Enumerative geometry and Geometric representation theory

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based on joint work with

Mina Aganagic, Roman Bezrukavnikov, Davesh Maulik, Nikita Nekrasov, Andrei Smirnov, ...

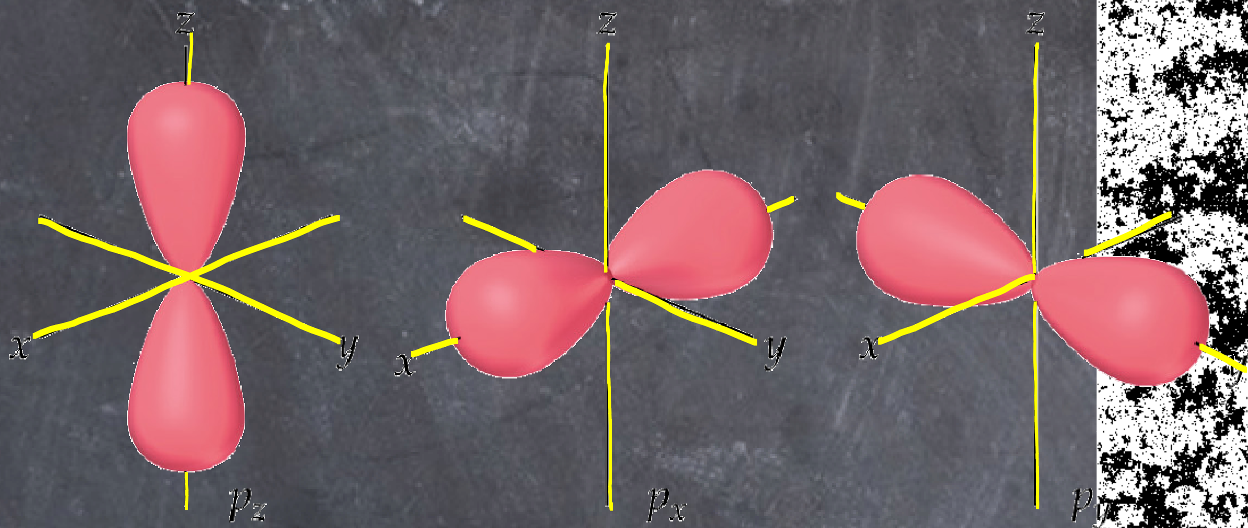
or, more precisely

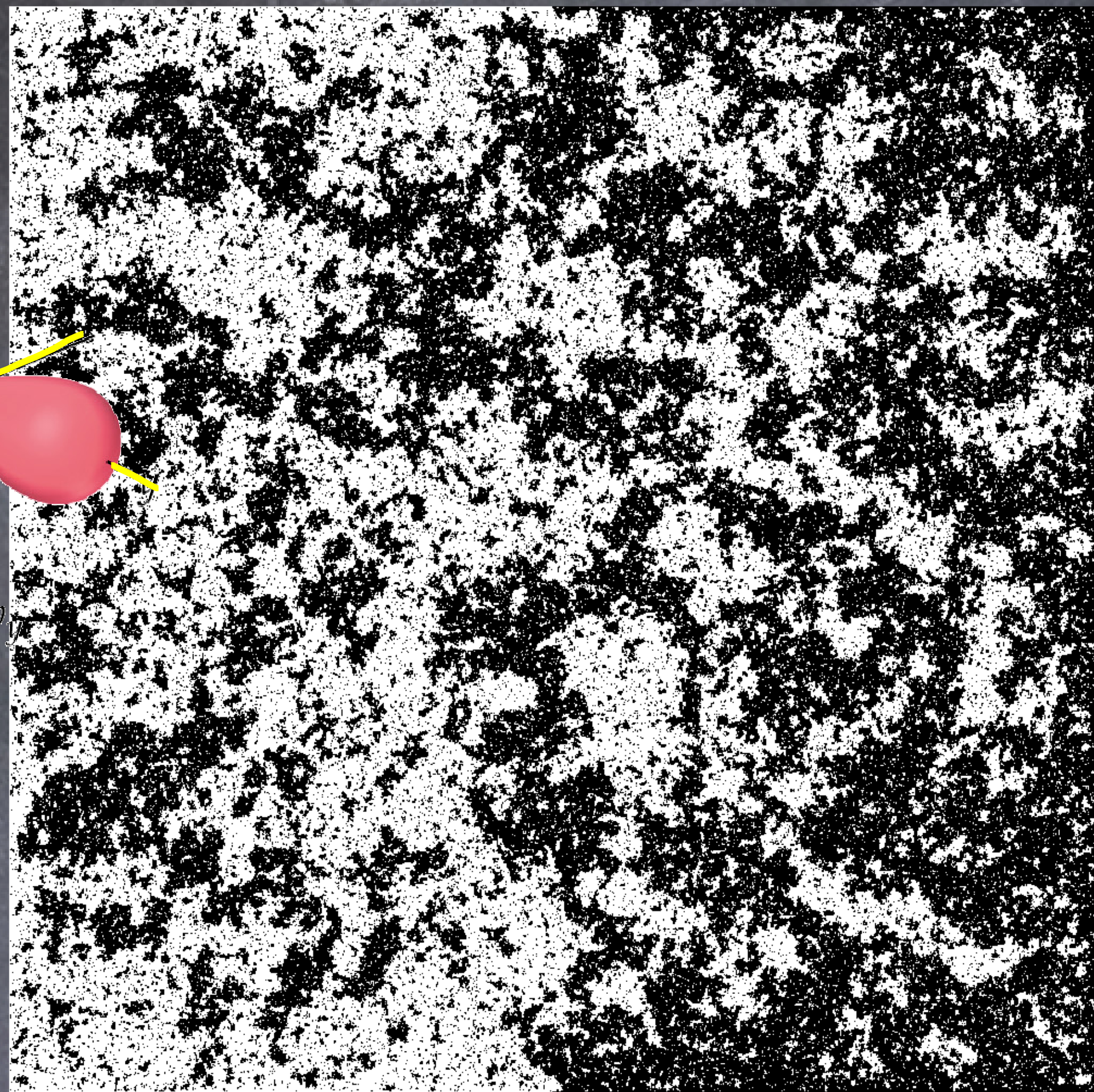
K-theoretic enumerative geometry

&

quantum groups

Representation theory has been successful in placing important operators of mathematical physics in the right context





In both representation theory and mathematical physics, linear operators are more and more replaced by correspondences, FM kernels, . . . , e.g.

$$K(X) \xrightarrow{\boxed{\text{a K-theory class on } X \times X}} K(X)$$

Enumerative geometry is a rich source of such

Equivariant K-theory is exactly the right generality for what follows

We will see how geometric representation theory places important operators of K-theoretic enumerative geometry in their proper context in the theory of quantum groups

Since the subject is quite technical,
I thought it might be a good idea
to start with potential

geometric applications

before plunging into technical depths

Lecture 1 Challenges in K-theoretic Donaldson-Thomas theories & related fields

DT theory is an enumerative theory of sheaves on 3-folds and of related/similar objects

To compare / contrast with Tom's lectures:

- work with arbitrary 3-folds, no assumptions on K_X

- work with coherent sheaves on moduli

- ...

Examples of DT moduli spaces $\mathbb{M}$

☆ Hilbert scheme of curves $\subset X$ = 3-fold

$$\mathcal{O}_X \twoheadrightarrow \mathcal{O}_{\text{subscheme}}$$

☆☆☆ Pandharipande-Thomas spaces

$$\mathcal{O}_X \longrightarrow \mathcal{F} \longrightarrow \text{Coker} \longrightarrow 0$$

pure dim = 1 dim = 0

+ many more

These possess a perfect obstruction theory and, hence

$$[\mathbb{M}]_{vir} \in H_{2(\deg \mathcal{F}, c_1(X))}(\mathbb{M})$$

independent of $\chi(\mathcal{F})$

$$\mathcal{O}^{vir} \in K_{Aut(X)}(\mathbb{M})$$

not quite the right object

forgotten by the map π

Basic object of study

$$\langle 1 \rangle = \pi_{PT \to Chow,\, *} \left((-q)^{\chi(\mathcal{F})} [\mathbb{M}]^{vir} \right)$$

$$\in H_{2 \cdot \mathrm{vir\,dim}} (Chow(X)) [[q]]$$

May replace $1 \in H^*(\mathbb{M})$ by a tautological class

Conjecturally [MNOP] equal to a similar cycle constructed from Gromov-Witten (X), with a tricky change of variable q

An even more basic conjecture

$$\langle 1 \rangle = \text{rational function of } q$$
$$\text{with poles at } 0, \infty, \sqrt{1}$$

Known for X = toric [MOOP], CY [Toda]

much, much more general [Pandharipande-Pixton]

... but it could be fair to say that there could be a better explanation of this rationality

with Nikita Nekrasov, we think that to understand this rational function we must

- work in equivariant K-theory
- interpret q as $q \in \mathrm{Aut}(Z)$ where Z is a certain Calabi-Yau 5-fold related to X

need not be CY

It was emphasized by Nekrasov, and is clear from math physics perspective, that in K-theory we must work not with $\mathcal{O}_{vir}$ but with its twist by $\sqrt{K_{vir}}$ where $K_{vir} = \det \mathrm{Obs} / \det \mathrm{Def}$

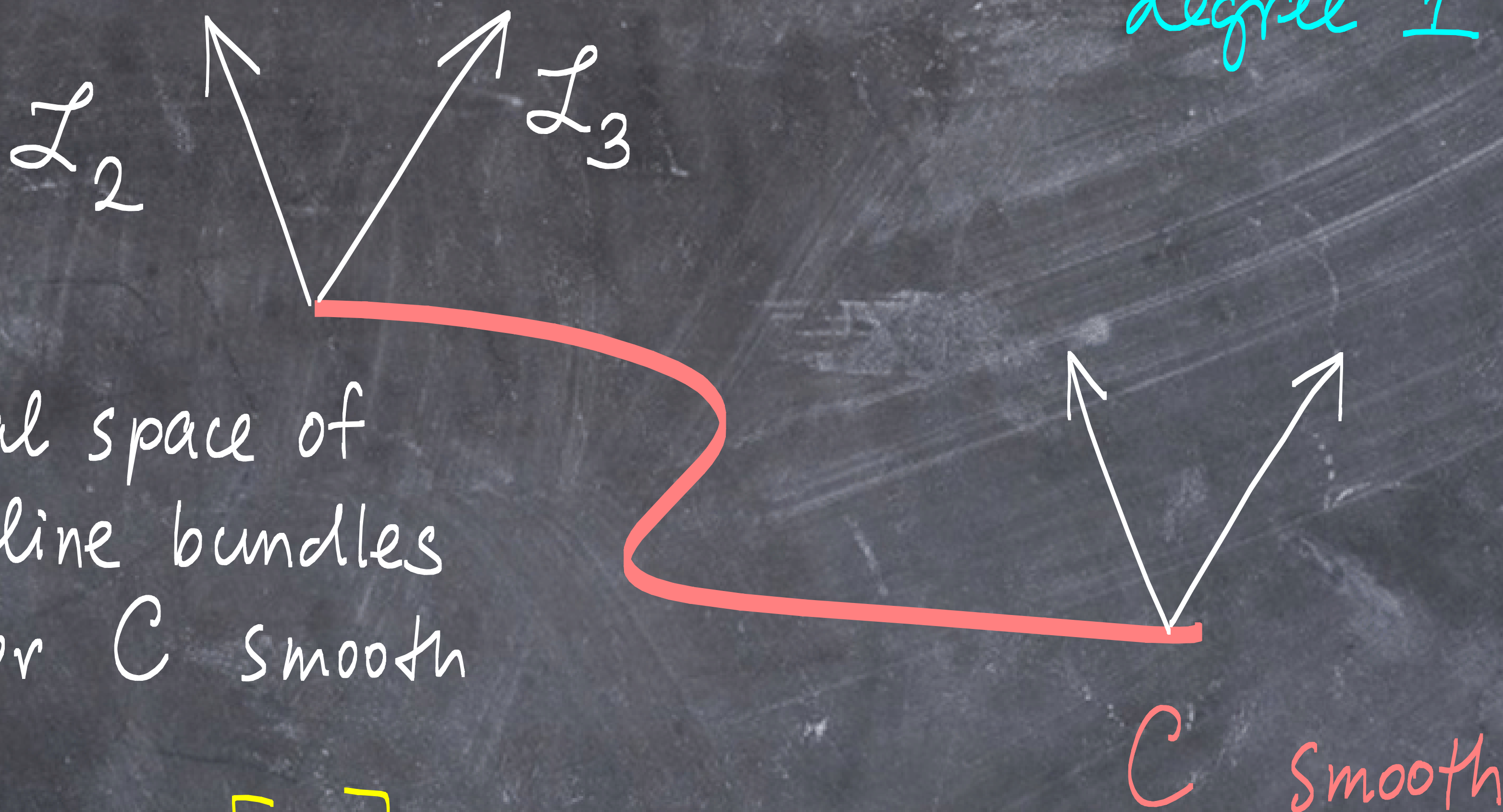
Like if $\mathbb{M}$ = a linear representation V of G

$$\mathcal{O}_{\mathbb{M}} \rightsquigarrow \hat{\mathcal{O}}_{\mathbb{M}} = \hat{S}^{\bullet} V^{*} = \sqrt{\det V} \otimes S^{\bullet} V^{*}$$

It is a theorem [NO] with suitable further twists, the required square roots exist

see below

First, a prototype, PT theory of a local curve
degree 1



X = total space of
two line bundles
over C smooth

Curve class = $[C]$

$$
\begin{tikzcd}
D \in \bigsqcup_n S^n C \arrow[r, hook] \arrow[d] & \mathbb{M} = PT(X, [C]) \arrow[d] \\
0 \arrow[r, hook] & \text{Sections of } \mathcal{L}_2 \oplus \mathcal{L}_3 = \mathrm{Chow}(X)
\end{tikzcd}
$$

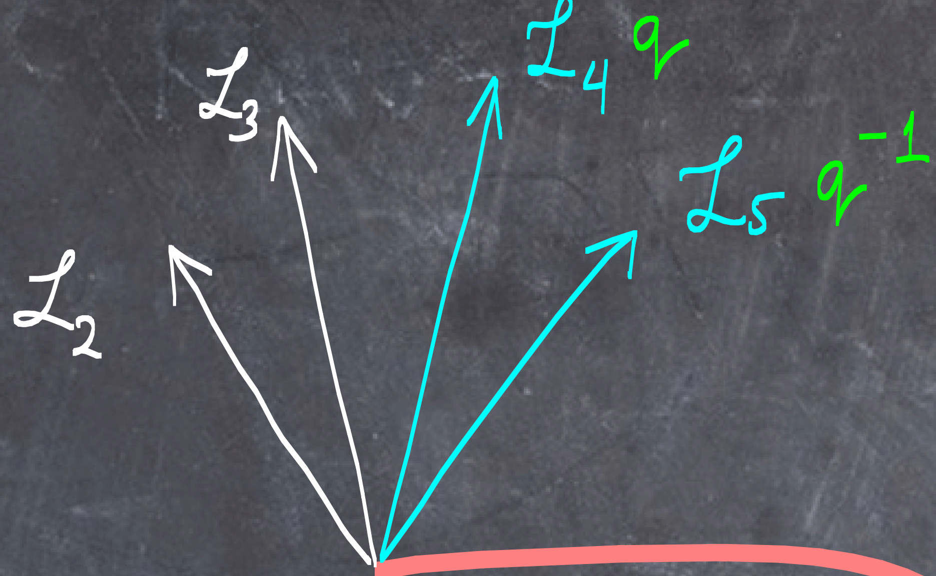
$$\text{Def} - \text{Obs} = H^\bullet(C, \mathcal{L}_2 \oplus \mathcal{L}_3)$$

$$+\ TS^nC - \text{twisted cotangent}$$

$$H^0(\mathcal{O}_D \otimes \mathcal{L}_2 \otimes \mathcal{L}_3)$$

in twisting $\mathcal{O}^{\mathrm{vir}}$ there is an additional degree of freedom which we can phrase as a choice of 4 line bundles on C such that

$$\bigotimes_{i=2}^{5} \mathcal{L}_i = K_C$$



we make q act by

$$\begin{pmatrix} q & \\ & q^{-1} \end{pmatrix} \text{ on } \mathcal{L}_4 \oplus \mathcal{L}_5$$

means that the total space Z of $\bigoplus_2^5 \mathcal{L}_i$ is a CY 5-fold

We define

$$\widehat{\mathcal{O}}^{\mathrm{vir}} = \mathcal{O}^{\mathrm{vir}} \otimes \left(K_{\mathrm{vir}} \otimes \det H^{\bullet}(\mathcal{F} \otimes (\mathcal{L}_4 - \mathcal{L}_5)) \right)^{1/2}$$

really $(\mathcal{L}_4/\mathcal{L}_5)^{\boxtimes n} \otimes q^{n+\dots}$ for $S^n C$

there is a simple

Theorem

$$\pi_{\mathbb{M} \to \mathrm{Chow},*} \, \widehat{\mathcal{O}}^{\mathrm{vir}} \propto \hat{S}^{\bullet} H^{\bullet}\left(\mathcal{L}_2 \oplus \mathcal{L}_3 \oplus q\mathcal{L}_4 \oplus q^{-1}\mathcal{L}_5\right)^{*}$$

up to $(-1)^{\cdots} q^{\cdots}$ which is best included into the definition of $\widehat{\mathcal{O}}^{\mathrm{vir}}$

deformations of C in Z !

cf. Macdonald

For the general conjecture we will need:

- a nonsingular 5-fold Z with $K_Z \simeq \mathcal{O}_Z$
- an action $\mathbb{C}_q^\times \to \mathrm{Aut}(Z, \Omega^5)$

  such that $X = \bigsqcup X_i = Z^{\mathbb{C}_q^\times}$

  has pure dimension $= 3$

Example 1

$$Z = q\,\mathcal{L}_4 \oplus \mathcal{L}_5\, q^{-1}$$

$$\downarrow$$

$$X$$

$$\mathcal{L}_4 \otimes \mathcal{L}_5 = K_X$$

Example 2

$$Z_n = \widetilde{Z/\mu_n} \qquad \mu_n \subset \mathbb{C}^\times_q$$

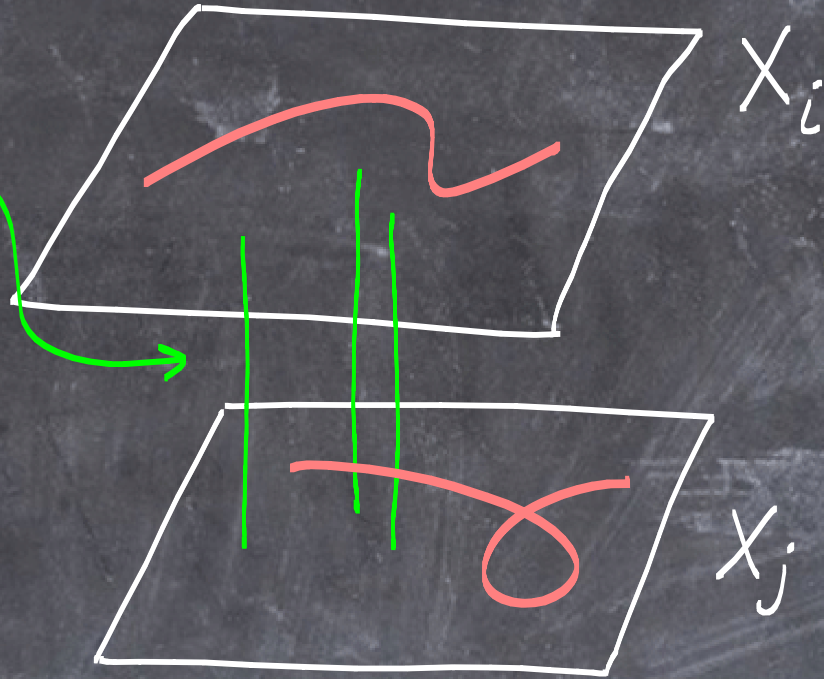
A_{n-1}-surface fibration over X

$$Z_n^{\mathbb{C}^\times_q} = n \text{ copies of } X$$

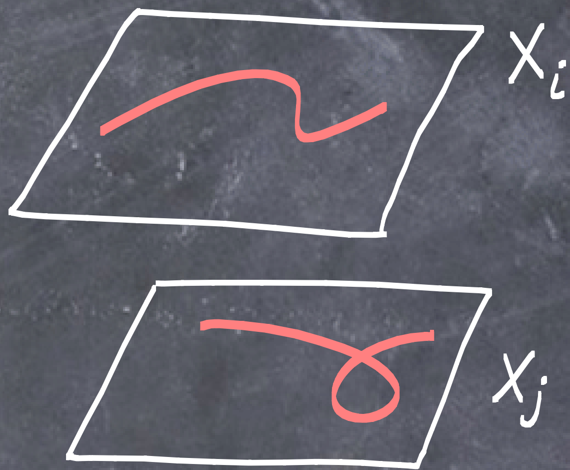
$\mathbb{C}_q^\times$ - fixed curves in Z look like this:

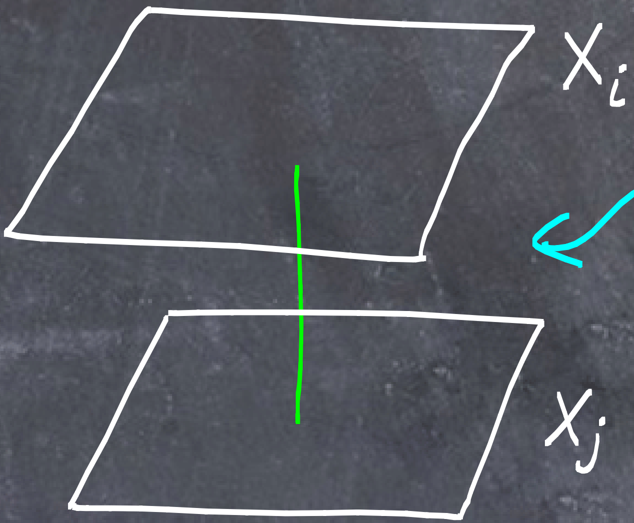
$\mathbb{C}_q^\times$ orbits discarded in Chow(X)

become equiv. variables



Chow(X)





gives

$\varphi_{ij} \in K(X_i \times X_j)$

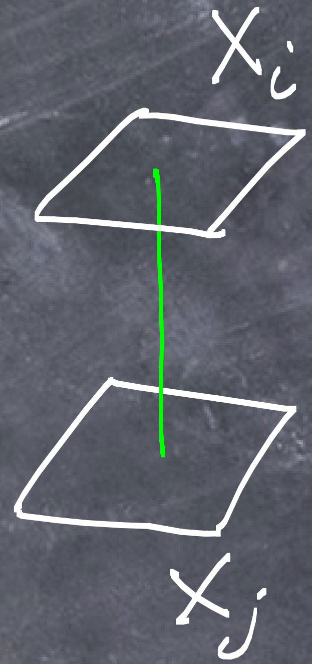
The general conjecture says

membranes side $\stackrel{?}{=}$ DT side

in $K_{\mathrm{equiv}}(\mathrm{Chow}(X))$

includes $\mathrm{Aut}(Z,\Omega^5)^{\mathbb{C}^\times_q}$, Q_{ij} = class

considered as a character of an additional torus



DT side

$$\pi_{DT \to Chow,*}\left((-q)^{\chi}\, \hat{\mathcal{O}}_{vir} \otimes \mathrm{Interaction} \right)$$

where

$$\mathrm{Interaction}_{ji} = S^{\bullet} Q_{ij}\; \chi\left(\mathcal{O}_{X_j} - \mathcal{F}_j, \Phi_{ji}(\mathcal{O}_{X_i} - \mathcal{F}_i)\right)$$

the normal bundle to each X_i determines its $\mathcal{L}_4^{(i)}$ and $\mathcal{L}_5^{(i)}$

Example for Z_n, $Z_n^{\mathbb{C}_q^\times} = n\, X$

the DT side becomes localization formula for

$$\mathcal{O}_X^{\oplus n} \longrightarrow \mathcal{F}$$

$$\circlearrowleft$$

w.r.t. $T \subset GL(n)$ $\qquad Q_{ij} \in T^\wedge$

In other words, for Z_n, the DT side computes rank n DT theory of X

Membranes side

$$\begin{array}{ccc} M2(Z) & \xleftarrow{\iota} & M2(Z)^{\mathbb{C}_q^\times} \\ & & \Big\downarrow \pi \\ & & \mathrm{Chow}(X) \end{array}$$

Semigroup

$$\text{Membrane side} = S^{\bullet}_{\mathrm{Chow}}\, \pi_*(\iota_*)^{-1}\, \hat{\mathcal{O}}_{\mathrm{vir}}$$

exist by loc.

where

$$M2(Z) = \text{maps } f: C \longrightarrow Z$$

1-dimensional scheme

such that

- f is an isomorphism away from a finite set in C
- slope stable, that is $\forall\, C' \subsetneq C$

$$\chi(C')\, f(C) > \chi(C)\, f(C')$$

bounded for given degree of f

this is important because there is no variable to keep track of $\chi(C)$

Finally, $\hat{\mathcal{O}}_{vir}$ for membanes is

- $\hat{\mathcal{O}}_{vir}$ provided by deformation theory wherever perfect

over an open set in Chow

- 

pushed forward from a certain blowup elsewhere ....

big construction, 2 people, need more

To summarize

$$
\begin{tikzcd}
M2(Z)^{\mathbb{C}^\times_q} \arrow[dr, "\text{proper}"'] & & DT(X) \arrow[dl, "\text{not proper, need } (-q)^{\chi}"] \\
& \mathrm{Chow}(X) &
\end{tikzcd}
$$

So the membrane side really <u>sums up</u> the DT series and is evidently a rational function

Checks in many numeric examples ...

Also checks for Hilb(X, points) where the following formula was conjectured by Nekrasov

<u>Theorem</u>

$$\sum_n (-q)^n \chi\left(\mathrm{Hilb}(X,n), \hat{\mathcal{O}}_{vir}\right) = S^{\bullet} \bigstar$$

where

$$\bigstar = \chi\left(X, \frac{q\mathcal{L}_4\left(T_X + K_X - T_X^\vee - K_X^\vee\right)}{(1-q\mathcal{L}_4)(1-q\mathcal{L}_5^{-1})}\right)$$

$$= \chi\left(Z, T_Z^\vee - T_Z\right) - \mathrm{const}(q)$$

this counts the fields of M-theory on Z

the conjecture predicts relations between different DT counts of the following form

$$\begin{tikzcd} & Z \arrow[dl, bend right, "\mathbb{C}^\times_q"'] \arrow[dr, bend left, "\text{a different action}"] & \\ X & & X' \end{tikzcd}$$

which interchange degree (Kähler) and equivariant var

for example, take $Z = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3 \oplus \mathcal{L}_4 \to C$

it is hard to see these dualities below equivariant K-theory

Some cases of such dualities are well-known in physics literature:

$$\zeta^n = 1 \qquad \gamma^m = 1$$

$$\mathcal{Z}_{n,m} = \overbrace{\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3 \oplus \mathcal{L}_4}^{} \Big/ \begin{pmatrix} \zeta & & & \\ & \zeta^{-1} & & \\ & & \gamma & \\ & & & \gamma^{-1} \end{pmatrix}$$

$$\downarrow$$

$$\mathbb{C}$$

PT

$\wr\wr$

says

$$K \text{ theory} \left(\begin{matrix}\text{quasi}\\ \text{maps}\end{matrix} : \mathbb{C} \to \text{rank } m \text{ bundles on } A_{n-1} \right)$$

$$\|$$

$$K \text{ theory} \left(\begin{matrix}\text{quasi}\\ \text{maps}\end{matrix} : \mathbb{C} \to \text{rank } n \text{ bundles on } A_{m-1} \right) \qquad (\star)$$

with an exchange of Kähler $\leftrightarrow$ equivariant variables this is a textbook case of a duality in 3-dimensional SUSY gauge theories known as

3D mirror / symplectic / Seiberg / Langlands / ...

both sides of $(\star)$ compute a certain index in these gauge theories on $M^3 = C \times S^1$

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from the representation-theoretic perspective $(\star)$ is a double-loop generalization of the classical level-rank duality

Status: • the membranes = DT conjecture, as well as equality of K-theoretic curve counts in symplectically dual varieties remain open as I speak

• geometric RT methods give as good a control over K-theoretic DT counts as we have in cohomology, and the principal challenge is to replicate those on the membane side

• Same tools gives us a much more refined information about K-theoretic curve counts in any Nakajima variety than just (☆), I expect (☆) for Nakajima varieties to follow from what we already know