Representation theory and homological stability

Benson Farb
Joint work with Tom Church
University of Chicago

Walterfest
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Introduction

Representation stability is a philosophy, definition, vocabulary.
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**Homological stability:** for certain sequences $X_n$,

$$H_i(X_n) \xrightarrow{\sim} H_i(X_{n+1}) \quad \text{once } n \gg i.$$  

(Examples: $\text{SL}_n \mathbb{Z}$, $S_n$, $\mathcal{M}_n$)
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Problem: If $X_n$ has symmetries $G_n$, and if $G_n$ grows, homological stability is almost impossible.

**Representation stability** (slogan form):

“The description of $H_i(X_n)$ in terms of the action of $G_n$ stabilizes.”
Pure braid groups

$\text{PConf}_n(\mathbb{C}) := \{(z_1, \ldots, z_n) | z_i \in \mathbb{C}, z_i \neq z_j\}$
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= configuration space of *n ordered* points in the plane
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Symmetric group \( S_n \) acts on \( \mathbb{C}^n \) and acts on \( P_{\text{Conf}_n}(\mathbb{C}) \) freely.
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Remarks

- These spaces are all aspherical, so \( H^*(\text{PConf}_n(\mathbb{C})) = H^*(P_n) \).
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- These spaces are all aspherical, so \( H^*(\text{PConf}_n(\mathbb{C})) = H^*(P_n) \).
- Distinction between homology and cohomology not relevant.
Homological stability for braid groups

**Theorem (Arnol’d 1968, F. Cohen 1972)**

The braid groups $B_n$ satisfy **homological stability**:

$$\text{for } n \gg i, \quad H_i(B_n; \mathbb{Z}) \xrightarrow{\sim} H_i(B_{n+1}; \mathbb{Z}).$$
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Over $\mathbb{Q}$:

$$H_0(B_n; \mathbb{Q}) = H_1(B_n; \mathbb{Q}) = \mathbb{Q}, \quad \text{while } H_i(B_n; \mathbb{Q}) = 0 \text{ for } i \geq 2.$$
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*False* for the pure braid groups $P_n$, even over $\mathbb{Q}$:

- $H_1(P_2; \mathbb{Q}) = \mathbb{Q}$
- $H_1(P_3; \mathbb{Q}) = \mathbb{Q}^3$
- $H_1(P_4; \mathbb{Q}) = \mathbb{Q}^6$
- $H_1(P_5; \mathbb{Q}) = \mathbb{Q}^{10}$
- $\vdots$
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False for the pure braid groups $P_n$, even over $\mathbb{Q}$:

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\begin{align*}
H_1(P_2; \mathbb{Q}) &= \mathbb{Q} \\
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H_1(P_5; \mathbb{Q}) &= \mathbb{Q}^{10} \\
\vdots
\end{align*}
\]

in fact for each $i \geq 1$,

\[
\dim H_i(P_n; \mathbb{Q}) \to \infty \quad \text{as } n \to \infty
\]
The action of the symmetric group on $H^*(P_n)$

$S_n$ action on $\text{PConf}_n(\mathbb{C})$
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$H^*(B_n; \mathbb{Q}) \approx \text{the } S_n\text{-invariants } H^*(P_n; \mathbb{Q})^{S_n}$
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Thus:

$H^i(B_n; \mathbb{Q}) = 0$ for $i \geq 2$ (Arnold)

$\iff$

$S_n$ action on $H^i(P_n; \mathbb{Q})$ has no fixed vectors for $i \geq 2$. 

Benson Farb (University of Chicago)

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The action of the symmetric group on $H^*(P_n)$

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Main question [Brieskorn, Stanley, Orlick, Lehrer-Solomon, ...]

What is $H^i(P_n; \mathbb{Q})$ as an $S_n$–representation?
The action of the symmetric group on $H^*(P_n)$

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What is $H^i(P_n; \mathbb{Q})$ as an $S_n$–representation?

Example

$H^1(P_n; \mathbb{Q}) \approx \text{Sym}^2 \mathbb{Q}^n / \mathbb{Q}^n.$
representation: $\mathbb{Q}$–vector space with linear action of a group
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irreducible: has no (nontrivial) invariant subspaces
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**Maschke’s Theorem:** Every rep. of a finite group decomposes as direct sum of irreducibles.
**Representation theory basics**

**representation:** \(\mathbb{Q}\)-vector space with linear action of a group

**irreducible:** has no (nontrivial) invariant subspaces

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**Goals of representation theory:**
representation: $\mathbb{Q}$-vector space with linear action of a group
irreducible: has no (nontrivial) invariant subspaces

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1. Classify all irreducible representations of a given group.
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Maschke’s Theorem: Every rep. of a finite group decomposes as direct sum of irreducibles.

Goals of representation theory:
1. Classify all irreducible representations of a given group.
2. Understand how to decompose a given representation into irreducibles.
Beautiful fact

There is a natural bijection:

\{ \text{irreducible representations of } S_n \} \leftrightarrow \{ \text{partitions of } n \}
**Beautiful fact**

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**Example**

- trivial representation $\mathbb{Q}$ $\longleftrightarrow$ $n$
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Example

- trivial representation \( \mathbb{Q} \) \( \leftrightarrow \) \( n \)
- standard representation \( \mathbb{Q}^n/\mathbb{Q} \) \( \leftrightarrow \) \( (n - 1) + 1 \)
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Example

- trivial representation \( \mathbb{Q} \) \( \leftrightarrow \) \( n \)
- standard representation \( \mathbb{Q}^n/\mathbb{Q} \) \( \leftrightarrow \) \( (n - 1) + 1 \)
- \( \Lambda^3 \) (standard representation) \( \leftrightarrow \) \( (n - 3) + 1 + 1 + 1 \)
Beautiful fact

There is a natural bijection:

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Example

- trivial representation $\mathbb{Q} \leftrightarrow n = V(0)$
- standard representation $\mathbb{Q}^n/\mathbb{Q} \leftrightarrow (n - 1) + 1 = V(1)$
- $\wedge^3(\text{standard representation}) \leftrightarrow (n - 3) + 1 + 1 + 1 = V(1, 1, 1)$

Notational convention:

$V(a_1, \ldots, a_k)_n$

$= \text{irreducible corresponding to partition } (n - \sum a_i) + a_1 + \cdots + a_k.$
$H^*(P_n)$ of the pure braid group, revisited

Back to the $S_n$-representation $H^*(P_n; \mathbb{Q})$. 
H*(P_n) of the pure braid group, revisited

Back to the S_n-representation H*(P_n; Q).

Main question, restated

Which irreducible representations (i.e. which partitions) appear in H^i(P_n; Q)? What are their multiplicities?
Back to the $S_n$-representation $H^*(P_n; \mathbb{Q})$.

**Main question, restated**

Which irreducible representations (i.e. which partitions) appear in $H^i(P_n; \mathbb{Q})$? What are their multiplicities?

**Example**

$H^1(P_n; \mathbb{Q}) = V(0) \oplus V(1) \oplus V(2)$ for $n \geq 4$
Back to the $S_n$-representation $H^*(P_n; \mathbb{Q})$.

**Main question, restated**

Which irreducible representations (i.e. which partitions) appear in $H^i(P_n; \mathbb{Q})$? What are their multiplicities?

**Example**

$H^1(P_n; \mathbb{Q}) = V(0) \oplus V(1) \oplus V(2)$ for $n \geq 4$

What about $H^2(P_n; \mathbb{Q})$?
\[ H^2 \] of the pure braid group \( P_n \)
$H^2$ of the pure braid group $P_n$

$$H^2(P_3; \mathbb{Q}) = V(1)$$
$H^2$ of the pure braid group $P_n$

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$H^2(P_4; \mathbb{Q}) = V(1)^\oplus 2 \oplus V(1, 1) \oplus V(2)$
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H^2(P_6; \mathbb{Q}) = V(1)^\oplus 2 \oplus V(1, 1)^\oplus 2 \oplus V(2)^\oplus 2 \oplus V(2, 1)^\oplus 2 \oplus V(3)
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H^2(P_7; \mathbb{Q}) = V(1)^2 \oplus V(1, 1)^2 \oplus V(2)^2 \oplus V(2, 1)^2 \oplus V(3) \oplus V(3, 1)
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$H^2(P_8; \mathbb{Q}) = \text{more complicated}$
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\[\text{No stability (?!)}\]
$H^2$ of the pure braid group $P_n$

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Correct answer

$H^2(P_n; \mathbb{Q}) = V(1)^{\oplus 2} \oplus V(1, 1)^{\oplus 2} \oplus V(2)^{\oplus 2} \oplus V(2, 1)^{\oplus 2} \oplus V(3) \oplus V(3, 1)$

for $n = 8, 9, \ldots$
Definition of representation stability
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**Input:** Sequence \( \{W_n\} \) of \( S_n \)-representations with maps \( \phi_n: W_n \to W_{n+1} \) respecting action of \( S_n \).
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**Definition (Representation stability)**

The sequence \( \{W_n\} \) is called *representation stable* if:

I. Injectivity:

II. Surjectivity:

III. Multiplicities:
Definition of representation stability

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II. Surjectivity: the image \( \phi_n(W_n) \) is all of \( W_{n+1} \).

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Definition of representation stability

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The sequence \( \{ W_n \} \) is called *representation stable* if:

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Definition of representation stability

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II. Surjectivity: the \( S_{n+1} \)-span of the image \( \phi_n(W_n) \) is all of \( W_{n+1} \).

III. Multiplicities: decompose

\[
W_n = \bigoplus c_{\lambda,n} V(\lambda)_n
\]

into irreducibles. For each partition \( \lambda \), the multiplicities \( c_{\lambda,n} \) are eventually constant as \( n \to \infty \).
Representation stability for pure braid groups
Theorem (Church–Farb)

For each $i \geq 0$, the sequence of $S_n$–representations $\{H^i(P_n; \mathbb{Q})\}$ is representation stable, and in fact stabilizes once $n \geq 4i$. 
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Remark: Other approaches by Putman, Wiltshire-Gordon, Church-Ellenberg-Farb.
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Remark: Other approaches by Putman, Wiltshire-Gordon, Church-Ellenberg-Farb.

Remark: Easiest mechanism for stability fails.
$H^2$ of the pure braid group $P_n$

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H^2(P_3; \mathbb{Q}) = V(1)
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H^2(P_4; \mathbb{Q}) = \begin{bmatrix} V(1) \end{bmatrix} \oplus V(1, 1) \oplus V(2)
\]

\[
H^2(P_5; \mathbb{Q}) = \begin{bmatrix} V(1) \end{bmatrix} \oplus V(1, 1) \oplus V(2) \oplus V(2, 1)
\]

\[
H^2(P_6; \mathbb{Q}) = V(1) \oplus V(1, 1) \oplus V(2) \oplus V(2, 1) \oplus V(3)
\]

\[
H^2(P_7; \mathbb{Q}) = V(1) \oplus V(1, 1) \oplus V(2) \oplus V(2, 1) \oplus V(3) \oplus V(3, 1)
\]
$H^2$ of the pure braid group $P_n$

$$H^2(P_3; \mathbb{Q}) = V(1)$$

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The truth is more complicated.
\( H^2 \) of the pure braid group \( P_n \)

\[
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\]

\[
H^2(P_4; \mathbb{Q}) = V(1)^{\oplus 2} \oplus V(1, 1) \oplus V(2)
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\[
H^2(P_5; \mathbb{Q}) = \begin{bmatrix} V(1) \end{bmatrix}^{\oplus 2} \oplus V(1, 1)^{\oplus 2} \oplus V(2)^{\oplus 2} \oplus V(2, 1)^{\oplus 2}
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\]

\[
H^2(P_6; \mathbb{Q}) = \begin{array}{c}
V(1) \\
\downarrow \\
V(1) \\
\downarrow \\
V(1, 1) \\
\downarrow \\
V(2) \\
\downarrow \\
V(2, 1)
\end{array}^{\oplus 2} \oplus V(1, 1)^{\oplus 2} \oplus V(2)^{\oplus 2} \oplus V(2, 1)^{\oplus 2} \oplus V(3)
\]

\[
H^2(P_7; \mathbb{Q}) = \begin{array}{c}
V(1) \\
\downarrow \\
V(1, 1) \\
\downarrow \\
V(2)
\end{array}^{\oplus 2} \oplus V(1, 1)^{\oplus 2} \oplus V(2)^{\oplus 2} \oplus V(2, 1)^{\oplus 2} \oplus V(3) \oplus V(3, 1)
\]

The truth is more complicated.
$H^4$ of the pure braid group $P_n$
$H^4$ of the pure braid group $P_n$

$$H^4(P_n) = V(1)^{\oplus 2} \oplus V(2)^{\oplus 6} \oplus V(1, 1)^{\oplus 6} \oplus V(3)^{\oplus 8} \oplus V(1, 1, 1)^{\oplus 9} \oplus V(2, 1)^{\oplus 16} \oplus V(4)^{\oplus 6}$$
$$\oplus V(1, 1, 1, 1)^{\oplus 5} \oplus V(5)^{\oplus 2} \oplus V(2, 2)^{\oplus 12} \oplus V(3, 1)^{\oplus 19} \oplus V(2, 1, 1)^{\oplus 17} \oplus V(4, 1)^{\oplus 12} \oplus V(2, 1, 1, 1)^{\oplus 7} \oplus V(3, 2)^{\oplus 14} \oplus V(2, 2, 1)^{\oplus 10} \oplus V(5, 1)^{\oplus 3} \oplus V(3, 3)^{\oplus 4} \oplus V(3, 1, 1)^{\oplus 16} \oplus V(2, 2, 2)^{\oplus 2} \oplus V(4, 2)^{\oplus 7} \oplus V(4, 1, 1)^{\oplus 8} \oplus V(5, 2) \oplus V(2, 2, 1, 1)^{\oplus 2} \oplus V(3, 1, 1, 1)^{\oplus 5} \oplus V(5, 1, 1)^{\oplus 2} \oplus V(4, 3)^{\oplus 2} \oplus V(3, 2, 1)^{\oplus 9} \oplus V(4, 1, 1, 1)^{\oplus 2} \oplus V(3, 3, 1)^{\oplus 2} \oplus V(3, 2, 2) \oplus V(4, 2, 1)^{\oplus 3} \oplus V(3, 2, 1, 1) \oplus V(5, 1, 1, 1) \oplus V(4, 3, 1)$
(In)stability in group homology

The pure braid group $P_n$ is just the simplest case of a broader framework.
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homological stability holds
(In)stability in group homology

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Homological stability holds as $n \to \infty$

Dim goes to $\infty$ as $n \to \infty$
(In)stability in group homology

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homological stability holds

homological stability **fails**

dim goes to $\infty$ as $n \to \infty$
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homological stability holds  
homological stability **fails**  
group of Lie type  
dim goes to $\infty$ as $n \to \infty$
Application 1:
Configuration spaces on manifolds

For any space $M$:

$$\text{PConf}_n(M) := \{(z_1, \ldots, z_n) \in M^n | z_i \neq z_j\}.$$
Configuration spaces on manifolds

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$S_n$ acts on $\text{PConf}_n(M)$:

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Configuration spaces on manifolds

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Example

$\text{PConf}_n(\mathbb{C}) = \text{space } X_n$ associated to pure braid group $P_n$

$\text{Conf}_n(\mathbb{C}) = \text{space } Y_n$ associated to braid group $B_n$
Theorem (Church)

For any manifold $M$ and any $i \geq 0$, the cohomology $H^i(PConf_n(M); \mathbb{Q})$ is representation stable.
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Corollary: Homological stability (over $\mathbb{Q}$) for $\text{Conf}_n(M)$!
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Proof: Apply theorem to trivial representation $V(0)$. □
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- Previously known only for open manifolds $M$ (McDuff-Segal).
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- Previously known only for open manifolds $M$ (McDuff-Segal).
- False over $\mathbb{Z}$ for closed $M$. 
Application 2:
Puncture stability

Let $\mathcal{M}_{g,n}$ = moduli space of genus $g$ Riemann surfaces with $n$ marked points.
Puncture stability

Let $\mathcal{M}_{g,n}$ = moduli space of genus $g$ Riemann surfaces with $n$ marked points.

Theorem (Jimenez Rolland)

Let $g \geq 2$, $i \geq 1$. The sequence $\{H^i(\mathcal{M}_{g,n}; \mathbb{Q})\}$ is representation stable.
Let $\mathcal{M}_{g,n} =$ moduli space of genus $g$ Riemann surfaces with $n$ marked points.

**Theorem (Jimenez Rolland)**

Let $g \geq 2$, $i \geq 1$. The sequence $\{H^i(\mathcal{M}_{g,n}; \mathbb{Q})\}$ is representation stable.

- Apply to trivial rep. $\implies$ recent theorem of Hatcher-Wahl (over $\mathbb{Q}$).
Application 3:
Statistics for polynomials in $\mathbb{F}_q[T]$

Joint with Tom Church and Jordan Ellenberg.
Statistics for polynomials in $\mathbb{F}_q[T]$

Joint with Tom Church and Jordan Ellenberg.

$\text{Conf}_n(\mathbb{C}) = \text{space of sets of } n \text{ distinct points in } \mathbb{C}$
Statistics for polynomials in $\mathbb{F}_q[T]$

Joint with Tom Church and Jordan Ellenberg.

$\text{Conf}_n(\mathbb{C}) = \text{space of sets of } n \text{ distinct points in } \mathbb{C}$

$Z_n(\mathbb{C}) = \text{space of squarefree degree } n \text{ monic polynomials}$
Statistics for polynomials in $\mathbb{F}_q[T]$

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$$\text{Conf}_n(\mathbb{C}) \iff Z_n(\mathbb{C})$$
Statistics for polynomials in $\mathbb{F}_q[T]$

Joint with Tom Church and Jordan Ellenberg.

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$\text{Conf}_n(\mathbb{C}) \leftrightarrow Z_n(\mathbb{C})$

$\{\lambda_1, \ldots, \lambda_n\} \leftrightarrow (z - \lambda_1) \cdots (z - \lambda_n)$
Statistics for polynomials in $\mathbb{F}_q[T]$
Observation: there is a polynomial that tells us when $z^n + \cdots + a_1z + a_0$ has repeated roots.
Statistics for polynomials in $\mathbb{F}_q[T]$

**Observation:** there is a *polynomial* that tells us when $z^n + \cdots + a_1z + a_0$ has repeated roots.

$$Z_2(\mathbb{C}) = \left\{ z^2 + bz + c \quad \middle| \quad b^2 - 4c \neq 0 \right\}$$
Observation: there is a polynomial that tells us when $z^n + \cdots + a_1z + a_0$ has repeated roots.

$Z_2(\mathbb{C}) = \left\{ z^2 + bz + c \mid b^2 - 4c \neq 0 \right\}$

$Z_3(\mathbb{C}) = \left\{ z^3 + bz^2 + cz + d \mid b^2c^2 - 4c^3 - 4b^3d - 27d^2 + 18bcd \neq 0 \right\}$
Statistics for polynomials in $\mathbb{F}_q[T]$

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$Z_n(\mathbb{F}_q) =$ space of squarefree degree $n$ polynomials in $\mathbb{F}_q[T]$
Statistics for polynomials in $\mathbb{F}_q[T]$

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$Z_n(\mathbb{F}_q) =$ space of squarefree degree $n$ polynomials in $\mathbb{F}_q[T]$

**Idea:** Compute statistics about $Z_n(\mathbb{F}_q)$ from info on \[ V(\lambda) < H^*(Z_n(\mathbb{C})) = H^*(PConf_n(\mathbb{C})). \]
Statistics for polynomials in $\mathbb{F}_q[T]$

$\#$ of squarefree polys in $\mathbb{F}_q[T] = |\text{Conf}_n(\mathbb{A}^1(\mathbb{F}_q))|$
Statistics for polynomials in $\mathbb{F}_q[T]$

\[
\text{\# of squarefree polys in } \mathbb{F}_q[T] = \left| \text{Conf}_n(\mathbb{A}^1(\mathbb{F}_q)) \right|
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Statistics for polynomials in $\mathbb{F}_q[T]$

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\[= \# \text{ of fixed pts. of Frobenius on Conf}_n(\mathbb{A}^1(\mathbb{F}_q))
\]

[Lefschetz fixed pt. formula] \[= \text{alt. sum of traces of action on } H^i\]
Statistics for polynomials in $\mathbb{F}_q[T]$

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[Grothendieck-Lefschetz] $= \sum_{i \geq 0} (-1)^i \text{tr} (\text{Frob}_*|H^i(\text{Conf}_n(\mathbb{A}^1)))$
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$[H^i(B_n) = 0 \text{ for } i \geq 2]$ $= \text{tr} (\text{Frob}_*|H^0(\text{Conf}_n(\mathbb{A}^1)))$

$- \text{tr} (\text{Frob}_*|H^1(\text{Conf}_n(\mathbb{A}^1)))$
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$[H^0(B_n) = H^1(B_n) = \mathbb{Q}] = q^n - q^{n-1}$
Statistics for polynomials in $\mathbb{F}_q[T]$

\[
\text{# of squarefree polys in } \mathbb{F}_q[T] = | \text{Conf}_n(\mathbb{A}^1(\mathbb{F}_q)) | \\
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[Grothendieck-Lefschetz] \[= \sum_{i \geq 0} (-1)^i \text{tr} (\text{Frob}_* \mid H^i(\text{Conf}_n(\mathbb{A}^1)))
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$[H^0(B_n) = H^1(B_n) = \mathbb{Q}] = q^n - q^{n-1}$

**Method:** Compute other statistics using *twisted* cohomology of $\text{Conf}_n(\mathbb{A}^1(\mathbb{C})) = \text{Conf}_n(\mathbb{C})$. 
<p>| Irred. $S_n$-representation $V(\lambda)$ | Counting theorem in $\mathbb{F}_q[T]$ |</p>
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Application 4:
Statistics for tori in $\text{GL}_n(F_q)$

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<td>the # of factors is more likely to be even than odd,</td>
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<td>with bias $\sqrt{# \text{ of tori}}$</td>
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Benson Farb (University of Chicago)  
Representation stability  
June 7, 2011  
20 / 25
Final remarks: the ubiquity of representation stability
Final remarks: the ubiquity of representation stability

**Note:** representation stability can be defined for representations of $\text{SL}_n \mathbb{Q}$, $\text{GL}_n \mathbb{Q}$, $\text{Sp}_{2n} \mathbb{Q}$, other finite groups of Lie type, etc.
Group homology
Group homology

- String motion group (J. Wilson)
- Conjectures for the Torelli group and $\text{IAut}(F_n)$.
- Congruence subgroups of arithmetic groups, mapping class groups, and $\text{Aut}(F_n)$. [Uses modular representation theory. A new phenomenon arises: stable periodicity.]
Lie algebras and their homology

- Graded components of:
  - Free Lie algebras and free nilpotent Lie algebras
  - Malcev completions of surface groups
  - Malcev completions of pure braid and Torelli groups (conj.).
- Homology of various families of Lie algebras
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**Theorem:** \( \{\mathcal{L}_n\} \) is representation stable

\[ \iff \]

\( \{H_i(\mathcal{L}_n)\} \) is representation stable for all \( i \geq 0 \).
More examples of representation stability

Enumerative geometry
- Cohomology of flag varieties.
- Equivariant cohomology of Schubert varieties.

Algebraic combinatorics
- Lefschetz representations associated to rank-selected posets.
- Relates to the $(n+1)^{(n-1)}$ Conjecture (Haiman Theorem)
The Fundamental Theorem

Theorem (Neumann-Neumann, 1946):
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