An Introduction to the Volume Conjecture, III
Generalizations

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Complexification

**Conjecture (Volume Conjecture, R. Kashaev, J. Murakami + H.M.)**

\[
2\pi \lim_{N \to \infty} \frac{\log |J_N(K; \exp(2\pi \sqrt{-1}/N))|}{N} = \text{Vol}(S^3 \setminus K).
\]

**Conjecture (Complexification of VC, J. Murakami, M. Okamoto, T. Takata, Y. Yokota, + H.M.)**

\[
2\pi \lim_{N \to \infty} \frac{\log J_N(K; \exp(2\pi \sqrt{-1}/N))}{N} = \text{Vol}(S^3 \setminus K) + \sqrt{-1} \text{CS}(S^3 \setminus K) \pmod{\pi^2 \sqrt{-1}\mathbb{Z}}.
\]

*Here CS is the SL(2; \mathbb{C}) Chern–Simons invariant.*

We may regard the left hand side as the definition of the Chern–Simons invariant for general knots.
Deform the parameter $2\pi \sqrt{-1}$

- In VC, the limit corresponds to the complete hyperbolic structure of $S^3 \setminus K$ (if it is hyperbolic).
- The complete structure can be deformed to incomplete ones.
- If we deform the parameter $2\pi \sqrt{-1}$, does the limit corresponds to an incomplete hyperbolic structure?
- Let us consider the limit

$$\lim_{N \to \infty} \frac{\log J_N(K; \exp((u + 2\pi \sqrt{-1})/N))}{N}$$

When $u = 0$, we have the (complexified) Volume Conjecture.
Deformation of the parameter

Generalization for

**Theorem (Yokota+H.M.)**

\[ \exists \mathcal{O} \subset \mathbb{C}: \text{neighborhood of } 0. \text{ If } u \in \mathcal{O} \setminus \pi \sqrt{-1} \mathbb{Q}, \text{ the following limit exists} \]

\[
\lim_{N \to \infty} \frac{\log J_N(\mathcal{O}; \exp((u + 2\pi \sqrt{-1})/N))}{N}
\]

Put

\[ H(u) := (u + 2\pi \sqrt{-1}) \times \text{(the limit above)}. \]

- \( H(u) \) is differentiable,
- \( \nu(u) := 2 \frac{d}{du} \frac{H(u)}{u} - 2\pi \sqrt{-1} \) satisfies the following.

\[
\text{Vol}(\mathcal{O} u) + \sqrt{-1} \text{CS}(\mathcal{O} u)
\]

\[ \equiv -\sqrt{-1}H(u) - \pi u + u \nu(u)\sqrt{-1}/4 - \pi \kappa(\gamma_u)/2 \text{ (mod } \pi^2 \sqrt{-1} \mathbb{Z}). \]
Deformation of the hyperbolic structure

- $\mathcal{H}_u$ is the closed hyperbolic three-manifold defined by $u$, that is, it is defined by the following representation of $\pi_1 \left( S^3 \setminus \mathcal{H} \right) \to SL(2; \mathbb{C})$:

\[
\begin{align*}
\text{meridian} & \mapsto \begin{pmatrix} \exp(u/2) & * \\ 0 & \exp(-u/2) \end{pmatrix}, \\
\text{longitude} & \mapsto \begin{pmatrix} \exp(v(u)/2) & * \\ 0 & \exp(-v(u)/2) \end{pmatrix}.
\end{align*}
\]

Here the meridian goes around $\mathcal{H}$, and the longitude goes along $\mathcal{H}$.

- When $u = 0$ this gives the holonomy representation, that is, each loop in $\pi_1 \left( S^3 \setminus \mathcal{H} \right)$ is identified with a deck transformation of the universal cover of $S^3 \setminus \mathcal{H}$, which is $\text{Isom}_+ (\mathbb{H}^3) \cong PSL(2; \mathbb{C}) = SL(2; \mathbb{C})/\pm$.

- For $u \neq 0$, the hyperbolic structure is incomplete.
Dehn surgery

- If $\mathcal{U}_u$ is incomplete, we can complete it by attaching either a point or a circle.
- $\gamma_u$ is the attaching circle.
- If $pu + qv(u) = 2\pi\sqrt{-1}$, this is the $(p, q)$-Dehn surgery.

$$\kappa(\gamma_u) := \text{length}(\gamma_u) + \sqrt{-1}\text{torsion}(\gamma_u),$$

where
- length is its length,
- torsion measures how the circle is twisted (mod $2\pi$).
Precise expression of the limit

\[ J_N \left( \bigotimes; q \right) = \sum_{j=0}^{N-1} q^{jN} \prod_{k=1}^{j} \left( 1 - q^{-N-k} \right) \left( 1 - q^{-N+k} \right). \]

Put

\[ H(z, w) := \text{Li}_2(z^{-1}w^{-1}) - \text{Li}_2(zw^{-1}) + \log z \log w, \]

where

\[ \text{Li}_2(x) := - \int_0^x \frac{\log(1-t)}{t} dt. \]

If \( \theta \) is near \( 2\pi \sqrt{-1} \in \mathbb{C} \) and not a rational multiple of \( 2\pi \sqrt{-1} \), then

\[ \theta \lim_{N \to \infty} \frac{\log J_N \left( \bigotimes; \exp(\theta/N) \right)}{N} = H\left( y, \exp(\theta) \right), \]

where \( y \) satisfies

\[ y + y^{-1} = \exp(\theta) + \exp(-\theta) - 1. \]
Approximation of the summand by dilogarithm

\[ q := \exp(\theta / N) \]

\[ \log \left( \prod_{k=1}^{j} \left( 1 - q^{-N\pm k} \right) \right) \]

\[ = \sum_{k=1}^{j} \log \left( 1 - \exp(\pm k\theta / N - \theta) \right) \]

\[ \sim N \int_{0}^{j/N} \log(1 - \exp(\pm \theta s - \theta)) \, ds \]

\[ = \frac{N}{\pm \theta} \int_{\exp(-\theta)}^{\exp(\pm j\theta / N - \theta)} \frac{\log(1 - t)}{t} \, dt \]

\[ = \frac{N}{\pm \theta} \left( \text{Li}_2(\exp(-\theta)) - \text{Li}_2(\exp(\pm j\theta / N - \theta)) \right). \]
Approximation of $J_N$ by an integral

$$J_N \left( \bigcirc; \exp(\theta/N) \right)$$

\[
N \to \infty \sum_{j=0}^{N-1} \exp(j\theta) \exp \left[ \frac{N}{\theta} \left( \text{Li}_2(\exp(-j\theta/N - \theta)) - \text{Li}_2(\exp(j\theta/N - \theta)) \right) \right]
\]

\[
= \sum_{j=0}^{N-1} \exp \left[ \frac{N}{\theta} H(\exp(j\theta/N), \exp(\theta)) \right]
\]

\[
\approx \int_C \exp \left[ \frac{N}{\theta} H(x, \exp(\theta)) \right] \, dx
\]

for a suitable contour $C$.

To find the ‘maximum’ of $\{H(x, \exp(\theta))\}$, we will find a solution $y$ to the equation $\frac{dH}{dx}(x, \exp(\theta)) = 0$, which is

$$\frac{\log [\exp(\theta) + \exp(-\theta) - x - x^{-1}]}{x} = 0.$$
Saddle point method

- Choose $y$ so that

\[ y + y^{-1} = \exp(\theta) + \exp(-\theta) - 1, \]

then

\[ J_N \left( \bigotimes; \exp(\theta / N) \right) \sim \exp \left( \frac{N}{\theta} H(y, \exp(\theta)) \right) \]

\[ \Rightarrow \]

\[ \theta \lim_{N \to \infty} J_N \left( \bigotimes; \exp(\theta / N) \right) = H(y, \exp(\theta)). \]

Putting $u := \theta - 2\pi \sqrt{-1}$, we have

\[ (u + 2\pi \sqrt{-1}) \lim_{N \to \infty} \frac{J_N \left( \bigotimes; \exp((u + 2\pi \sqrt{-1}) / N) \right)}{N} = H(u) \]

with $H(u) := H(y, \exp(\theta))$.

Note that this can be done rigorously.
Calculation of the volume using dilogarithm

- $\Delta(z)$, $\Delta(w)$: ideal hyperbolic tetrahedra parametrized by complex numbers $z$ and $w$, respectively.

- $S^3 \setminus \mathcal{K} = \Delta(z) \cup \Delta(w)$ if $z(z - 1)w(w - 1) = 1$. (This is just the glueing condition. The hyperbolic structure may not be complete. The completion condition is $w(1 - z) = 1$.)

- Introduce parameters $u$ and $y$ so that

  \[
  \exp u = w(1 - z), \quad (\text{meridian})
  \]

  \[
  y + y^{-1} = \exp(u) + \exp(-u) - 1.
  \]

  Note that $z$, $w$ and $y$ are defined by $u$.

- Use the formula:

  \[
  \text{Vol}(\Delta(z)) = \text{Im} \, \text{Li}_2(z) + \log |z| \, \arg(1 - z).
  \]
Calculation of the volume by $H$ function

$$\text{Vol}(S^3 \setminus \bigcirc) = \text{Im} H(u) - \pi \text{Re} u - \text{Re} u \text{Im} \log(z(1 - z))$$

Since \( \frac{d H(u)}{d u} = \log(z(z - 1)) \),

$$\text{Vol}(S^3 \setminus \bigcirc) = \text{Im} H(u) - \pi \text{Re} u - \frac{1}{2} \text{Re} u \text{Im} v(u)$$

putting \( v(u) := 2 \frac{d H(u)}{d u} - 2\pi \sqrt{-1} \).

Indeed, \( \exp(v(u)) \) corresponds to the longitude \( z^2(1 - z)^2 \).

We will show:

$$\text{length } \gamma_u = -\frac{1}{2\pi} \text{Im} \left( \frac{uv(u)}{u} \right).$$
Length of the geodesic $\gamma_u$ (W. Neumann and D. Zagier)

On $\partial \mathbb{H}^3 = S^2_\infty = \mathbb{C} \cup \{\infty\}$:

- $\mu := \text{meridian} \mapsto [z \mapsto \exp(u)z + c \exp(u/2)]$
- $\lambda := \text{longitude} \mapsto [z \mapsto \exp(v)z + d \exp(v/2)]$.

- When $u = 0$, we have the complete structure.
  - $\Rightarrow$ the corresponding representation is a parallel transport.
- When $u \neq 0$, we have an incomplete structure.
  - $\Rightarrow$ Since the meridian and the longitude commute, their images have the same two fixed points; \[
  \frac{c \exp(u/2)}{1 - \exp(u)} = \frac{d \exp(v/2)}{1 - \exp(v)} \quad \text{and} \quad \infty.
  \]
  Changing the coordinate, the fixed points are assumed to be $O$ and $\infty$.
  - $\Rightarrow$
    - $\mu \mapsto [z \mapsto \exp(u)z]$
    - $\lambda \mapsto [z \mapsto \exp(v)z]$. 
Calculation of the complex length

- Choose \((p, q)\) so that \(pu + qv = 2\pi \sqrt{-1}\) \((p, q \in \mathbb{R})\).
- Assume \(p\) and \(q\) are coprime integers.
- \(u\) defines an incomplete structure whose completion is the \((p, q)\)-Dehn surgery.
- \(\gamma_u = r\mu + s\lambda \in H_1(\partial(S^3 \setminus \bigcirc))\).
  \((\because\) the meridian of the attached solid torus is identified with \(p\mu + q\lambda\), and the meridian and \(\gamma_u\) make a basis of \(H_1(\partial(S^3 \setminus \bigcirc))\).\)
Calculation of length and torsion

- $\gamma_u$ corresponds to the multiplication by $\exp(ru + sv)$, and so
  \[ \exp(\text{length} + \sqrt{-1} \text{torsion}) = \exp(\pm (ru + sv)). \]
- In $H^3$, this defines $\text{Im}(\pm (ru + sv))$-rotation, and an upward shift by $\exp(\Re(\pm (ru + sv)))$ in coordinate, which has length $\Re(\pm (ru + sv))$.
- 
  \[
  \begin{cases}
  pu + qv = 2\pi \sqrt{-1}, \\
  ru + sv = \pm (\text{length} + \sqrt{-1} \text{torsion}).
  \end{cases}
  \]
- $\text{length } \gamma_u = -\frac{1}{2\pi} \text{Im} (u\overline{v})$.

  (Here we choose the negative sign since $v = u \times \frac{|v|^2}{u\overline{v}}$ and the orientation of $(u, v)$ should be positive on $\mathbb{C}$.)
Conclusion

\[
\text{length } \gamma_u = -\frac{1}{2\pi} \text{Im} \left( u\overline{v} \right) = -\frac{1}{2\pi} \text{Im} u \text{Re} v + \frac{1}{2\pi} \text{Re} u \text{Im} v.
\]

\[
\Rightarrow 
\]

\[
\text{Vol}(S^3 \setminus \natural 1) = \text{Im} H(u) - \pi \text{Re} u - \frac{1}{2} \text{Re} u \text{Im} \nu(u) \\
= \text{Re} \left( -\sqrt{-1} H(u) - \pi u + uv(u)\sqrt{-1}/4 - \pi \kappa(\gamma_u)/2 \right),
\]

The Chern–Simons invariant is obtained by T. Yoshida’s formula.
Conjecture

For any hyperbolic knot $K$, the following limit exists

$$\lim_{N \to \infty} \log J_N(K; \exp((u + 2\pi \sqrt{-1})/N)) \cdot \frac{1}{N}$$

for small $u$. Put

$$H(K; u) := (u + 2\pi \sqrt{-1}) \times \text{(the limit above)}.$$

- $H(K; u)$ is differentiable,
- $v(K; u) := 2 \frac{d}{du} H(K; u) - 2\pi \sqrt{-1}$ satisfies the following.

$$\Vol(K_u) = \text{Im } H(K; u) - \pi \text{ Re } u - \text{ Re } u \text{ Im } v(K; u)/2.$$
Small parameter

The previous conjecture should be compared with:

**Theorem (S. Garoufalidis and T. Lê)**

*For any* $K$, $\exists \varepsilon$ s.t. if $|a| < \varepsilon$

$$\lim_{N \to \infty} J_N(K; \exp(a/N)) = \frac{1}{\Delta(K; \exp a)},$$

*where* $\Delta(K; t)$ *is the Alexander polynomial.*

What happens between $2\pi\sqrt{-1}$ and 0?
FAQs

Q1. Is Jun Murakami your relative?
A1. No!

Q2. How about Haruki Murakami?
A2. Never!