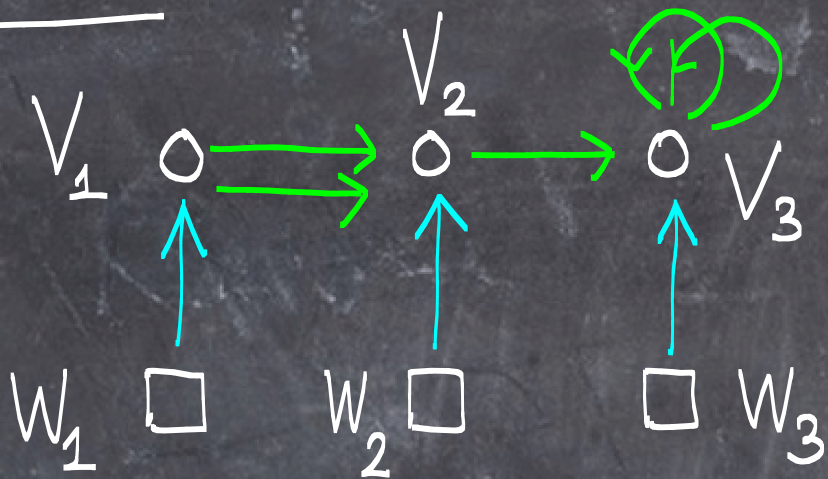
# Lecture 3 Stable envelopes and applications

recall the definition of

Nakajima quiver varieties

# Nakajima varieties

## quiver




edge mult

$$\mathrm{Rep}\, Q = \prod \mathrm{Hom}(V_i, V_j \otimes E_{ij}) \times \prod \mathrm{Hom}(W_i, V_i)$$

$$\mathcal{M}(v,w) = T^* \mathrm{Rep}\, Q /\!\!/\!\!/ GL(v)$$
$$= \mu^{-1}(0) /\!\!/ GL(V)$$

$$GL(W) \times GL(E) \times \mathbb{C}^\times_\hbar$$ ← scaling $T^*$ directions

Our main goal for today is to construct an action of a quantum group

$$\mathcal{U}_\hbar(\widehat{\mathfrak{g}}_Q) = \text{Hopf algebra deformation of}$$
$$\mathcal{U}(\mathfrak{g}_Q \otimes \mathbb{C}[t^{\pm 1}]) + \text{center} + \text{“loop rotations”}$$
$$t \mapsto qt$$

on K-theory of $\mathcal{M}(w) = \bigsqcup_v \mathcal{M}(v,w)$

weight spaces for Cartan $\mathfrak{h} \subset \mathfrak{g}_Q$

Main feature of quantum groups: the coproduct

$$\Delta : \mathcal{U}_\hbar \to \mathcal{U}_\hbar \otimes \mathcal{U}_\hbar$$

is not cocommutative $\quad \Delta \neq \Delta_{21} = (12)\,\Delta\,(12)$

instead nontrivial intertwiner

$$F_1 \otimes F_2 \xrightarrow{\;R_{V_1,V_2}\;} F_2 \otimes F_1$$

$\leftarrow$ two $\mathcal{U}_\hbar$-modules

which is moreover rational in parameters of $F_i$,

because $F_1 \otimes F_2 \not\cong F_2 \otimes F_1$ for certain $F_i$

In our case, $F = K_{eq}(M(w))$ will be a highest weight module with highest weight $w$ gets an addional parameter $a \in \mathbb{C}^\times$ via "loop rotation"

$$\mathcal{U}_\hbar \xrightarrow[t \mapsto at]{} \mathcal{U}_\hbar \hookrightarrow F(a)$$

so that

$$R(a_1/a_2) \qquad \begin{matrix} F_1(a_1) \\ \otimes \\ F_2(a_2) \end{matrix} \longrightarrow \begin{matrix} F_2(a_2) \\ \otimes \\ F_1(a_1) \end{matrix}$$

is a rational function of $a_1/a_2$

"spectral parameter"

remarkably (by general yoga of quantum groups) the whole package ($U_\hbar$ itself, module structure etc.) may be reconstructed from R-matrices alone

e.g. we can take matrix elements in any given $F_0$ to get operators in an arbitrary module

$$F_0 \otimes \underline{\quad} \xrightarrow{R} \underline{\quad} \otimes F_0$$

Braid = Yang-Baxter $\Rightarrow$ commutation relations we will construct R-matrices geometrically as a composition of two operators on equivariant K-theories:

$$\begin{tikzcd} M(w) \times M(w') \arrow[r, "\mathrm{Stab}_+"] \arrow[rr, bend right, "R"'] & M(w+w') & M(w') \times M(w) \arrow[l, "\mathrm{Stab}_-"'] \end{tikzcd}$$

called stable envelopes

From definitions

$$M(w) \times M(w') = M(w+w')^{\mathbb{C}_u^\times}$$

where

$$\mathbb{C}_u^\times \longrightarrow \left\{ \begin{pmatrix} \overbrace{u \ \ddots \ u}^{w_i} & \\ & \overbrace{1 \ \ddots \ 1}^{w_i'} \end{pmatrix} \right\} \subset GL(W_i \oplus W_i')$$

$$\circlearrowleft$$

$$M(w+w')$$

The variable $u$ will be the spectral parameter for the R-matrices

Special instance of the following general problem

$(X, \omega)$ = holo symplectic, maybe symplectic resolution

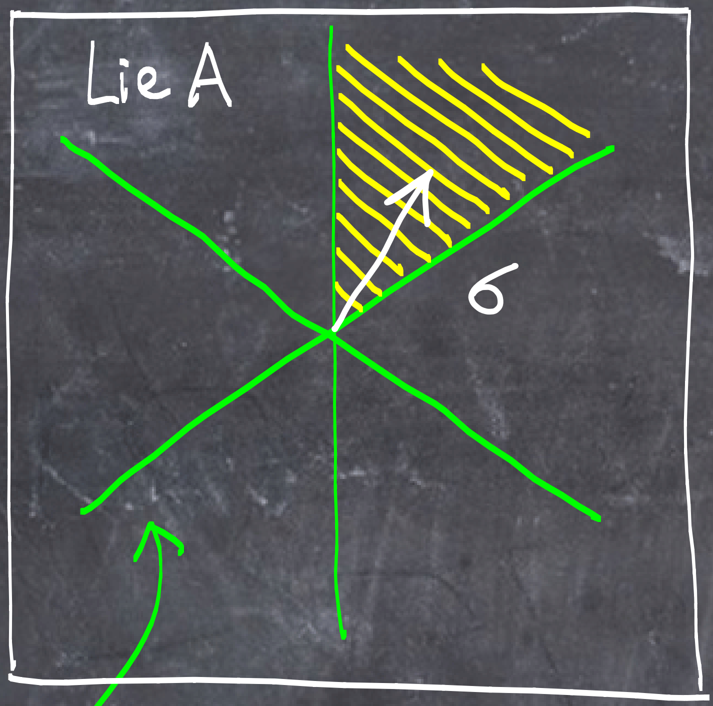
$T \circlearrowright X$, scales $\omega$ with weight $\hbar$, $A \subset \operatorname{Ker} \hbar$

Want: a correspondence

$$\mathrm{Stab} \in K_T(X \times X^A) \quad \text{proper over } X$$

that depends on certain further choices, called

chamber, slope, and polarization

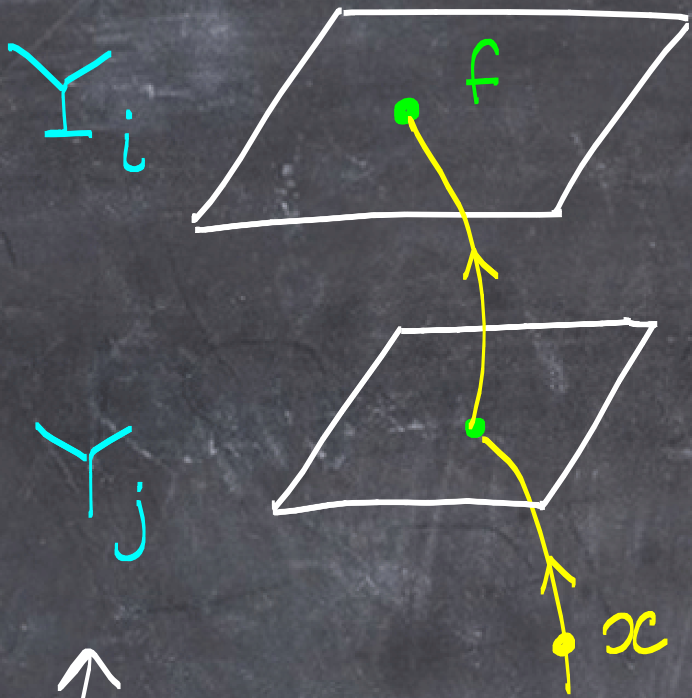



A-weights in $N_{X/X^A}$

a chamber $C \subset \operatorname{Lie} A$ determines the attracting manifold

$$\mathrm{Attr} = \left\{ (x,f),\ \lim_{u \to 0} \sigma(u) \cdot x = f \right\}$$

for $\sigma : \mathbb{C}^\times \to A$ with $d\sigma \in C$




$\widehat{\mathrm{Attr}}$ = transitive closure of Attr

a singular Lagrangian

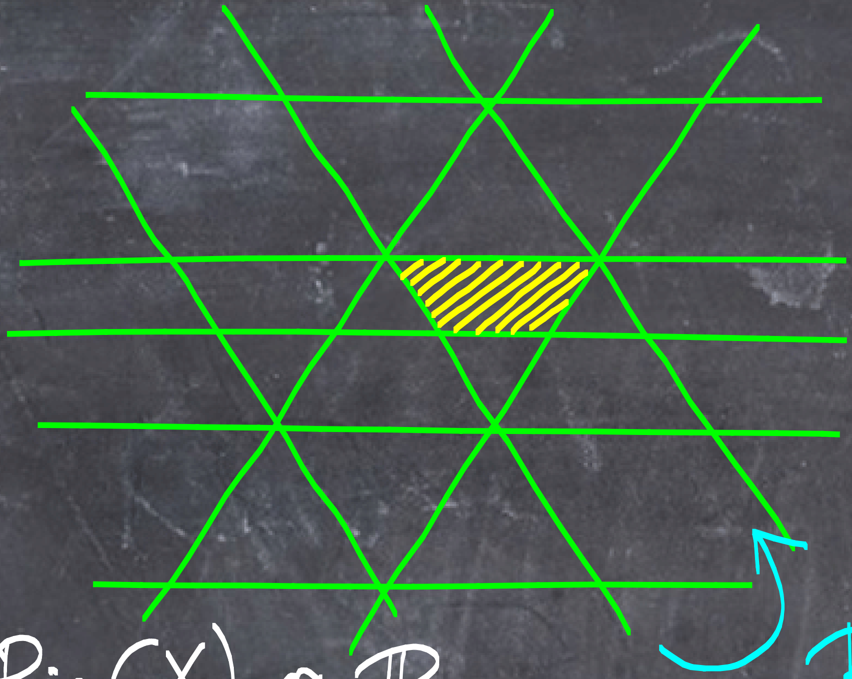
in $X \times X^A$

Stab will be an

improved version of $\mathcal{O}_{\widehat{\mathrm{Attr}}}$

components of $X^A = \bigsqcup Y_i$

ordered



$\mathrm{Pic}(X)\otimes\mathbb{R}$ $\qquad$ $\mathrm{Pic}(X)$

the second choice is the slope $\mathcal{L}\in\mathrm{Pic}(X)\otimes\mathbb{R}$ away from a periodic locally finite set of hypeplanes $(\mathcal{L},\underset{H_2(X,\mathbb{Z})}{\underset{\cap}{\alpha}})\in\mathbb{Z}$

Stable envelope depends only on the alcove of $\mathcal{L}$ and is twisted by line bundles under shifts by $\mathrm{Pic}(X)$

**Definition** $\mathrm{Stab}_{C,\mathcal{L},\text{pol}} \in K_T(X \times X^A)$ is the unique K-theory class such that

from polarization

(1) its support is $\widehat{\mathrm{Attr}}$

(2) $\mathrm{Stab} = \mathcal{O}_{\mathrm{Attr}} \otimes$ specific line bundle near diagonal

(3) $\deg_A \mathrm{Stab}\big|_{Y' \times Y} \subset \deg_A \mathrm{Stab}\big|_{Y' \times Y'} + \ell(Y') - \ell(Y)$

where $\ell(Y) = A\text{-weight of } \mathcal{L}\big|_Y \in A^\wedge \otimes \mathbb{R}$

Here

$$\deg_A \sum c_\mu a^\mu = \text{Newton polygon}$$

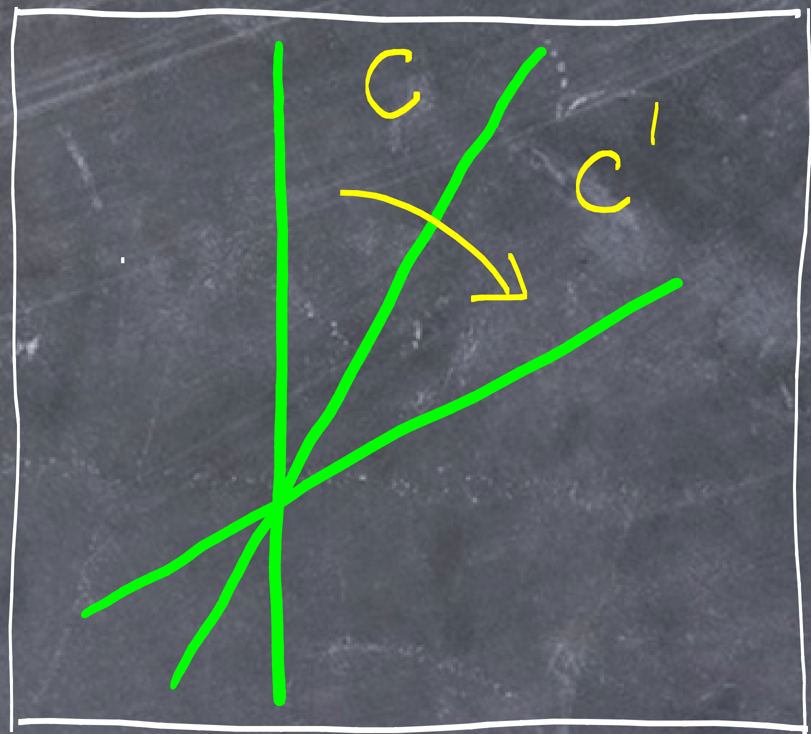
$$= \mathrm{Conv}(\{\mu\}) \in A^\wedge \otimes \mathbb{R}$$

Existence elementary for $\dim A = 1$ but not in general

Need $\dim A = 2$ to have

$$R_{C',C} = \mathrm{Stab}_{C'}^{-1} \circ \mathrm{Stab}_C$$

satisfy braid relations ($\Rightarrow$ YB)



Thm (Maulik - O.) Stable envelopes exist for Nakajima varieties

Extends the abelianization approach of D. Shenfeld to K-theory

replacemt of resolution of singularities,
smooth Lagrangian correspondence with hypertoric ($\Rightarrow$ snc) situation

going back to

$$M(w)\times M(w') = M(w+w')^{\mathbb{C}^\times_u} \quad \text{where}$$

$$\mathbb{C}^\times_u \longrightarrow \left\{ \begin{pmatrix} \overbrace{u \;\ddots\; u}^{w_i} & \\ & \overbrace{1 \;\ddots\; 1}^{w_i'} \end{pmatrix} \right\} \subset GL(W_i \oplus W_i')$$

$$\circlearrowleft$$

$$M(w+w')$$

we get

$$R(u) \circlearrowleft K(\mathcal{M}(w)) \otimes K(\mathcal{M}(w')) \quad + \; YB$$

and hence a quantum loop algebra $\mathcal{U}_\hbar(\hat{\mathfrak{g}}_Q)$

## Weights of $\hat{\mathfrak{g}}_Q$

1) $w \in \mathbb{Z}^I$ $\leftarrow$ vertices$(Q)$, center

2) $v \in \mathbb{Z}^I \longleftarrow H_2(X, \mathbb{Z})$

characters $z^v$ = degree variables in quantum K-theory

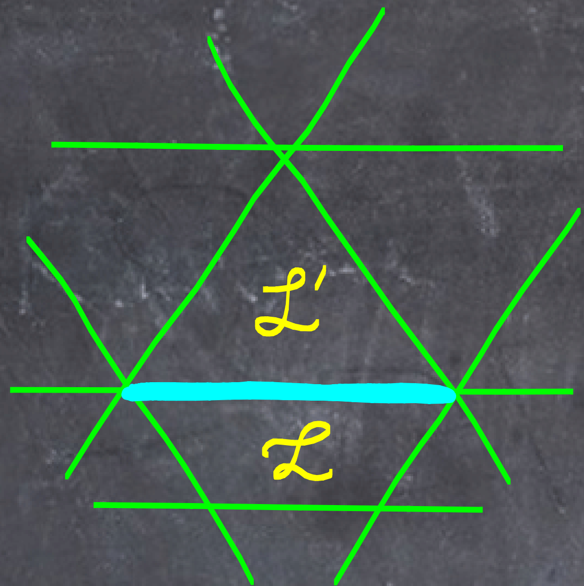
3) $k \in \mathbb{Z} = (\mathbb{C}^\times_q)^\wedge$ loop rotations

act by $R(u) \mapsto R(qu)$, where $q$

is also the shift in the quantum difference

connection

also have wall R-matrices




$$R_{\text{wall}} = \text{Stab}_{\mathfrak{C},\mathcal{L}'}^{-1} \circ \text{Stab}_{\mathfrak{C},\mathcal{L}}$$

these depend on the spectral parameter in a trivial way

We have $R(u) = \overrightarrow{\prod}$ wall R-matrices

which results in $\mathcal{U}_\hbar(\mathfrak{g}_{\text{wall}}) \hookrightarrow \mathcal{U}_\hbar(\hat{\mathfrak{g}}_Q)$

rank 1 subalgebra

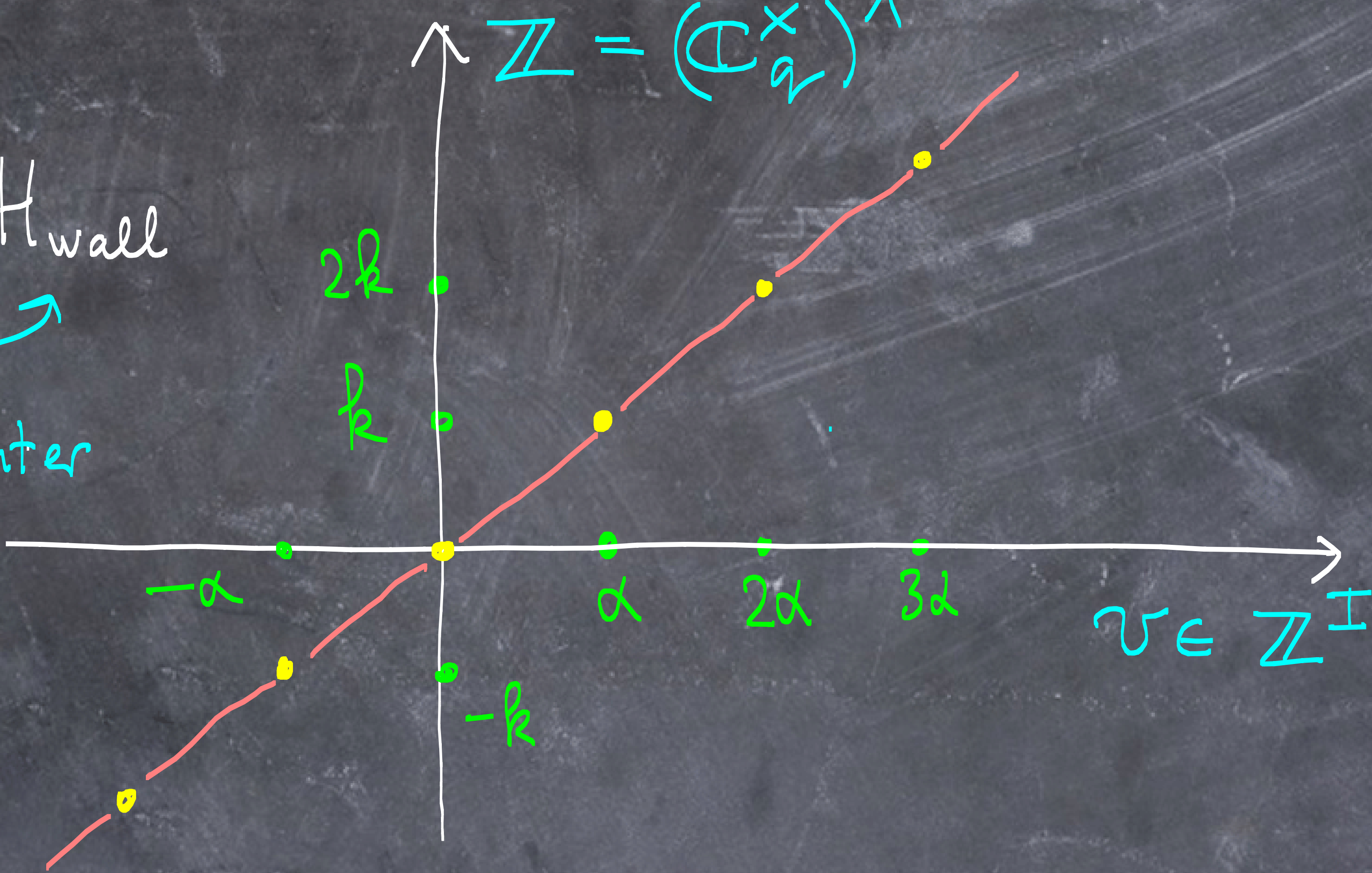
## Roots of $\hat{\mathfrak{g}}_Q$

To a wall of the form $(\cdot, \alpha) = k$, with $\alpha$ in $H_2(X,\mathbb{Z})$,

corresponds $\mathfrak{g}_{\text{wall}}$ with roots

$$H_Q \to H_{\text{wall}}$$

$-k\,1$ + center



# Dynamical quantum Weyl group

for every wall $\gamma$ we take a universal tensor

$$B_\gamma = m\,(S \otimes 1)\,J \in \hat{U}_\hbar(\mathfrak{g}_\gamma)[H_\gamma]$$

product   antipode   fusion $\approx$ universal solution of qKZ

$\|$

a map $H_\gamma \longrightarrow U_\hbar(\mathfrak{g}_\gamma) \hat{\otimes} U_\hbar(\mathfrak{g}_\gamma)$

Cartan torus

and we embed $B_\gamma \mapsto \hat{U}_\hbar(\hat{\mathfrak{g}}_Q)[H_Q]$ using

$$U_\hbar(\mathfrak{g}_\gamma) \hookrightarrow U_\hbar(\mathfrak{g}_Q)\ , \quad H_Q \to H_\gamma$$

Examples of walls $\gamma = \{(\alpha, \cdot) = k\}$

① real root, $(\alpha,\alpha) = 2$, $\mathfrak{g}_\gamma \simeq \mathfrak{sl}_2$

$B_\gamma$ = dynamical reflection of Etingof and Varchenko

② imaginary root, $(\alpha,\alpha) = 0$, $\mathfrak{g}_\gamma$ spanned by Heisenberg algebras

$$[e,f] = c - c^{-1}, \quad c \text{ central and } \Delta c = c \otimes c$$

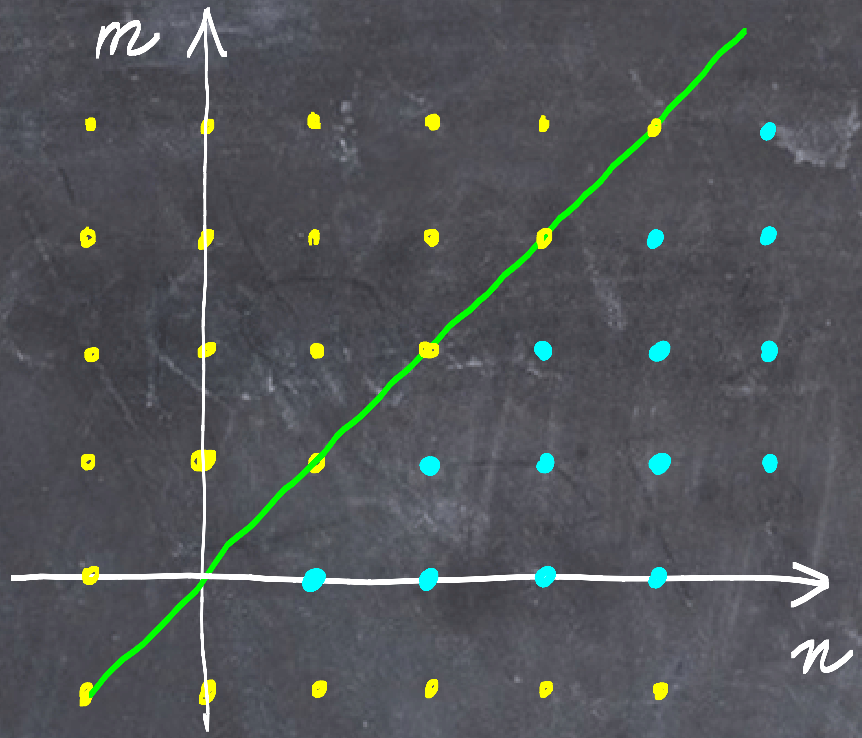
$$\mathrm{Ad}: (e, f) \to (Ae, A^{-1}f)$$

$$B = :\exp\left(\frac{c}{1-A} fe\right):$$

For example:

Roots of $\mathcal{U}_\hbar(\widehat{\widehat{\mathfrak{gl}(1)}})$ are indexed by $\mathbb{Z}^2$, all imaginary

and the purely quantum part of the difference connection




$$= \overrightarrow{\prod_{0 \le \frac{m}{n} \le 1}} B_{n,m}$$

substitute $e_{n,m}$, $f_{n,m}$, $c_{n,m}$ and

$$A_{n,m} = z^n q^m$$

K-theoretic lift of [OP] for $\mathrm{Hilb}(\mathbb{C}^2)$ and [MO] for sheaves of higher rank on $\mathbb{C}^2$

# Stable envelopes beyond K-theory

(1) FM functor in a joint work with D. Halpern-Leistner and Maulik, we are lifting Stab to $D^b\mathrm{Coh}(X \times X^A)$, with the same conditions.

For $\dim A = 1$, the construction, with many applications appears in DHL's thesis.

Should produce a categorification of $\mathcal{U}_\hbar(\hat{\mathfrak{g}}_Q)$

## (2) Quantization

Stable envelopes should specialize to parabolic induction under

$$D^b \operatorname{Coh} X \approx D^b \operatorname{Mod}_{\text{graded}} . \text{Quantization}(X)$$

which is currently being pursued by several groups of researchers including Bezrukavnikov, Braden, Finkelberg, Kaledin, Licata, Losev, McGerty, Nevins, Proudfoot, Webster, ...

Slope $\mathcal{L} \rightsquigarrow$ parameter of the quantization

Cone $\mathfrak{C} \rightsquigarrow$ choice of parabolic for Levi $=$ Quant$(X^A)$

(3) **Elliptic cohomology** fix $|q|<1$, $E = \mathbb{C}^*/q^{\mathbb{Z}}$

$T$-equivariant elliptic cohomology / $\mathbb{C}$ is a covariant functor

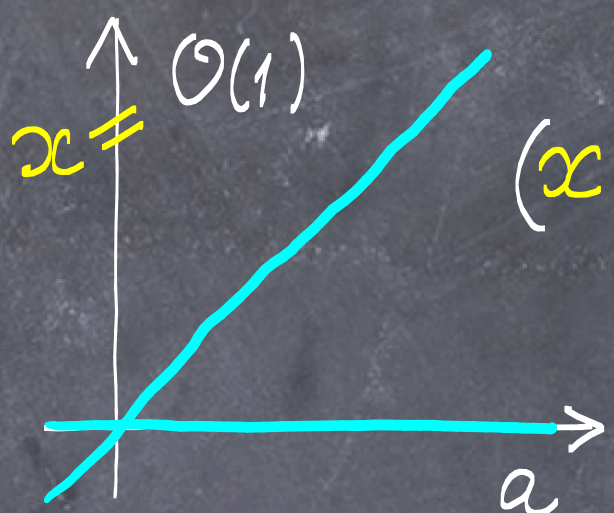
$$T\text{-spaces} \longrightarrow \text{schemes over}$$

$$\mathrm{Ell}_T(pt) = T/q^{\mathrm{cochar}\, T}$$

for Nakajima varieties, is very close to reducing

$$\mathrm{Spec}\, K_T(X) \otimes \mathbb{C} \longrightarrow T \quad \text{mod } q^{\mathbb{Z}}$$

e.g $X = \mathbb{P}^1$ ↻ $a \in T$




$$(x-a)(x-1) = 0$$

slope $\mathcal{L}$

Over $\mathrm{Ell}_T(X) \times (\operatorname{Pic} X \otimes_{\mathbb{Z}} E)$ lives

a universal line bundle $\mathcal{U}_X$, a relative of Poincaré line bundle. Elliptic stable envelope is a map

$$\text{shift \& twist}\,(\mathcal{U}_{X^A}) \longrightarrow \mathcal{U}_X$$

necessary for existence

of $\mathcal{O}_{\mathrm{Ell}_T(pt)\times(\operatorname{Pic}(X)\otimes E)}$-modules

satisfying the support & normalization of stable envelopes in $K_T(X)$

the degree condition is hidden in the behavior along $\mathrm{Ell}_A(pt)$

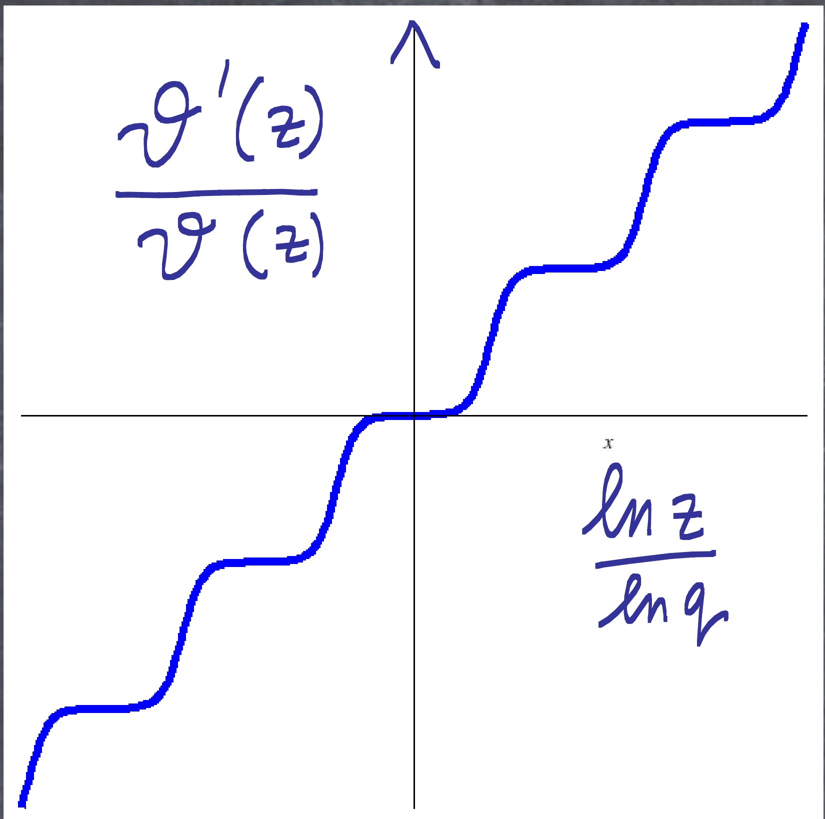
**Theorem** (Aganagic - O.) Elliptic stable envelope exist for Nakajima varieties

They define an action of elliptic quantum group on $\mathrm{Ell}_T(X)$ and have important applications to monodromy of the quantum difference connection and to the correspondence between quantum K-theories of symplectic dual pairs

I find the following point very interesting :

The dependence of elliptic stable envelopes on the slope $\mathcal{L}$ is analytic. Alcoves appear in the limit

because of the piecewise polynomial asymptotics of theta-functions as the elliptic curve degenerates




$\mathbb{C}^*/q^{\mathbb{Z}}$ $\qquad$ $q \to 0$

it would be really interesting to categorify the elliptic theory

Thank you and see you at