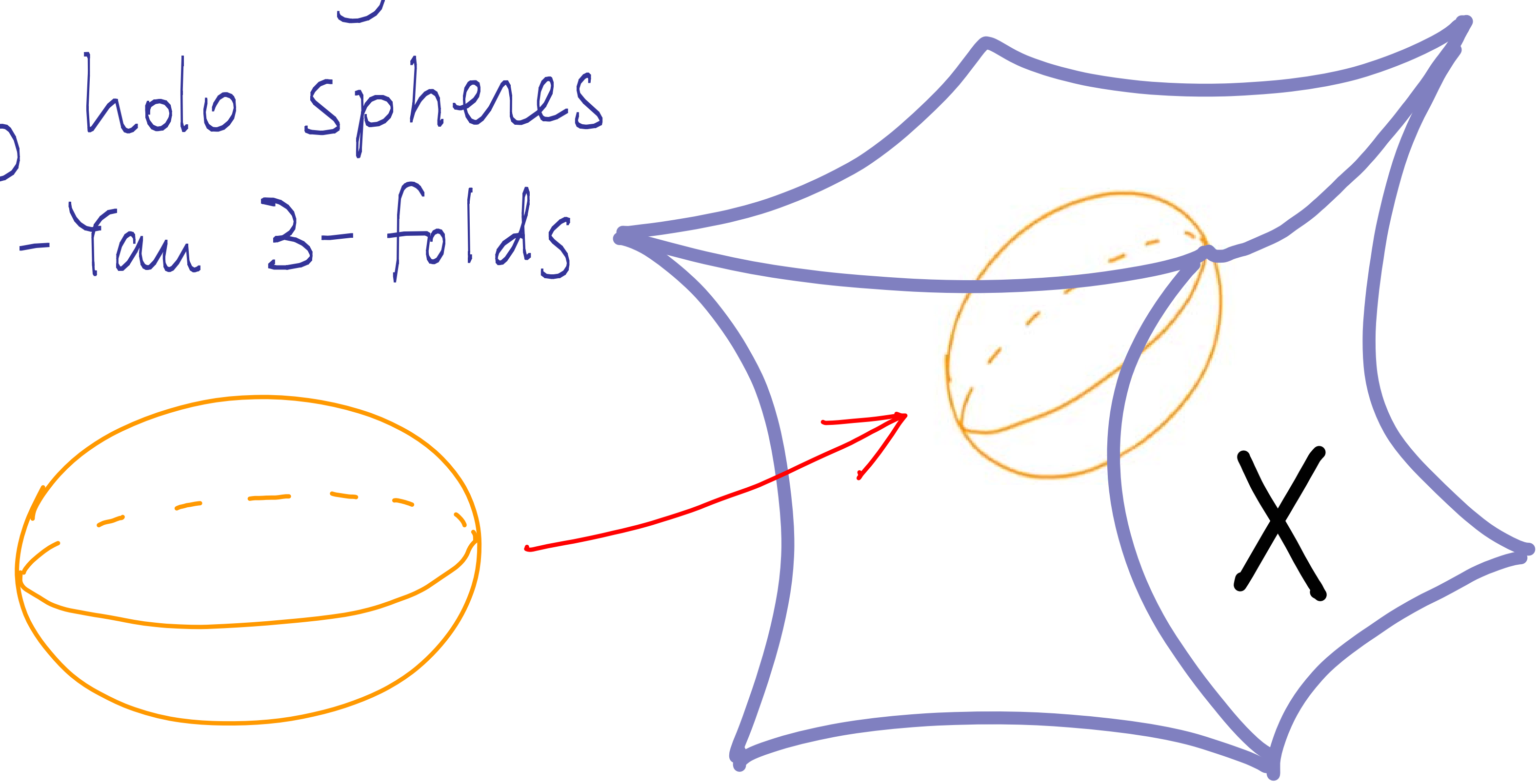
Towards probabilistic mirror symmetry?



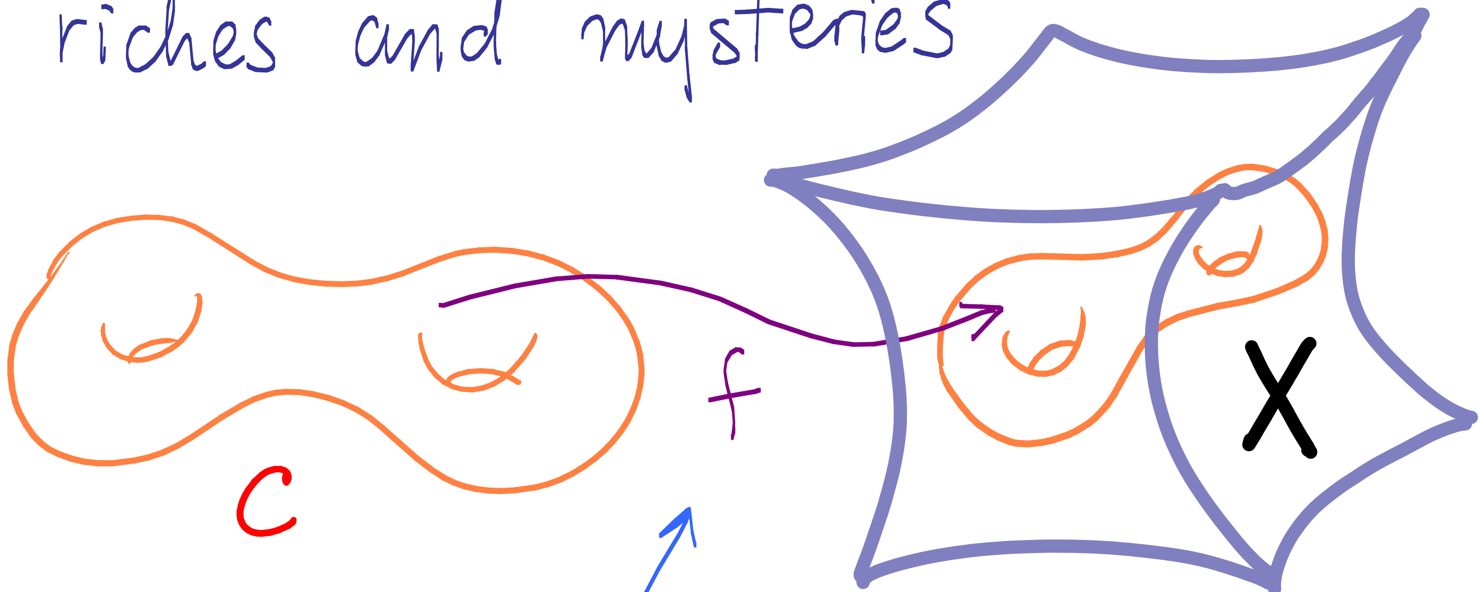
For a mathematician, mirror symmetry can mean many different things.

One of its early successes had to do with counting holo spheres in Calabi-Yau 3-folds

$c_1(X)=0$



Enumerative geometry of curves in 3-folds has great riches and mysteries



In fact, there are many competing/collaborating theories, depending on the definition of "curve"

For example, one can study modulo reparametrization

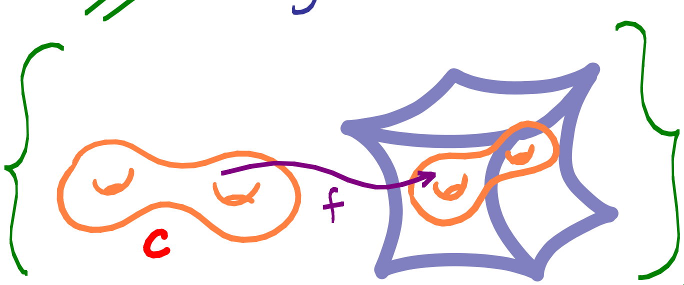
This would give the Gromov-Witten / topological A-string curve counts. They are defined using the virtual fundamental class of

$$\overline{\mathcal{M}}_g(X,d) = \left\{ f: C \to X \text{ stable map of } \deg = f_*[C] = d \right\} \Big/ \simeq$$

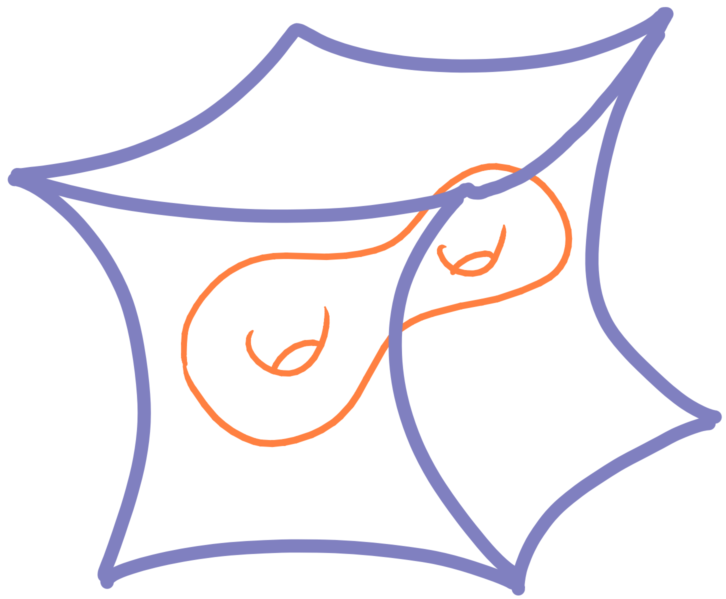
at worst nodal, genus g

$H_2(X,\mathbb{Z}) \ni d$

$=$



$/\simeq$



Instead of parametrization, one can describe curves $C \subset X$ by the equations that cut them out (plot vs. implicit plot)

These equation, i.e. functions that vanish on C, form a sheaf of ideals $\mathcal{I}_C \subset \mathcal{O}$ on X

One can also use torsion sheaves to describe a curve, like we did earlier for surfaces

Correspondences with various sheaf-based enumerative theories and GW have been proposed

⋮

[Aganagic - Klemm - Mariño - Vafa] ← topological vertex

⋮

[O. - Reshetikhin - Vafa]

⋮

[Maulik - Nekrasov - O. - Pandharipande]

⋮

[Pandharipande - Thomas]

⋮

In all cases, while morally counting the same things, various enumerative theories agree only after very nontrivial transformations of the answer. For example

Gromov-Witten i.e. parametrized count $\xleftrightarrow[\text{[MNOP]}]{?}$ Donaldson-Thomas i.e. equation-based count

goes as follows

Define the following generating function

$$Z_{GW}(d,u) = \exp \sum_{g=0}^{\infty} u^{2g-2}$$ # of degree d genus g curves according to GW

$$Z_{DT}(d,q) = \sum_{\chi} q^{\chi}$$ # of degree d genus $g = 1-\chi$ as counted by equations,

holo Euler char

i.e. on $\mathrm{Hilb}(X; 0, 0, d, \chi)$

if $c_1(X) \cdot d > 0$
put in insertions

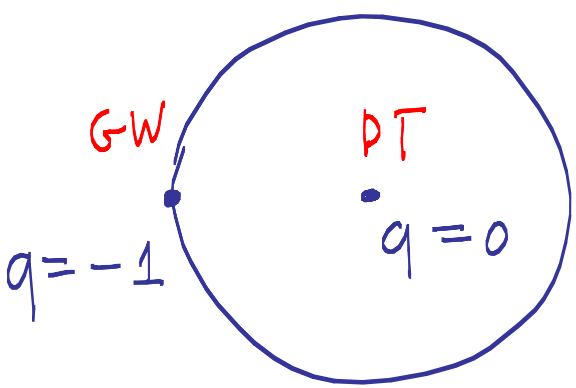
The following is the $c_1 = 0$ specialization of more general conj.

Conj [MNOP] Substitution $\boxed{q = -e^{iu}}$ gives

$$Z_{GW}(d,u) = \frac{Z_{DT}(d,q)}{\prod_{n>0} \left(1-(-q)^n\right)^{n\chi(X)}}$$

a theorem here

conjecturally, rational in q



Thm [Maulik-Oblomkov-O-Pandharipande]

This (and its $c_1 \neq 0$ generalization) is true for any toric X.

means that $T \simeq (\mathbb{C}^*)^3$ acts on X with an open orbit, e.g. $\mathbb{P}^3$ or $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$

For toric X, the Donaldson-Thomas curve counts may be done by counting T-fixed points on $\mathrm{Hilb}(X)$

Let's see what these look like ....

Warm-up: $\mathrm{Hilb}(\mathbb{C}^2; d, \chi)$

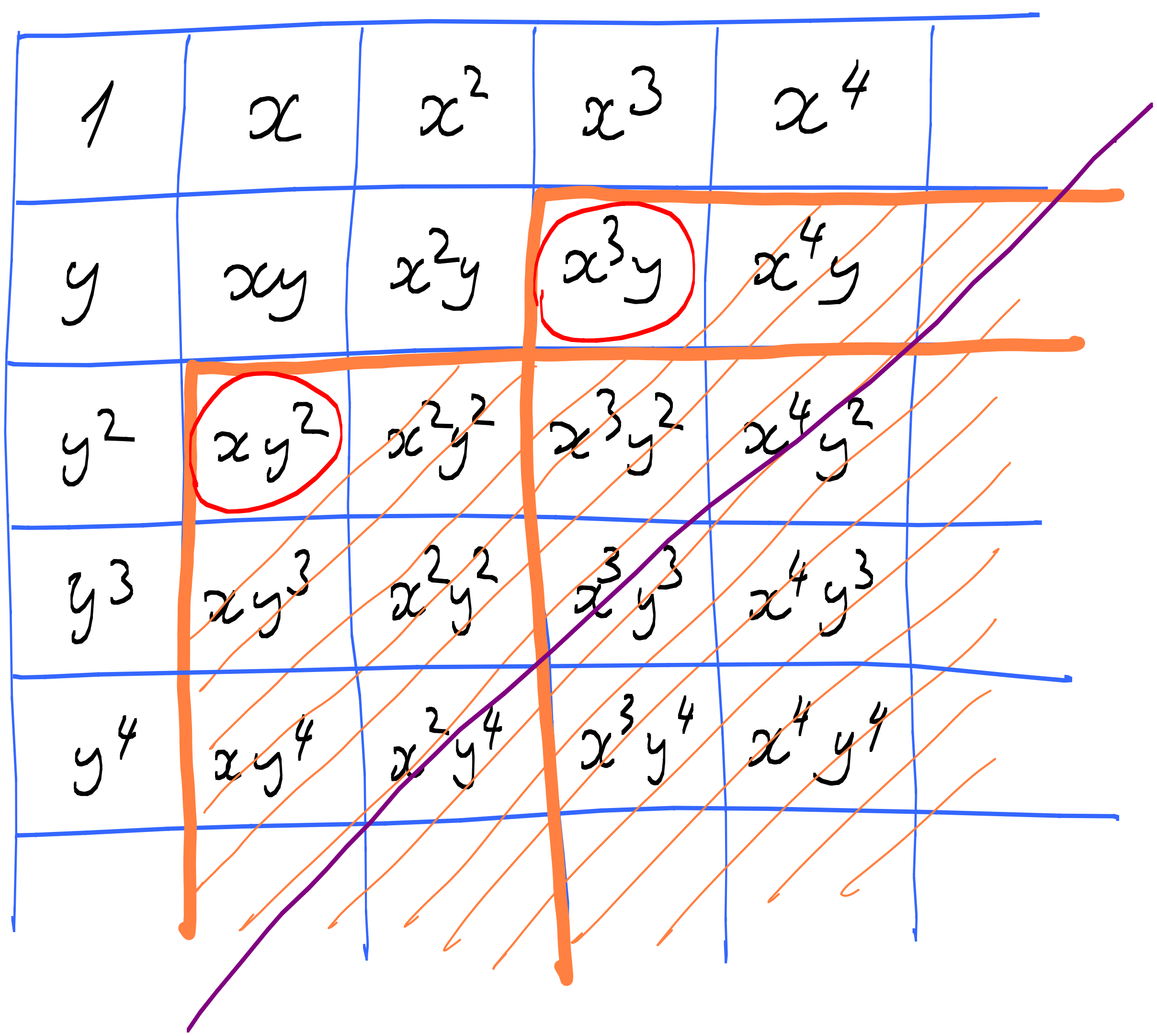
this is

$$\left\{ \begin{array}{l} \text{ideals } I \subset \mathbb{C}[x,y] \text{ such that} \\ \dim\left(\mathbb{C}[x,y]/I\right)_{\deg \leq m} = dm + \chi, \quad m \gg 0 \end{array} \right\}$$

the torus T acts by rescaling x and y

Fixed points = monomial ideals

Fixed points = staircases / partitions



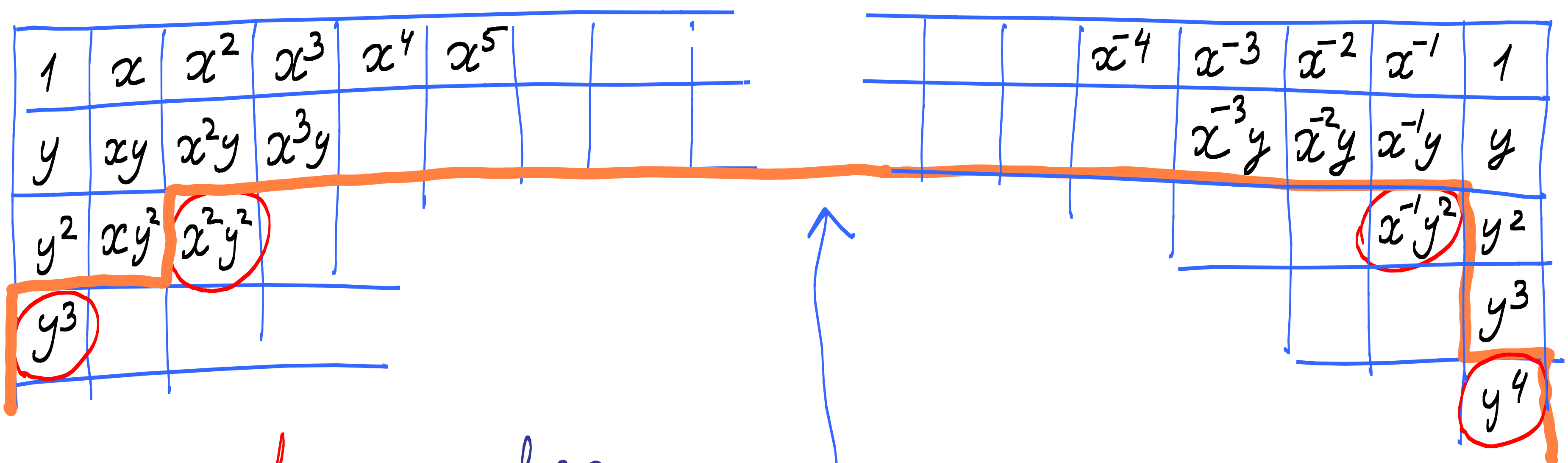
d = total width of legs

(here, $d = 2$)

χ = renormalized area

Next example: $\mathrm{Hilb}(\underset{x}{\underset{\uparrow}{\mathbb{P}^1}} \times \underset{y}{\underset{\uparrow}{\mathbb{C}}} ; d, \chi)$

chart at $x=0$ chart at $x=\infty$



d = degree

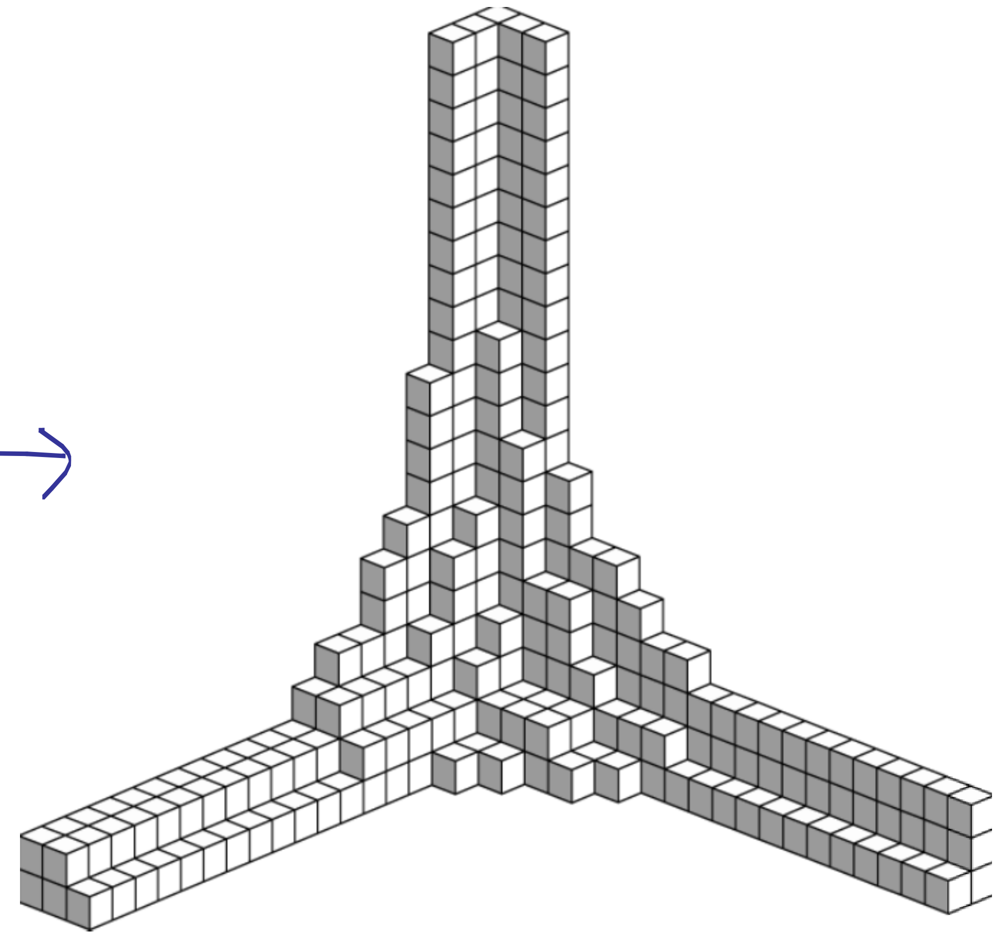
= width, here 2

χ = renormalized area

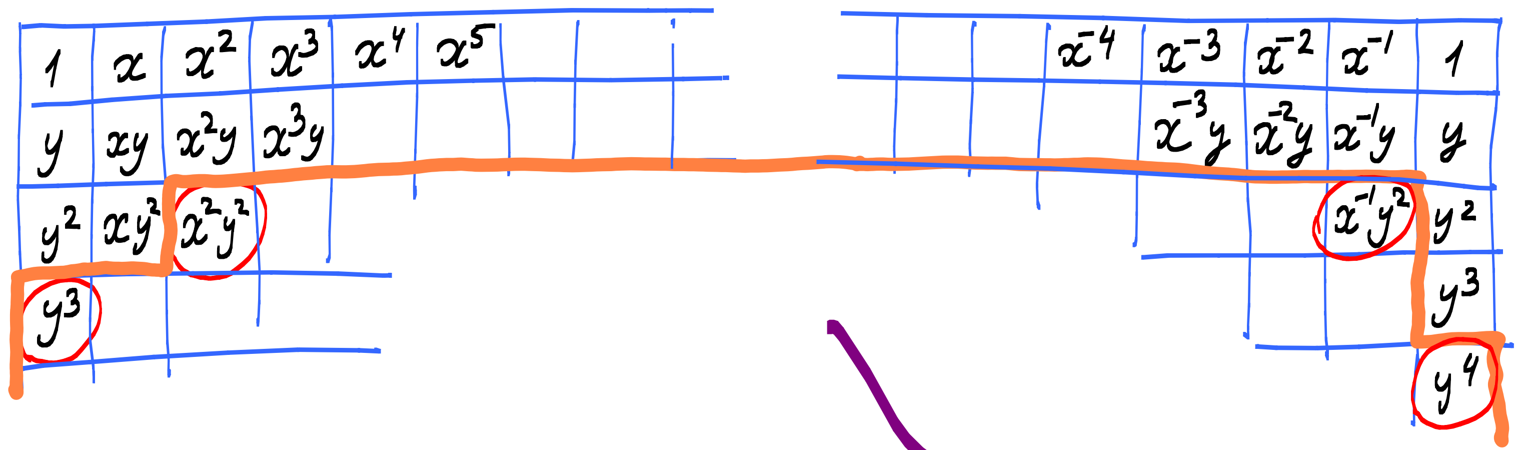
now we put on our 3D glasses,

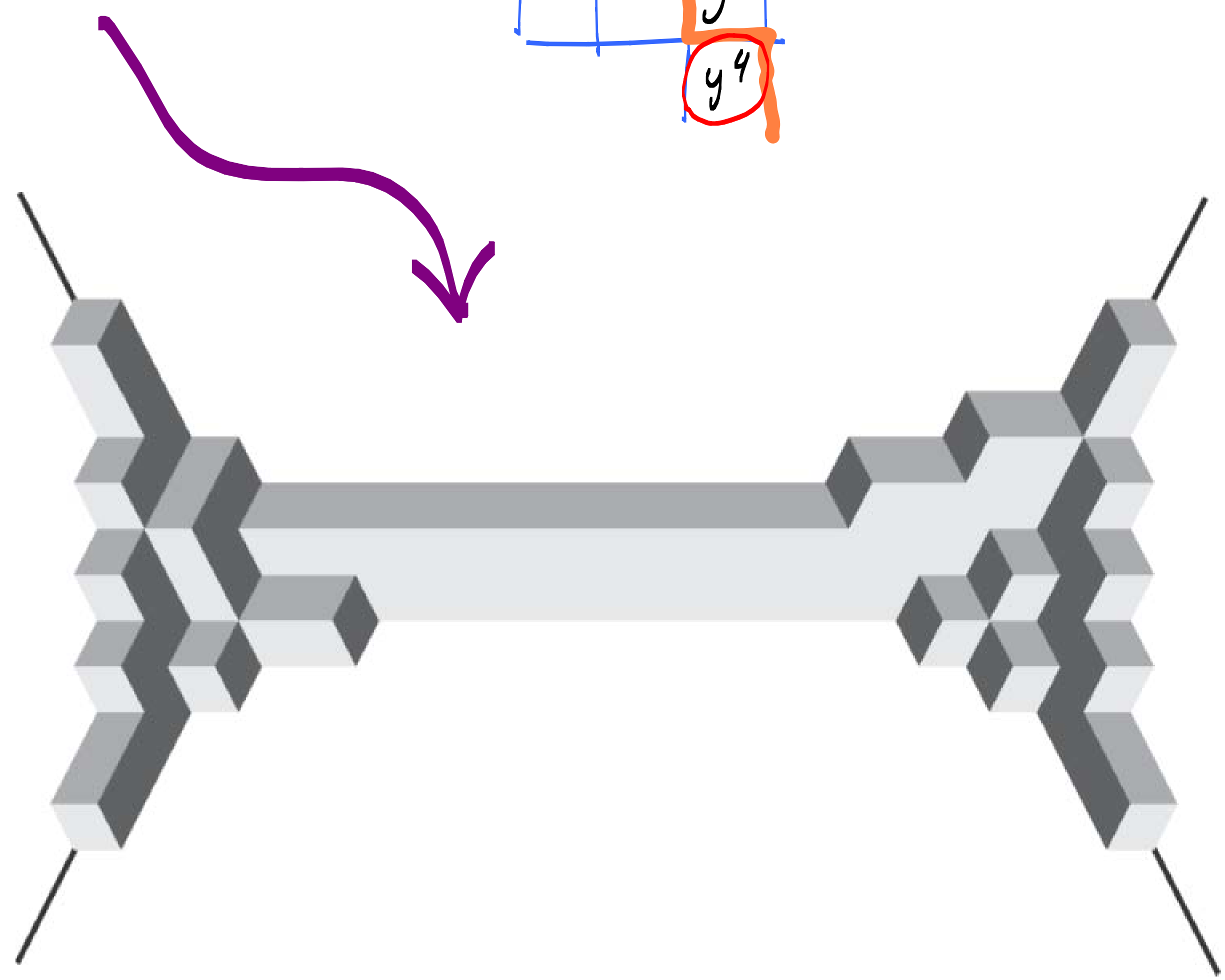
1	x	x^2	x^3	x^4
y	xy	x^2y	x^3y	x^4y
y^2	xy^2	x^2y^2	x^3y^2	x^4y^2
y^3	xy^3	x^2y^2	x^3y^3	x^4y^3
y^4	xy^4	x^2y^4	x^3y^4	x^4y^4

$\longrightarrow$



$\mathrm{Hilb}(\mathbb{C}^3)$





picture of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$

$\downarrow$

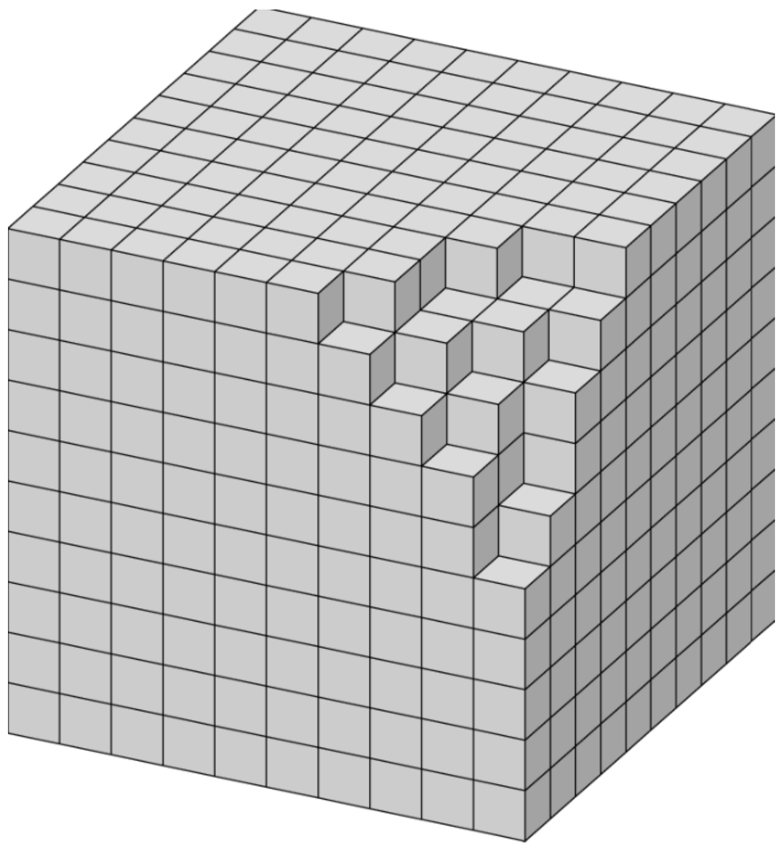
$\mathbb{P}^1$.

Conclusion: For toric X, Donaldson-Thomas curve counts = box counting

in the tropical limit

the weight q^{χ}, responsible for the genus of the curve, becomes q^{vol}

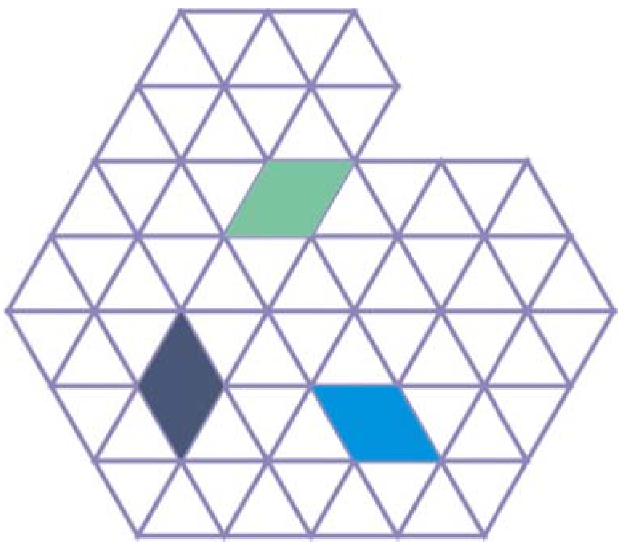
(sign issues in the real world)



We've been doing q^{vol} boxcounting for 2 days

Brief summary of what we discussed ....

First, boxcounting is a dimer problem

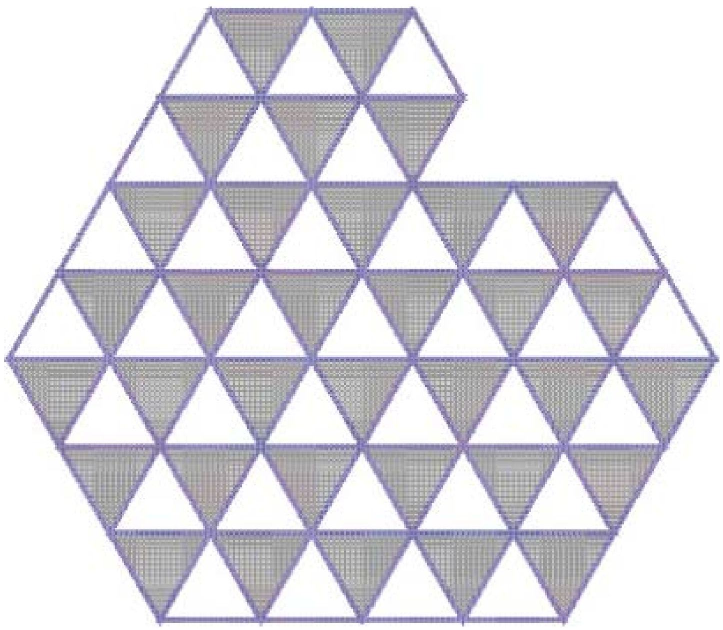


Ω

Dimer covers of Ω may described completely using discrete holo functions

or q-holomorphic for q^{vol}

$$A = \mathbb{C}\langle x_1, x_2, x_3 \rangle \big/ x_j x_i = q_{ij} x_i x_j$$ graded



$\blacktriangledown$ = monomials in A_0

$\triangle$ = monomials in A_1

A-bimodule

$$F = \sum_{\blacktriangledown \in \Omega} A \blacktriangledown \Big/ \sum_{\triangle \notin \Omega} A \triangle$$

Kasteleyn operator

$$0 \to F \xrightarrow{\cdot (x_1+x_2+x_3)} F(1) \to 0$$

like Dolbeault

Cohomology

$$0 \to \widehat{Q} \longrightarrow 0 \to 0$$

discrete holo functions in Ω as left A-module

$$Z = \sum_{\text{configurations}} q^{\text{vol}}$$

$$= \det \text{Kasteleyn}$$

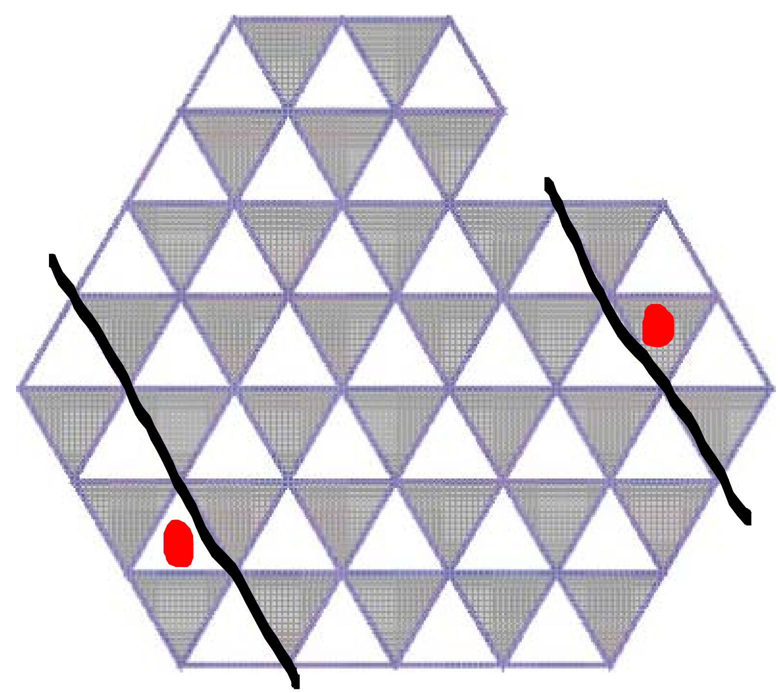
$$\approx \det \bar{\partial}_q : \hat{Q} \to \hat{Q} \otimes (0,1)\text{-forms}$$

very much like Θ function, e.g. because

- vanishes when an unexpected section appears

- satisfies

  Plücker-Fay-Hirota-Kasteleyn-...



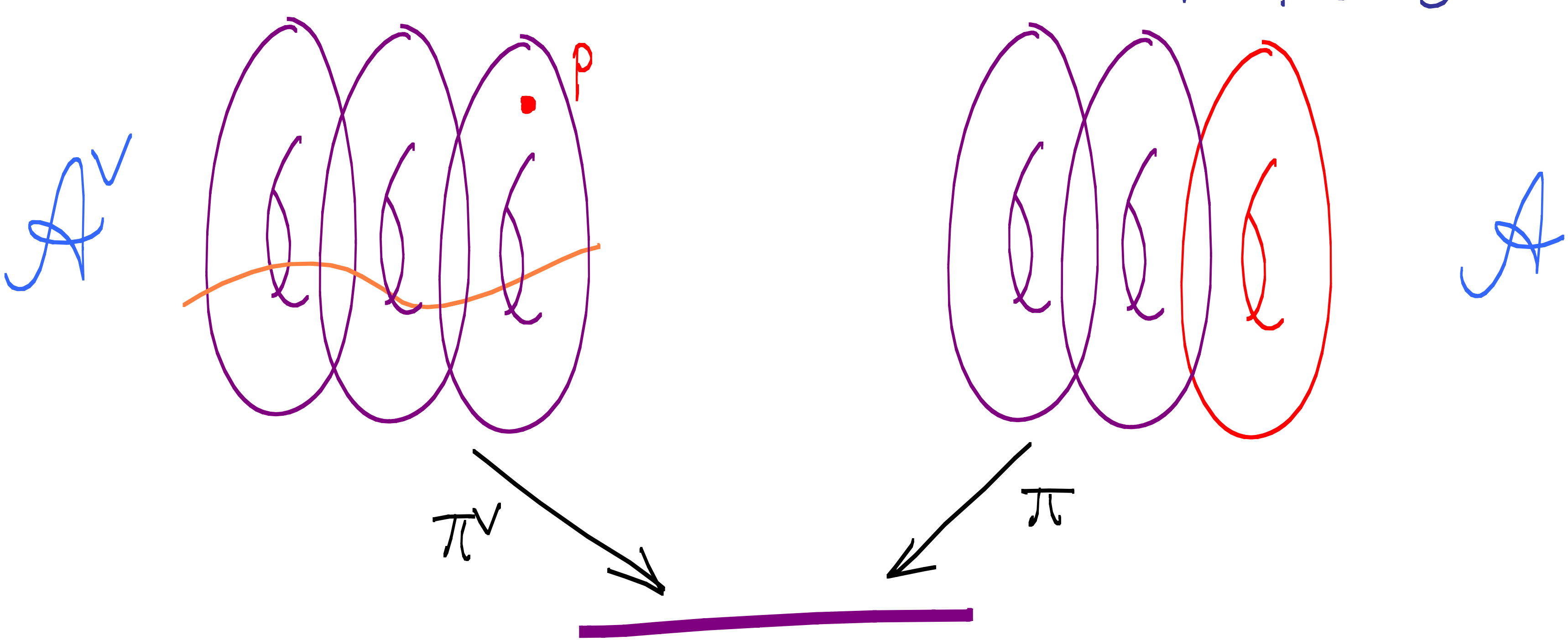
One can try to define something like this for a general SYZ dual pair of integrable systems

$$
\begin{tikzcd}
\mathcal{A} \arrow[dr, "\pi"'] & & \mathcal{A}^\vee \arrow[dl, "\pi^\vee"] \\
& \mathcal{B} &
\end{tikzcd}
$$

Abelian varieties over $\mathcal{B}$

Lagrangian fibrations

points $p \in \mathcal{A}^\vee$ $\sim$ rank 1 sheaves F_p on fibers



Lagrangian sections of $\pi^\vee$ $\sim$ line bundles $\mathcal{L}$ on $\mathcal{A}$

or in some more natural category

Upon quantization of $\mathcal{A}$

F_p = a module over some noncomm. algebra A

$\det \bar{\partial}\big|_{F_p}$ = an interesting section of some line bundle that

satisfies Hirota-type equations w.r.t. $F_p \to \mathcal{L} \otimes F_p$

Painlevé

perhaps, a construction like this
already exists ...

Then <a wild guess> if you can make sense of it for the Donagi-Markman system, shouldn't it give us Z(CY - 3 fold)?