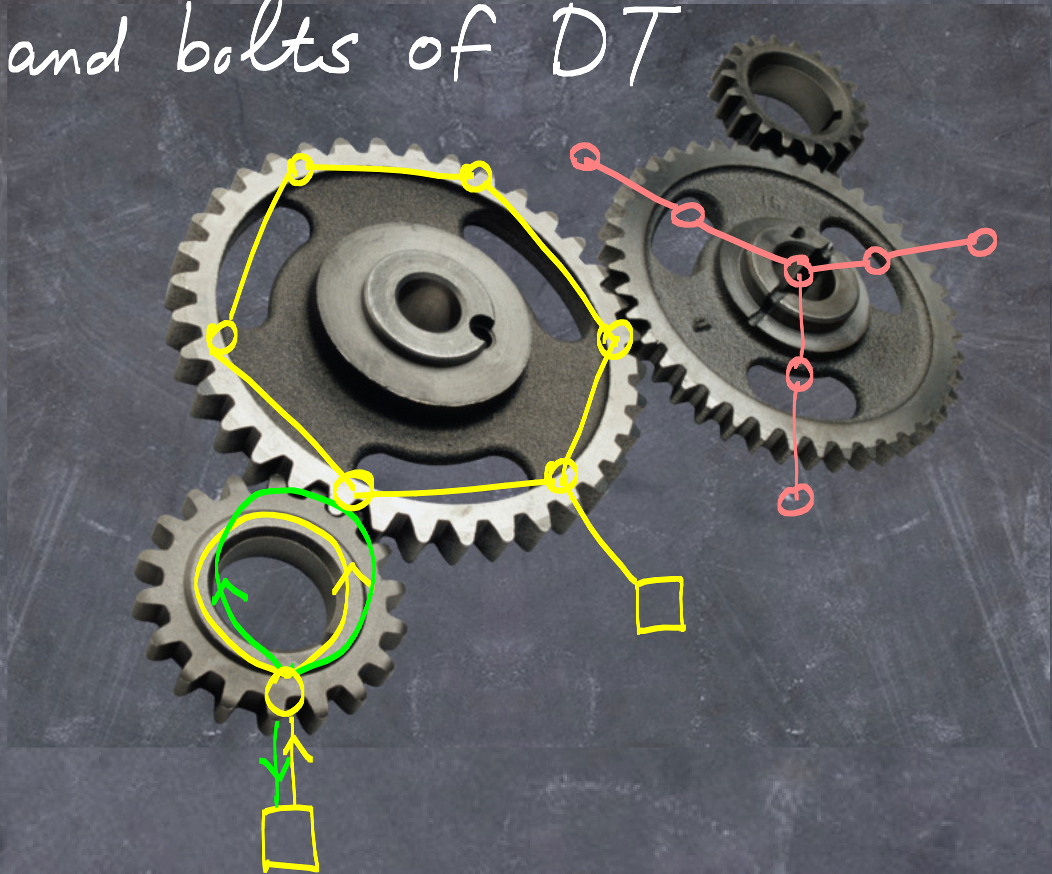
Lecture 2 Nuts and bolts of DT



Let X = nonsingular quasiprojective 3 fold

no assumptions about K_X

connected reductive $G \circlearrowright X$

$\mathbb{M}$ = one of DT moduli spaces, such as the

Pandharipande-Thomas moduli space

$$\left\{ \mathcal{O}_X \to \mathcal{F} \to \mathrm{Coker} \to 0 \right\}$$

$\mathcal{F}$: pure 1 dim

Coker: 0-dimensional

$\hat{\mathcal{O}}_{\mathrm{vir}}$ = symmetrized virtual structure sheaf of $\mathbb{M}$

We are interested in equivariant

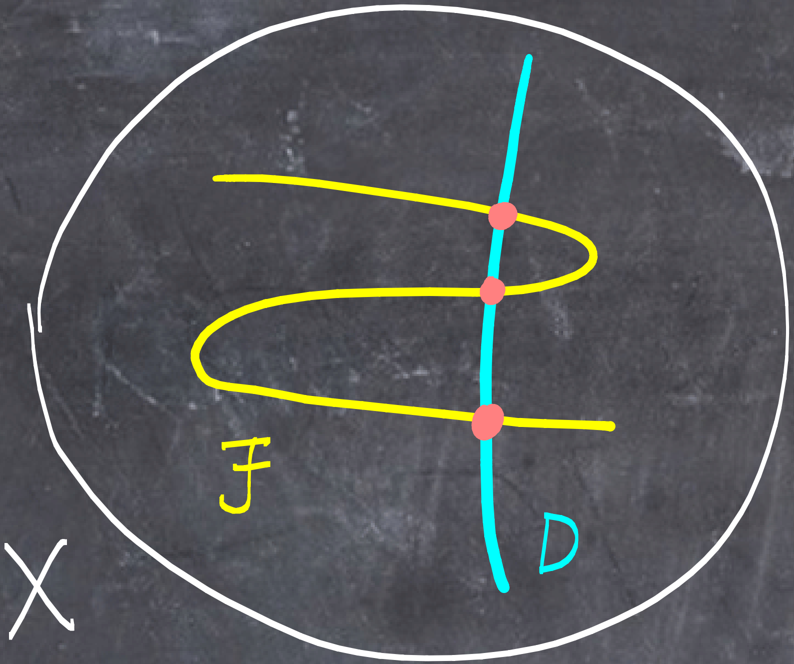
K-theoretic DT counts of the kind

$$\chi\left(\mathbb{M},\ \widehat{\mathcal{O}}_{\mathrm{vir}} \otimes \text{tautological} \cdot z^{[\mathcal{F}]}\right)$$

constructed from the universal sheaf on $\mathbb{M} \times X$

class of $\mathcal{F}$ in $K_{Top}(X)$

More generally, if $D \subset X$ is a nonsingular divisor there exist a resolution of the rational map



$$ev: \mathcal{F} \longmapsto \mathcal{F} \otimes \mathcal{O}_D \in \mathrm{Hilb}(D)$$

$$\begin{tikzcd}
 & \mathbb{M}_{\text{relative } D} \arrow[dr, "ev", "\text{proper}"'] & \\
\mathbb{M}_{\text{transverse}} \arrow[ur, hook] \arrow[rr] & & \mathrm{Hilb}(D)
\end{tikzcd}$$

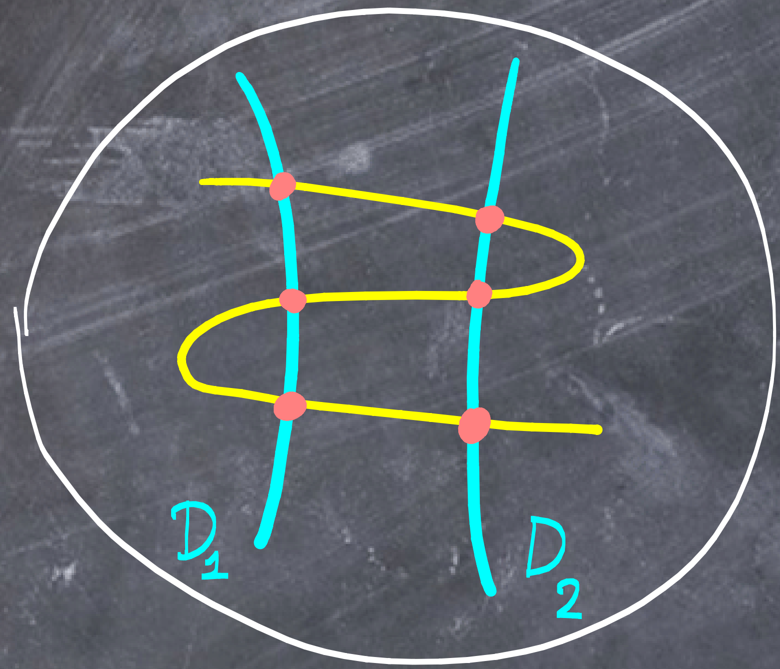
Known as the moduli space relative D [Li-Wu, ...]

using it, we can define relative DT counts

$$ev_*(\text{ same as before }) \in K_{equiv}(\mathrm{Hilb}(D, pts))$$

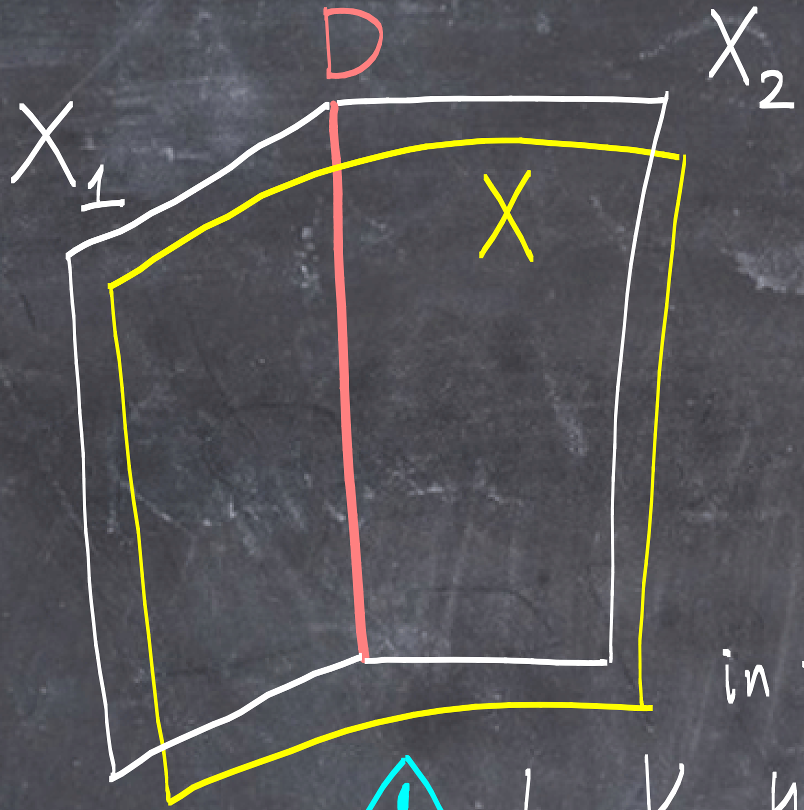
if $D = D_1 \sqcup D_2$ this will look like an operator

$$K(\mathrm{Hilb}(D_1)) \longrightarrow K(\mathrm{Hilb}(D_2))$$



The basic moves:

- degeneration
- trading relative conditions for tautological
- equivariant localization



Degeneration formula [Li-Wu,...]

$$X \rightsquigarrow X_1 \cup_D X_2$$

$\uparrow$ nonsingular

lets us compute counts in X in terms of relative counts for (X_i, D)

⚠ In K-theory, there is a nontrivial

gluing operator $\in \operatorname{Aut}(K(\operatorname{Hilb}(D)))[[z]]$

similar to the one discussed by Givental in Gromov-Witten context

The work of [Maulik - Pandharipande,
Levine - Pandharipande,
Pandharipande - Pixton, ...]

strongly suggest that the iterations of the loop

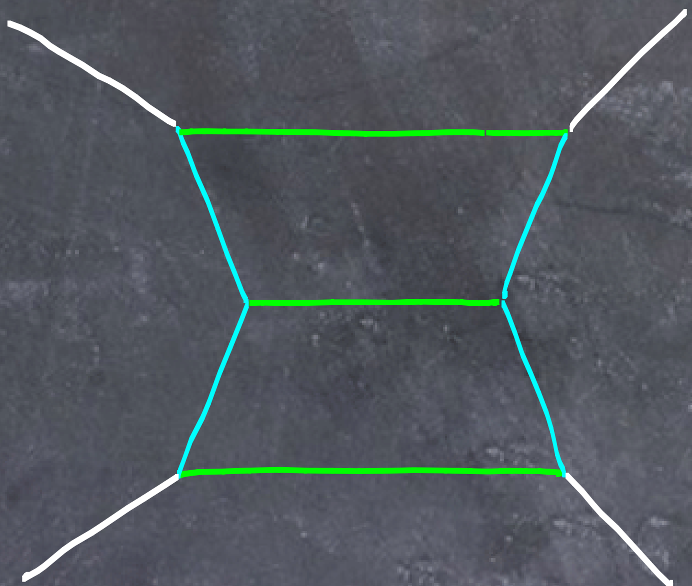
$$X \underset{\text{replace relative by tautological}}{\overset{\text{degeneration}}{\rightleftarrows}} (X, D)$$

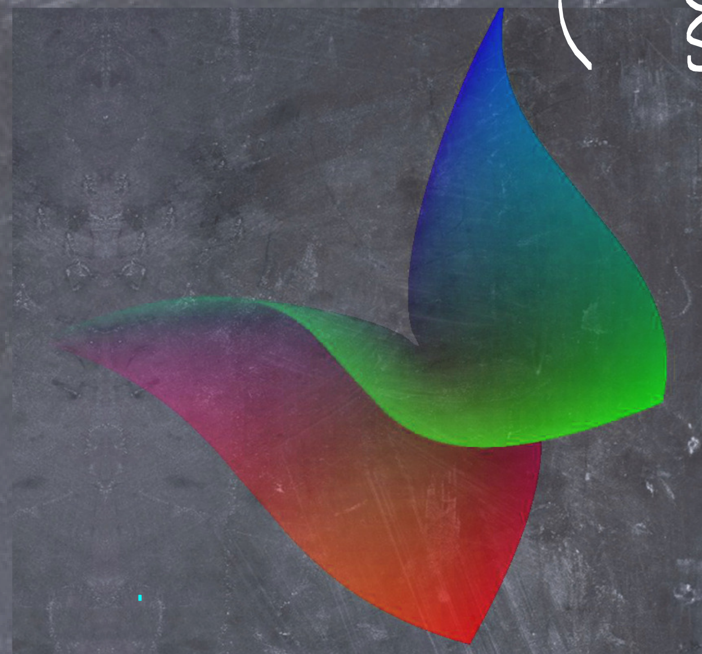
to be revisited below

express DT counts for arbitrary X in terms of those for toric X

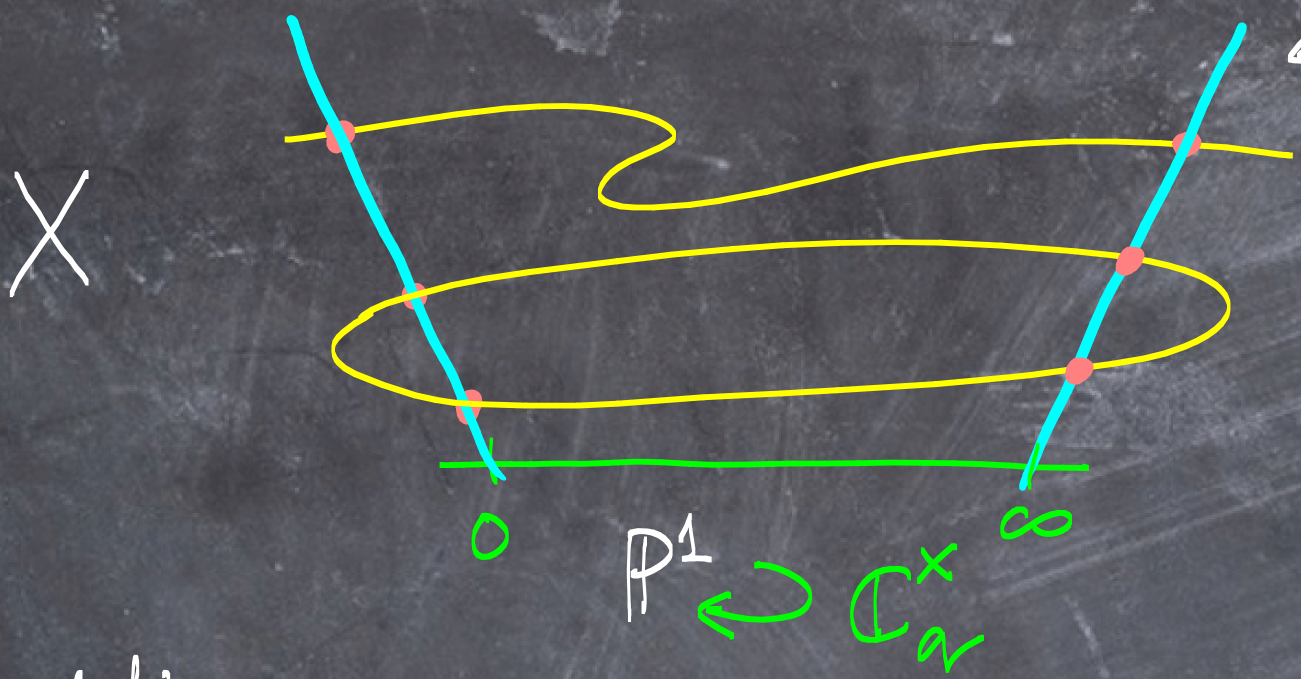
(in finite, but possibly very large # of steps) for toric X, localization is added to the toolbox and it reduces [MOOP] everything to the case

$$\begin{array}{ccc} A_n \text{ surface} \hookrightarrow & X & \\ & \downarrow & \\ & \mathbb{P}^1 & \end{array} = \widetilde{\mathcal{O}(m_1)\oplus\mathcal{O}(m_2)} \Big/ \begin{pmatrix} \zeta & \\ & \zeta^{-1} \end{pmatrix} \qquad \zeta^n = 1$$

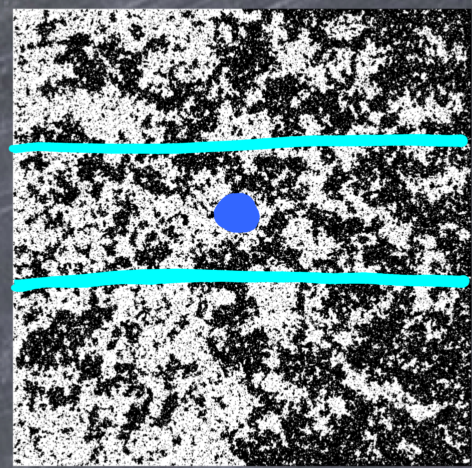




We can work equivariantly relative fibers over $0, \infty \in \mathbb{P}^1$



A_n fibers



and this defines operators

$$S_{m_1, m_2} \circlearrowright K_{eq}\left(\mathrm{Hilb}(A_n)[q]\right)_{loc}[[z]]$$

that form a representation of $(m_1, m_2) \in \mathbb{Z}^2$ by degeneration by work of [..., Nakajima, ...]

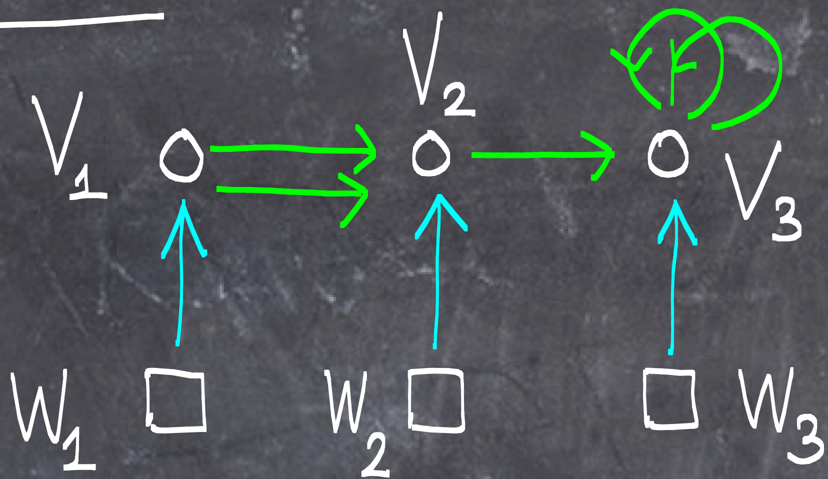
$$K_{eq}(\mathrm{Hilb}(A_n))$$

is a very important space in geometric representation theory and one of my goals in this lecture is to explain how operators like S_{m_1, m_2} are placed in GRT context in fact, instead of $\mathrm{Hilb}(A_n)$ it is natural to work in the generality of

<u>Nakajima varieties</u>

Nakajima varieties

quiver



edge mult

$$\mathrm{Rep}\,Q = \prod \mathrm{Hom}(V_i, V_j \otimes E_{ij}) \times \prod \mathrm{Hom}(W_i, V_i)$$

$$M(v,w) = T^* \mathrm{Rep}\,Q /\!/\!/ GL(v)$$

$$= \mu^{-1}(0) /\!/ GL(v)$$

$GL(W) \times GL(E) \times \mathbb{C}^\times_\hbar \leftarrow$ scaling T^* directions

Among Nakajima varieties for 

we find $\mathrm{Hilb}(A_5)$ as well as moduli of higher rank sheaves on A_5 surface

General Nakajima varieties share many properties of moduli of sheaves on symplectic surfaces and counting curves in them is a problem very much akin to DT theory of 3-folds.

Nakajima varieties form the largest class of equivariant symplectic resolutions, which are rapidly gaining importance in representation theory

Nakajima constructs an action

$$\mathcal{U}_{\hbar}(\hat{\mathfrak{g}}_{KM}) \hookrightarrow K_{eq}(\mathcal{M}(w))$$

$$\mathcal{M}(w) = \bigsqcup \mathcal{M}(v,w)$$

deformation parameter

where $\mathfrak{g}_{KM}$ is the Kac-Moody Lie algebra with

$$\text{Cartan matrix} = 2 - Q - Q^T$$

adjacency matrix

geometric applications require a larger algebra

$\mathcal{U}_{\hbar}(\hat{\mathfrak{g}}_Q)$, where $\mathfrak{g}_Q \supset \mathfrak{g}_{KM}$ whose construction will be explained in Lecture 3 the neccesity & sufficience of the larger algebra is evident from the following formula in cohomology

Consider

$$\mathrm{ev}_* \left([\overline{\mathcal{M}}_{0,2}(X)]_{\mathrm{vir}}\, z^{\deg} \right) \circlearrowleft H^{\bullet}_{eq}(X)$$

Theorem (Maulik-O.) $\|$ for X = Nakajima variety

$$\boxed{} = \hbar \sum \ln(1 - z^{\alpha}) \sum e^{i}_{-\alpha}\, e^{i}_{\alpha}$$

over all $\alpha \in H_2(X,\mathbb{Z})_{\mathrm{eff}}$ which are roots of $\mathfrak{g}_Q$

dual bases of $\mathfrak{g}_{Q,\alpha}$ and $\mathfrak{g}_{Q,-\alpha}$

"Casimir"

The set

$$\text{roots of } \mathfrak{g}_Q \in H_2(X,\mathbb{Z}),$$

and the multiplicities of these roots, comes up in many interrelated questions, such as

- Automorphisms of $D^b \operatorname{Coh} X$ [Bezrukavnikov, Kaledin, ...]

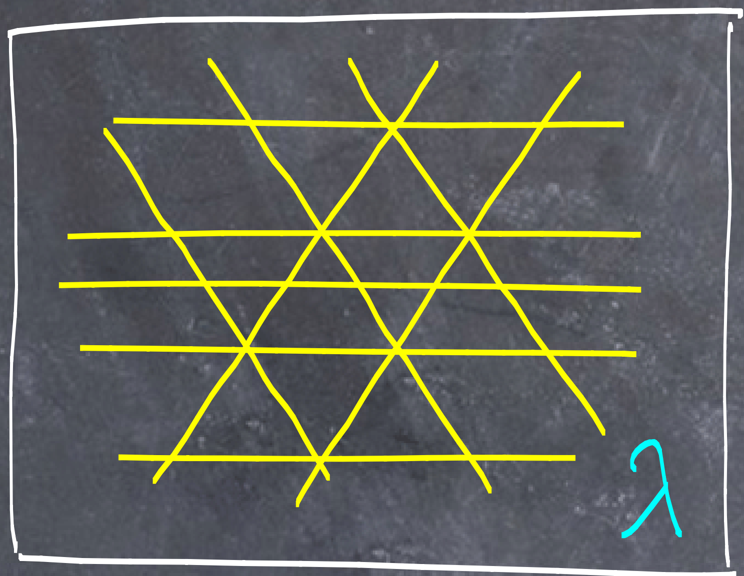
$\lambda \in H^2(X,\mathbb{C})$

- Monodromy of the "quantum differential equation"

$$\nabla = \frac{d}{d\lambda} - \left(\lambda \cup + \frac{\partial \Xi}{\partial \lambda}\right) \qquad \frac{\partial z^\alpha}{\partial \lambda} = (\alpha,\lambda)\, z^\alpha$$

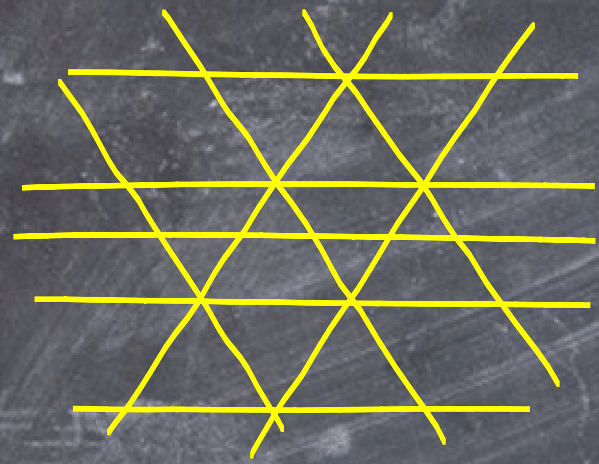
Casimir connection

In particular, in the work in progress with Bezrukavnikov we identify the monodromy of ∇ with the action of the quantum Weyl group of $\mathcal{U}_\hbar(\hat{\mathfrak{g}}_Q)$, extending results and conjectures of V. Toledano-Laredo, A. Varchenko, and others. As an abstract group, this is the braid group of a periodic arrangement in $H^2(X) =$ Cartan subalgebra



$(\lambda, \alpha) \in \mathbb{Z}$ $\qquad \alpha \in \{\text{roots}\}$

for $X = \mathcal{M}(v,w)$ only roots α with $\alpha \le v$ actually enter the relation to $\mathrm{Aut}\, D^b\mathrm{Coh}\, X$ is that the alcoves of the same arrangement parametrize t-structures on $D^b\mathrm{Coh}\, X$ constructed by Bezrukavnikov via quantization in char $p \gg 0$



With some more progress in our work, we should be able to show

quantum Weyl group = monodromy of ∇ =

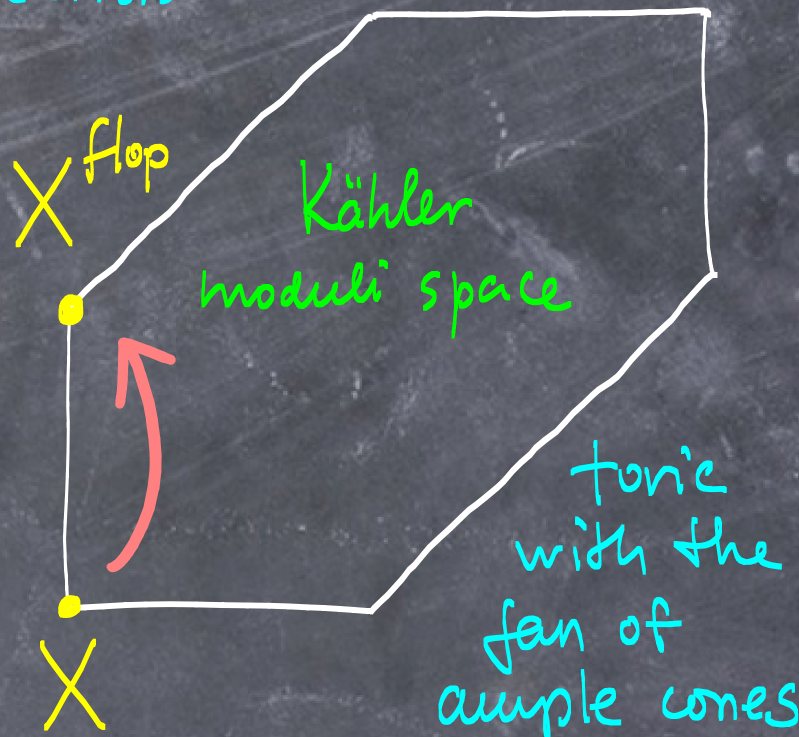
= action of Bezrukavnikov's braid group on $K_{eq}(X)$

The operator Ξ, the connection ∇, ... is about equivariant quantum cohomology of Nakajima varieties.

In K-theory, we go one level up:

- ∇ is replaced by a flat difference connection, the quantum difference connection
- its monodromy is an enriched version of flop, elliptic in both Kähler and equivariant variables

categorical interpretation?



. . .

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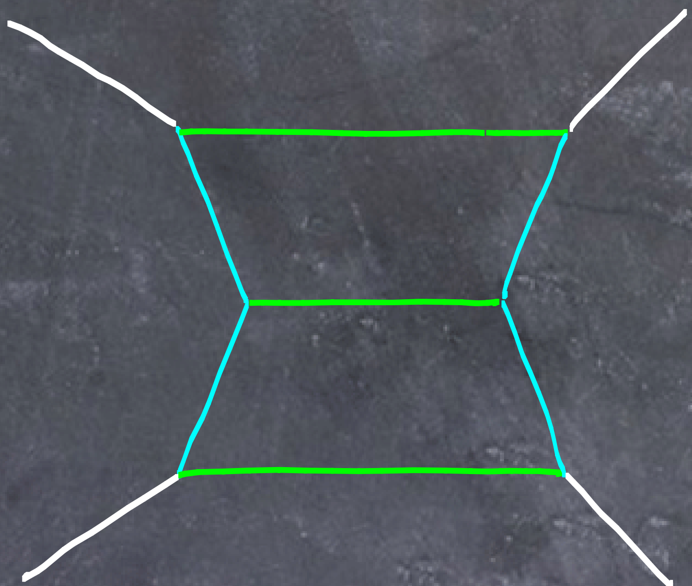
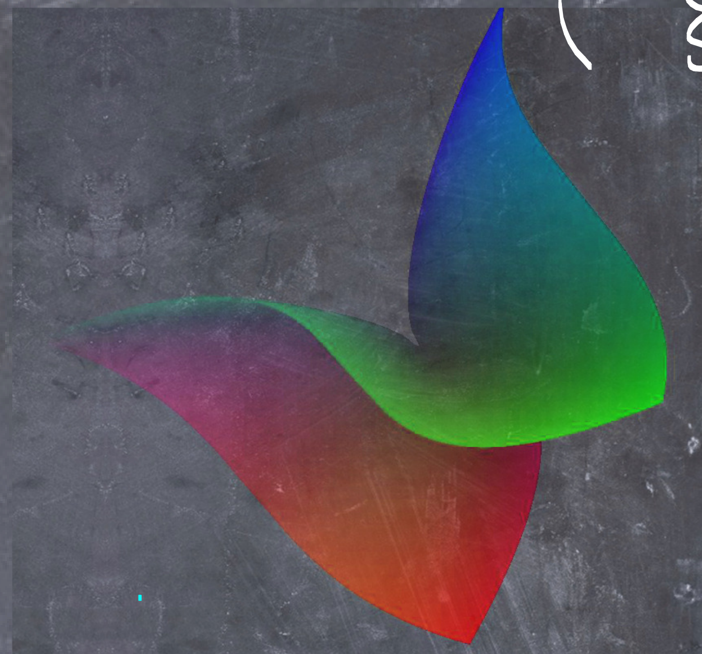
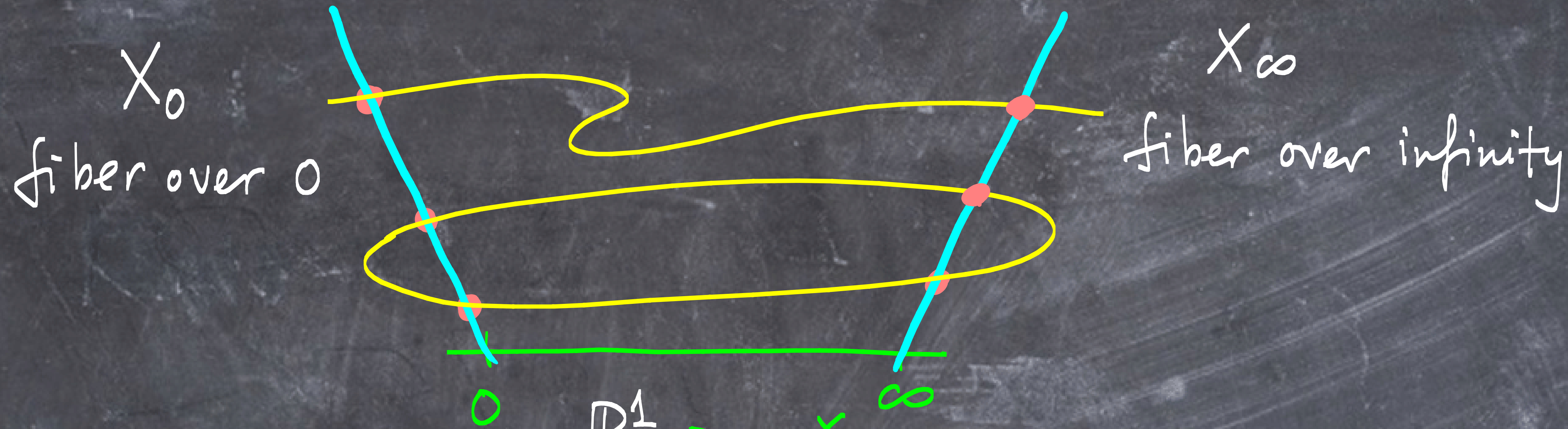




We can work equivariantly relative fibers over $0, \infty \in \mathbb{P}^1$



$$\mathbb{P}^1 \circlearrowleft \mathbb{C}^\times_q$$

$$S_{m_1,m_2} : K_{eq}(X_0) \longrightarrow K_{eq}(X_\infty)_{loc}[[z]]$$

the action of $\mathbb{C}^\times_q \times \mathrm{Aut}(X)$ on two sides differs

precisely by the map $\mathbb{C}^\times_q \to \mathrm{Aut}(X)$ that defines the X-bundle in parallel to what happens in cohomology, we have

<u>theorem</u> The shift operators S_{m_1, m_2} are uniquely characterized as part of a flat difference connection that shifts Kähler variables by $z \longmapsto q^{\mathcal{L}} z$, $\mathcal{L} \in \mathrm{Pic}(X)$, and equivariant variables by $\sigma : \mathbb{C}^\times \to \mathrm{Aut}(X)$

$=$

In contrast to cohomology, Kähler and equivariant parts of this connection are objects of the same nature and, in fact, exchanged under symplectic duality the Kähler part of the quantum connection comes from the action of action of $\mathcal{U}_\hbar(\hat{\mathfrak{g}}_Q)$ on $K_{eq}(X)$

and namely

Nakajima variety

Theorem* [O-Smirnov] it is given by the action of

$$\mathrm{Pic}(X) = \text{lattice in the}$$

dynamical quantum Weyl group of $\mathcal{U}_\hbar(\hat{\mathfrak{g}}_Q)$

take parameters in the Cartan torus $\ni z$

For finite-dimensional $\mathfrak{g}_Q$, dqWg defined by Etingof & Varchenko the identification of the quantum connection determines the correspondence between

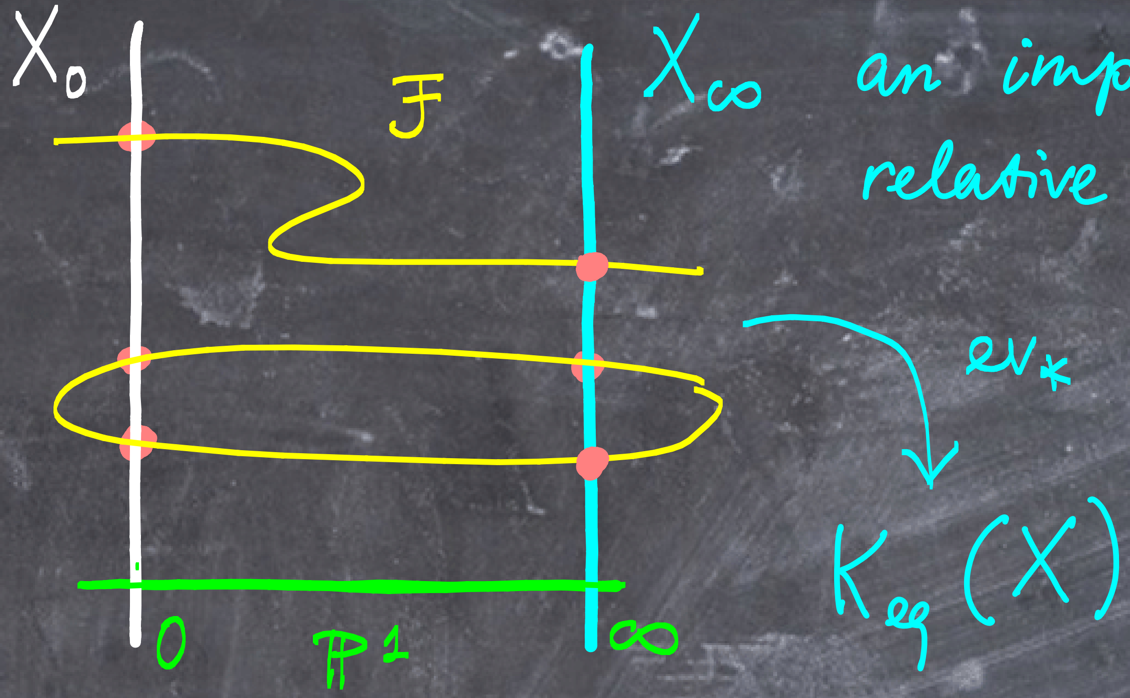
$$\text{relative} \longleftrightarrow \text{tautological}$$

conditions in enumerative K-theory via the following construction. Let

$\mathbb{M}(\mathbb{P}^1 \to X)$ = moduli space of quasimaps to X in the sense of [Ciocan-Fontanine-Kim-Maulik]

which is an analog of PT space for general $X = \mathcal{M}(v, w)$ and enters the geometric definition of the connection we take a tautological class

$$\mathcal{G} = \mathcal{G}(\mathcal{O}_{X_0} \overset{L}{\otimes} \mathcal{F})$$



an impose a relative condition at X_∞

Theorem For a Nakajima variety $X = \mathcal{M}(v, w)$

$$ev_*\left(\widehat{\mathcal{O}}_{vir} \otimes \mathcal{G}\right) = 0 \qquad \deg \neq 0$$

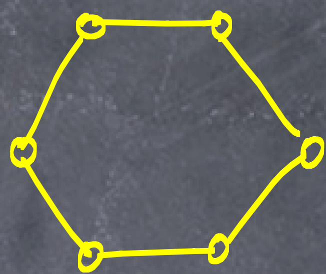
for any $\mathcal{G}$ and all $w \gg 0$

Concretely, for $\mathrm{Hilb}(A_n)$ this means that looking at sheaves of sufficiently high rank any given tautological $\Rightarrow$ relative computation

<u>Conclusion</u> for <u>today</u> :

K-theoretic DT counts for threefolds are built from representation theory of $\mathcal{U}_\hbar(\widehat{\mathfrak{gl}(n)})$

$$\widehat{\mathfrak{gl}(n)} = \mathfrak{g}_Q$$

Q 

Plan for tomorrow:

- construction of $\mathcal{U}_\hbar(\hat{\mathfrak{g}}_Q)$ and of the dynamical quantum Weyl group
- discussion of extensions to

$$\begin{tikzcd}
D^b \mathrm{Coh}\, X \arrow[dr] & & \mathrm{Ell}_{eq}(X) \arrow[dl] \\
& ??? &
\end{tikzcd}$$