Clifford Algebras and Spin Groups

Math G4344, Spring 2012

We'll now turn from the general theory to examine a specific class class of groups: the orthogonal groups. Recall that $O(n, \mathbf{R})$ is the group of n by n orthogonal matrices (the group preserving the standard inner product on $\mathbf{R}^n$). This group has two components, with the component of the identity $SO(n, \mathbf{R})$, the orthogonal matrices of determinant 1. We'll mostly restrict attention to $SO(n, \mathbf{R})$.

An important feature of $SO(n, \mathbf{R})$ is that it is not simply-connnected. One has

$$\pi_1(SO(n, \mathbf{R})) = \mathbf{Z}_2$$

and the simply-connected double cover is the group $Spin(n, \mathbf{R})$ (the simply-connected double cover of $O(n, \mathbf{R})$ is called $Pin(n, \mathbf{R}$. It is this group for which we want to find an explicit construction. The Lie algebras $\mathfrak{spin}(n, \mathbf{R})$ and $\mathfrak{so}(n, \mathbf{R})$ are isomorphic, and the complex simple Lie algebra that corresponds to them is $\mathfrak{spin}(n, \mathbf{C})$ or $\mathfrak{so}(n, \mathbf{C})$. The group $Spin(n, \mathbf{C})$ will be the simply-connected complex Lie group corresponding to the Lie algebra $\mathfrak{spin}(n, \mathbf{R})$. It's compact real form is our $Spin(n, \mathbf{R})$.

Note that one can start more generally with a non-degenerate quadratic form Q over $\mathbf{R}$. Such quadratic forms can be diagonalized, with p +1s and q −1s as diagonal entries ($p + q = n$, the rank). There are corresponding orthogonal and spin groups $SO(p, q, \mathbf{R})$ and $Spin(p, q, \mathbf{R})$, which are isometry groups for these more general quadratic forms. One particular example of interest in physics is the Lorentz group $Spin(3, 1)$. We will however mostly restrict attention to the compact case $Spin(n, \mathbf{R}) = Spin(n, 0, \mathbf{R})$. The non-compact cases also fall outside what we have been studying, since they have no finite-dimensional unitary representations (although there are interesting infinite-dimensional unitary representations of these groups).

For quadratic forms over the complex numbers, the classification is simpler. All quadratic forms of the same rank are equivalent, so there is only once complex case to consider, $Spin(n, \mathbf{C})$.

The general theory of finite-dimensional representations shows that for compact simple Lie groups, we have fundamental representations, which can be associated to the nodes of the Dynkin diagram for the corresponding complex semi-simple Lie algebra. In the case of $SU(n)$ we saw that these representations were given by the defining representation on $\mathbf{C}^n$ and the other exterior algebra spaces $\Lambda^*(\mathbf{C}^n)$. In the orthogonal group case we again have such representations on $\Lambda^*(\mathbf{C}^n)$, but these are not the full story. In terms of Dynkin diagrams, for the odd-dimensional case $Spin(2n + 1)$ there is an extra node connected to the others by a double bond, and to this node will correspond an extra fundamental representation, the spinor representation. This is a representation of $Spin(2n + 1)$, but only a projective representation of $SO(2n + 1)$. For the even dimensional case $Spin(2n)$, there are two extra nodes at one end of the Dynkin diagram. These will correspond to two new fundamental representations, the half-spin representations. Again, these are only projective representations of $SO(2n)$

Some suggested references with more detail for the material covered here are [1], [2],[3][4],[5].

1 Low-dimensional Examples

One thing that makes the theory of spin groups and spin representations confusing is that for small values of the rank special phenomena occur, corresponding to the degenerate nature of the Dynkin diagrams. Here are some facts about the first few spin groups, the ones that behave in a non-generic way.

- $Spin(2)$ is a circle, double-covering the circle $SO(2)$.
- $Spin(3) = SU(2) = Sp(1)$, and the spin representation is the fundamental representation of $SU(2)$. The Dynkin diagram is a single isolated node.
- $Spin(4) = SU(2) \times SU(2)$, and the half-spin representations are the fundamental representations on the two copies of $SU(2)$. The Dynkin diagram is two disconnected nodes.
- $Spin(5) = Sp(2)$, and the spin representation on $\mathbf{C}^4$ can be identified with the fundamental $Sp(2)$ representation on $\mathbf{H}^2$. The Dynkin diagram has two nodes connected by a double bond.
- $Spin(6) = SU(4)$, and the half-spin representations on $\mathbf{C}^4$ can be identified with the fundamental $SU(4)$ representations on $\mathbf{C}^4$ and $\Lambda^3(\mathbf{C}^4)$. The Dynkin diagram has three nodes connected by two single bonds.
- The Dynkin diagram for $Spin(8)$ has three nodes, each connected to a fourth central node. The representations associated to the three notes are all on $\mathbf{C}^8$ and correspond to the two half-spin representations and the representation on vectors. There is a "triality" symmetry that permutes these representations, this is a 3! element group of outer automorphisms of $Spin(8)$.

2 Clifford Algebras

We'll construct the spin groups as groups of invertible elements in certain algebras. This will generalize what happens in the lowest non-trivial dimension (3), where one take $Spin(3)$ to be the unit length elements in the quaternion algebra $\mathbf{H}$. The double-cover homomorphism in this case takes

$$g \in Spin(3) \subset \mathbf{H} \to \{q \in \mathbf{H} \to g^{-1}qg \in \mathbf{H}\}$$

Here the right-hand side is a linear transformation preserving $Re(q)$ (the real part of $q \in \mathbf{H}$, and rotating the imaginary part of q (a 3-vector) by an orthogonal transformation, in $SO(3)$.

In the four dimensional case $Spin(4) = Spin(3) \times Spin(3)$, so an element of $Spin(4)$ is a pair (q_1, q_2) of unit quaternions. Here they act on 4-vectors identified with $\mathbf{H}$ as

$$(q_1, q_2) \in Spin(4) \to \{q \in \mathbf{H} \to q_1 q q_2\}$$

The generalization to higher dimensions will use generalizations of the quaternion algebra known as Clifford algebras.

A Clifford algebra is associated to a vector space V with inner product, in much the same way as the exterior algebra $\Lambda^* V$ is associated to V. The multiplication in the Clifford algebra is different, taking into account the inner product. One way of thinking of a Clifford algebra is as $\Lambda^* V$, with a different product, one that satisfies

$$v \cdot v = - < v, v > 1 = -||v||^2 1$$

for $v \in V$. More generally, one can define a Clifford algebra for any vector space V with a quadratic form $Q(\cdot)$. Note that I'll use the same symbol for the associated symmetric bilinear form

$$Q(v_1, v_2) = \frac{1}{2}(Q(v_1 + v_2) - Q(v_1) - Q(v_2))$$

Definition 1. *The Clifford algebra $C(V, Q)$ associated to a real vector space V with quadratic form Q can be defined as*

$$C(V, Q) = T(V)/I(V, Q)$$

where $T(V)$ is the tensor algebra

$$T(V) = \mathbf{R} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$$

and $I(V, Q)$ is the ideal in $T(V)$ generated by elements

$$v \otimes v + Q(v, v)1$$

(where $v \in V$).

The tensor algebra $T(V)$ is $\mathbf{Z}$-graded, and since $I(V, Q)$ is generated by quadratic elements the quotient $C(V, Q)$ retains only a $\mathbf{Z}_2$ grading

$$C(V, Q) = C_{\text{even}}(V, Q) \oplus C_{\text{odd}}(V, Q)$$

Note the following facts about the Clifford algebra:

- If $Q = 0$, one recovers precisely the definition of the exterior algebra, so

$$\Lambda^*(V) = C(V, Q = 0)$$

- Applying the defining relation for the Clifford algebra to a sum $v + w$ of two vectors gives

$$\begin{aligned}(v+w)\cdot(v+w) = v^2 + vw + wv + w^2 &= -Q(v+w, v+w) \\ &= -Q(v,v) - 2Q(v,w) - Q(w,w)\end{aligned}$$

  so the defining relation implies

$$vw + wv = -2Q(v,w)$$

  which could be used for an alternate definition of the algebra. Also note that in our case where $Q =< \cdot, \cdot >$, this means that two vectors v and w anticommute when they are orthogonal.

- About half of the math community uses the definition given here for the defining relation of a Clifford algebra, the other half uses the relation with the opposite sign

$$v \cdot v = ||v||^2 1$$

- Non-degenerate quadratic forms over a real vector space of dimension n can be put in by a change of basis into a canonical diagonal form with p $+1$'s and q -1's on the diagonal, $p+q=n$. We will mostly be interested in studying the Clifford algebra for the case of the standard positive definite quadratic form $p = n$, $q = 0$. Physicists are also quite interested in the case $p = 3$, $q = -1$, which corresponds to Minkowski space, four dimensional space-time equipped with this kind of quadratic form.

  Later on we will be considering the case of complex vector spaces. In this case there is only one non-degenerate Q up to isomorphism (all diagonal elements can be chosen to be $+1$).

- For the case of $V = \mathbf{R}^n$ with standard inner product ($p = n$), we will denote the Clifford algebra as $C(n)$. Choosing an orthonormal basis $\{e_1, e_2, \cdots, e_n\}$ of $\mathbf{R}^n$, $C(n)$ is the algebra generated by the e_i, with relations

$$e_i e_j + e_j e_i = -2\delta_{ij}$$

Clifford algebras are well-known to physicists as "gamma matrices" and were intoduced by Dirac in 1928 when he discovered what is known as the "Dirac" equation. Dirac was looking for a version of the Schrödinger equation of quantum mechanics that would agree with the principles of special relativity. One common guess for this was what is now known as the Klein-Gordon equation (units with the speed of light $c = 1$ are being used)

$$\Box\psi = (\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_4^2})\psi = 0$$

but the second-order nature of this PDE was problematic, so Dirac was looking for a first-order operator $\not{D}$ satisfying

$$\not{D}^2 = -\Box$$

Dirac found that if he defined

$$\not{D} = \sum_{i=1}^{4} \gamma_i \frac{\partial}{\partial x_i}$$

then this would work if the γ_i satisfied the relations for generators of a Clifford algebra on four dimensional Minkowski space. The Dirac operator remains of fundamental importance in physics, and over the last few decades its importance in mathematics in has become widely realized. For any space with a metric one can define a Dirac operator, which plays the role of a "square-root" of the Laplacian.

One can easily see that, as a vector space $C(n)$ is isomorphic to $\Lambda^*(\mathbf{R}^n)$. Any element of $C(n)$ is a linear combination of finite strings of the form

$$e_{i_1} e_{i_2} \cdots$$

and using the relations

$$e_i e_j = -e_j e_i$$

these can be put into a form where

$$i_1 < i_2 < \cdots$$

eliminating any repeated indices along the way with the relation $e_i^2 = -1$. So, just as for the exterior algebra, the 2^n elements

$$\begin{gathered} 1 \\ e_i \\ e_i e_j \quad i < j \\ \cdots \\ e_1 e_2 \cdots e_n \end{gathered}$$

form a basis.

More abstractly, the Clifford algebra is a filtered algebra

$$F_0 \subset F_1 \subset \cdots \subset F_n = C(n)$$

with F_i the part of $C(n)$ one gets from multiplying at most n generators. The associated graded algebra to the filtration is the exterior algebra

$$\begin{aligned} \mathrm{gr}_F C(n) &= F_1/F_0 \oplus F_2/F_1 \oplus \cdots \oplus F_n/F_{n-1} \\ &= \Lambda^*(\mathbf{R}^n) \end{aligned}$$

2.1 Examples

Let's now consider what these algebras $C(n)$ actually are for some small values of n.

For $n = 1$, $C(1)$ is the algebra of dimension 2 over $\mathbf{R}$ generated by elements $\{1, e_1\}$ with relation $e_1^2 = -1$. This is just the complex numbers $\mathbf{C}$, so $C(1) = \mathbf{C}$.

For $n = 2$, $C(2)$ is the algebra of dimension 4 over $\mathbf{R}$ generated by elements $\{1, e_1, e_2\}$ with relations

$$e_1^2 = -1,\ e_2^2 = -1,\ e_1e_2 = -e_2e_1$$

This turns out to be precisely the quaternion algebra $\mathbf{H}$ under the identification

$$i = e_1,\ j = e_2,\ k = e_1e_2$$

so $C(2) = \mathbf{H}$.

For higher values of n and for arbitrary signature of the quadratic form, see chapter 1 of [1] for a calculation of what all these real Clifford algebras are. We'll just quote the result here:

$$C(3) = \mathbf{H} \oplus \mathbf{H},\ C(4) = M(2, \mathbf{H}),\ C(5) = M(4, \mathbf{C})$$

$$C(6) = M(8, \mathbf{R}),\ C(7) = M(8, \mathbf{R}) \oplus M(8, \mathbf{R}),\ C(8) = M(16, \mathbf{R})$$

and for higher values of n, things are periodic with period 8 since

$$C(n + 8) = C(n) \otimes M(16, \mathbf{R})$$

3 Spin Groups

One of the most important aspects of Clifford algebras is that they can be used to explicitly construct groups called $Spin(n)$ which are non-trivial double covers of the orthogonal groups $SO(n)$.

There are several equivalent possible ways to go about defining the $Spin(n)$ groups as groups of invertible elements in the Clifford algebra.

1. One can define $Spin(n)$ in terms of invertible elements $\tilde{g}$ of $C_{\text{even}}(n)$ that leave the space $V = \mathbf{R}^n$ invariant under conjugation

$$\tilde{g}V\tilde{g}^{-1} \subset V$$

2. One can show that, for $v, w \in V$,

$$v \to v - 2\frac{Q(v, w)}{Q(w, w)}w = -wvw/Q(w, w) = wvw^{-1}$$

   is reflection in the hyperplane perpendicular to w. Then $Pin(n)$ is defined as the group generated by such reflections with $||w||^2 = 1$. $Spin(n)$ is the subgroup of $Pin(n)$ of even elements. Any rotation can be implemented as an even number of reflections (Cartan-Dieudonné) theorem.

3. One can define the Lie algebra of $Spin(n)$ in terms of quadratic elements of the Clifford algebra. This is what we will do here.

The Lie algebra of $SO(n)$ consists of n by n antisymmetric real matrices. A basis for these is given by

$$L_{ij} = E_{ij} - E_{ji}$$

for $i < j$. The L_{ij} generate rotations in the $i - j$ plane. They satisfy the commutation relations

$$[L_{ij}, L_{kl}] = \delta_{il}L_{kj} - \delta_{ik}L_{lj} + \delta_{jl}L_{ik} - \delta_{jk}L_{il}$$

The generators of e_i of the Clifford algebra $C(n)$ satisfy the relations

$$e_ie_j + e_je_i = -2\delta_{ij}$$

and one can use these to show that the $\frac{1}{2}e_ie_j$ satisfy the same commutation relations as the L_{ij}.

$$[\frac{1}{2}e_ie_j, \frac{1}{2}e_ke_l] = \delta_{il}(\frac{1}{2}e_ke_j) - \delta_{ik}(\frac{1}{2}e_le_j) + \delta_{jl}(\frac{1}{2}e_ie_k) - \delta_{jk}(\frac{1}{2}e_ie_l)$$

This shows that the vector space spanned by quadratic elements of $C(n)$ of the form e_ie_j, $i < j$, together with the operation of taking commutators, is isomorphic to the Lie algebra $\mathfrak{so}(n)$. To get the group $Spin(n)$, we can exponentiate these quadratic elements of $C(n)$. Since (to show this, just use the defining relation of $C(n)$)

$$(\frac{1}{2}e_ie_j)^2 = -\frac{1}{4}$$

one can calculate these exponentials to find

$$e^{\theta(\frac{1}{2}e_ie_j)} = \cos(\frac{\theta}{2}) + e_ie_j\sin(\frac{\theta}{2})$$

As θ goes from 0 to 4π this gives a $U(1)$ subgroup of $Spin(n)$. One can check that, acting on vectors by

$$v \to e^{\theta(\frac{1}{2}e_ie_j)}v(e^{\theta(\frac{1}{2}e_ie_j)})^{-1}$$

rotates the vector v by an angle θ in the $i-j$ plane. As we go around this circle in $Spin(n)$ once, we go around the the circle of $SO(n)$ rotations in the $i - j$ plane twice. This is a reflection of the fact that $Spin(n)$ is a double-covering of the group $SO(n)$.

Just as the adjoint action of the Lie algebra of $Spin(n)$ on itself is given by taking commutators, the Lie algebra representation on vectors is also given by taking commutators in the Clifford algebra. One can check that an infinitesimal rotation in the $i - j$ plane of a vector v is given by

$$v \to [e_ie_j, v]$$

This is the infinitesimal version of the representation at the group level

$$v \to \tilde{g}v(\tilde{g})^{-1}$$

where $\tilde{g} \in Spin(n)$ is gotten by exponentiating e_ie_j.

4 Maximal Tori

For the even-dimensional case of $Spin(2n)$, one can proceed as follows to identify its maximal torus, which we'll call $\tilde{T}$. Fixing an identification $\mathbf{C}^n = \mathbf{R}^{2n}$, we have

$$T \subset U(n) \subset SO(2n)$$

where T is a maximal torus of both $U(n)$ and $SO(2n)$. T can be taken to be the group of diagonal n by n complex matrices with k-th diagonal entry $e^{i\theta_k}$. As an element of $SO(2n)$ these become 2 by 2 block diagonal real matrices with blocks

$$\begin{pmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{pmatrix}$$

These blocks rotate by an angle θ_k in the $(2k-1)-2k$ plane, and all commute. For the odd-dimensional case of $SO(2n+1)$, which is of the same rank, the same T can be used, but one has to add in another diagonal entry, 1 as the $(2n+1)$'th entry, to embed this in $2n+1$ by $2n+1$ real matrices.

The double covering of $U(n)$ that is the restriction of the double covering of $SO(2n)$ by $Spin(2n)$ can be described in various ways. One is to define it as

$$\tilde{U}(n) = \{(A, u) \in U(n) \times S^1 : u^2 = \det A\}$$

The maximal torus $\tilde{T}$ of $Spin(2n)$ can be given explicitly in terms of n angles $\tilde{\theta}_k$ as

$$\prod_k (\cos(\tilde{\theta}_k) + e_{2k-1}e_{2k}\sin(\tilde{\theta}_k))$$

and is a double cover of the group T.

5 $Spin^c(n)$

A group related to $Spin(n)$ that has turned out to be of great interest in topology is the group $Spin^c(n)$. This can be defined as

$$Spin(n) \times_{\{\pm 1\}} S^1$$

i.e. by considering pairs $(A, u) \in Spin(n) \times S^1$ and identifying (A, u) and $(-A, -u)$. This group can also be defined as the subgroup of invertible elements in the complexified Clifford algebra $C(n) \otimes \mathbf{C}$ generated by $Spin(n)$ and $S^1 \subset \mathbf{C}$.

A Riemannian manifold M of dimension $2n$ comes with a bundle of orthonormal frames. This is a principal bundle with group $SO(2n)$. Locally it is possible to choose a double-cover of this bundle such that each fiber is the $Spin(2n)$ double cover, but globally there can be a topological obstruction to the continuous choice of such a cover. When such a global cover exists M is said to have a spin-structure. If M is Kähler, its frame bundle can be chosen to be a $U(n)$ bundle, but such an M will often not have a spin structure and one can't consider the spinor geometry of M. However, one reason for the importance of considering $Spin^c(2n)$ is that Kähler manifolds will always have a

$Spin^c$ geometry, i.e the obstruction to a spin-structure can be unwound within $Spin^c(2n)$. This is because while there is no homomorphism

$$U(n) \subset SO(2n) \to Spin(2n)$$

there is a homomorphism

$$f : U(n) \to Spin^c(2n)$$

that covers the inclusion

$$A \in U(n) \to (A, \det A) \in SO(2n) \times U(1)$$

given on diagonal matrices in T by

$$f(\mathrm{diag}(e^{i\theta_1}, \cdots, e^{i\theta_n})) = \prod_k (\cos(\frac{\theta_k}{2}) + e_{2k-1}e_{2k}\sin(\frac{\theta_k}{2})) \times e^{i\sum_k \frac{\theta_k}{2}}$$

References

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