PROPERTIES OF SCHEMES

Contents

1. Introduction

In this chapter we introduce some absolute properties of schemes. A foundational reference is [DG67].

2. Constructible sets

Constructible and locally construcible sets are introduced in Topology, Section 10. We may characterize locally constructible subsets of schemes as follows.

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Lemma 2.1. *Let X be a scheme. A subset E of X is locally constructible in X if and only if $E \cap U$ is constructible in U for every affine open U of X.*

Proof. Assume E is locally constructible. Then there exists an open covering $X = \bigcup U_i$ such that $E \cap U_i$ is constructible in U_i for each i. Let $V \subset X$ be any affine open. We can find a finite open affine covering $V = V_1 \cup \ldots \cup V_m$ such that for each j we have $V_j \subset U_i$ for some $i = i(j)$. By Topology, Lemma 10.4 we see that each $E \cap V_j$ is construcible in V_j. Since the inclusions $V_j \to V$ are quasi-compact (see Schemes, Lemma 19.2) we conclude that $E \cap V$ is constructible in V by Topology, Lemma 10.5. The converse implication is immediate. □

Lemma 2.2. *Let X be a quasi-separated scheme. The intersection of any two quasi-compact opens of X is a quasi-compact open of X. Every quasi-compact open of X is retrocompact in X.*

Proof. If U and V are quasi-compact open then $U \cap V = \Delta^{-1}(U \times V)$, where $\Delta : X \to X \times X$ is the diagonal. As X is quasi-separated we see that Δ is quasi-compact. Hence we see that $U \cap V$ is quasi-compact as $U \times V$ is quasi-compact (details omitted; use Schemes, Lemma 17.4 to see $U \times V$ is a finite union of affines). The second assertion follows from the first and the definitions. □

Lemma 2.3. *Let X be a quasi-compact and quasi-separated scheme. Any locally constructible subset of X is constructible.*

Proof. As X is quasi-compact we can choose a finite affine open covering $X = V_1 \cup \ldots \cup V_m$. As X is quasi-separated each V_i is retrocompact in X by Lemma 2.2. Hence by Topology, Lemma 10.5 we see that $E \subset X$ is constructible in X if and only if $E \cap V_j$ is constructible in V_j. Thus we win by Lemma 2.1. □

3. Integral, irreducible, and reduced schemes

Definition 3.1. Let X be a scheme. We say X is *integral* if it is nonempty and for every nonempty affine open $\text{Spec}(R) = U \subset X$ the ring R is an integral domain.

Lemma 3.2. *Let X be a scheme. The following are equivalent.*

1. *The scheme X is reduced, see Schemes, Definition 12.1.*
2. *There exists an affine open covering $X = \bigcup U_i$ such that each $\Gamma(U_i, \mathcal{O}_X)$ is reduced.*
3. *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is reduced.*
4. *For every open $U \subset X$ the ring $\mathcal{O}_X(U)$ is reduced.*

Proof. See Schemes, Lemmas 12.2 and 12.3. □

Lemma 3.3. *Let X be a scheme. The following are equivalent.*

1. *The scheme X is irreducible.*
2. *There exists an affine open covering $X = \bigcup_{i \in I} U_i$ such that I is not empty, U_i is irreducible for all $i \in I$, and $U_i \cap U_j \neq \emptyset$ for all $i, j \in I$.*
3. *The scheme X is nonempty and every nonempty affine open $U \subset X$ is irreducible.*

Proof. Assume (1). By Schemes, Lemma 11.1 we see that X has a unique generic point η. Then $X = \overline{\{\eta\}}$. Hence η is an element of every nonempty affine open $U \subset X$. This implies that $U = \overline{\{\eta\}}$ and that any two nonempty affines meet. Thus (1) implies both (2) and (3).

Assume (2). Suppose $X = Z_1 \cup Z_2$ is a union of two closed subsets. For every i we see that either $U_i \subset Z_1$ or $U_i \subset Z_2$. Pick some $i \in I$ and assume $U_i \subset Z_1$ (possibly after renumbering Z_1, Z_2). For any $j \in I$ the open subset $U_i \cap U_j$ is dense in U_j and contained in the closed subset $Z_1 \cap U_j$. We conclude that also $U_j \subset Z_1$. Thus $X = Z_1$ as desired.

Assume (3). Choose an affine open covering $X = \bigcup_{i \in I} U_i$. We may assume that each U_i is nonempty. Since X is nonempty we see that I is not empty. By assumption each U_i is irreducible. Suppose $U_i \cap U_j = \emptyset$ for some pair $i, j \in I$. Then the open $U_i \coprod U_j = U_i \cup U_j$ is affine, see Schemes, Lemma 6.8. Hence it is irreducible by assumption which is absurd. We conclude that (3) implies (2). The lemma is proved. ☐

Lemma 3.4. *A scheme X is integral if and only if it is reduced and irreducible.*

Proof. If X is irreducible, then every affine open $\text{Spec}(R) = U \subset X$ is irreducible. If X is reduced, then R is reduced, by Lemma 3.2 above. Hence R is reduced and (0) is a prime ideal, i.e., R is an integral domain.

If X is integral, then for every nonempty affine open $\text{Spec}(R) = U \subset X$ the ring R is reduced and hence X is reduced by Lemma 3.2. Moreover, every nonempty affine open is irreducible. Hence X is irreducible, see Lemma 3.3. ☐

Example 3.5. We give an example of an affine scheme $X = \text{Spec}(A)$ which is connected, all of whose local rings are domains, but which is not integral. Connectedness for A means A has no nontrivial idempotents, see Algebra, Lemma 19.3. Integrality means A is a domain (see above). Local rings being domains means that whenever $fg = 0$ in A, every point of X has a neighborhood where either f or g vanishes.

Roughly speaking, the construction is as follows: let X_0 be the cross (the union of coordinate axes) on the affine plane. Then let X_1 be the (reduced) full preimage of X_0 on the blow-up of the plane (X_1 has three rational components forming a chain). Then blow up the resulting surface at the two singularities of X_1, and let X_2 be the reduced preimage of X_1 (which has five rational components), etc. Take X to be the inverse limit. The only problem with this construction is that blow-ups glue in a projective line, so X_1 is not affine. Let us correct this by glueing in an affine line instead (so our scheme will be an open subset in what was described above).

Here is a completely algebraic construction: For every $k \geq 0$, let A_k be the following ring: its elements are collections of polynomials $p_i \in \mathbf{C}[x]$ where $i = 0, \ldots, 2^k$ such that $p_i(1) = p_{i+1}(0)$. Set $X_k = \text{Spec}(A_k)$. Observe that X_k is a union of 2^k+1 affine lines that meet transversally in a chain. Define a ring homomorphism $A_k \to A_{k+1}$ by

$$(p_0, \ldots, p_{2^k}) \longmapsto (p_0, p_0(1), p_1, p_1(1), \ldots, p_{2^k}),$$

in other words, every other polynomial is constant. This identifies A_k with a subring of A_{k+1}. Let A be the direct limit of A_k (basically, their union). Set $X = \text{Spec}(A)$. For every k, we have a natural embedding $A_k \to A$, that is, a map $X \to X_k$. Each A_k is connected but not integral; this implies that A is connected but not integral. It remains to show that the local rings of A are domains.

Take $f, g \in A$ with $fg = 0$ and $x \in X$. Let us construct a neighborhood of x on which one of f and g vanishes. Choose k such that $f, g \in A_{k-1}$ (note the $k-1$ index). Let y be the image of x in X_k. It suffices to prove that y has a neighborhood on which either f or g viewed as sections of $\mathcal{O}_{X_k}$ vanishes. If y is a smooth point of X_k, that is, it lies on only one of the $2^k + 1$ lines, this is obvious. We can therefore assume that y is one of the 2^k singular points, so two components of X_k pass through y. However, on one of these two components (the one with odd index), both f and g are constant, since they are pullbacks of functions on X_{k-1}. Since $fg = 0$ everywhere, either f or g (say, f) vanishes on the other component. This implies that f vanishes on both components, as required.

4. Types of schemes defined by properties of rings

In this section we study what properties of rings allow one to define local properties of schemes.

Definition 4.1. Let P be a property of rings. We say that P is *local* if the following hold:

(1) For any ring R, and any $f \in R$ we have $P(R) \Rightarrow P(R_f)$.
(2) For any ring R, and $f_i \in R$ such that $(f_1, \ldots, f_n) = R$ then $\forall i, P(R_{f_i}) \Rightarrow P(R)$.

Definition 4.2. Let P be a property of rings. Let X be a scheme. We say X is *locally* P if for any $x \in X$ there exists an affine open neighbourhood U of x in X such that $\mathcal{O}_X(U)$ has property P.

This is only a good notion if the property is local. Even if P is a local property we will not automatically use this definition to say that a scheme is "locally P" unless we also explicitly state the definition elsewhere.

Lemma 4.3. *Let X be a scheme. Let P be a local property of rings. The following are equivalent:*

(1) *The scheme X is locally P.*
(2) *For every affine open $U \subset X$ the property $P(\mathcal{O}_X(U))$ holds.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ satisfies P.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is locally P.*

Moreover, if X is locally P then every open subscheme is locally P.

Proof. Of course (1) $\Leftrightarrow$ (3) and (2) $\Rightarrow$ (1). If (3) $\Rightarrow$ (2), then the final statement of the lemma holds and it follows easily that (4) is also equivalent to (1). Thus we show (3) $\Rightarrow$ (2).

Let $X = \bigcup U_i$ be an affine open covering, say $U_i = \text{Spec}(R_i)$. Assume $P(R_i)$. Let $\text{Spec}(R) = U \subset X$ be an arbitrary affine open. By Schemes, Lemma 11.6 there exists a standard covering of $U = \text{Spec}(R)$ by standard opens $D(f_j)$ such that each ring R_{f_j} is a principal localization of one of the rings R_i. By Definition 4.1 (1) we get $P(R_{f_j})$. Whereupon $P(R)$ by Definition 4.1 (2). $\square$

Here is a sample application.

Lemma 4.4. *Let X be a scheme. Then X is reduced if and only if X is "locally reduced" in the sense of Definition 4.2.*

Proof. This is clear from Lemma 3.2. □

Lemma 4.5. *The following properties of a ring R are local.*

(1) *(Cohen-Macauley.) The ring R is Noetherian and CM, see Algebra, Definition 97.6.*
(2) *(Regular.) The ring R is Noetherian and regular, see Algebra, Definition 103.6.*
(3) *(Absolutely Noetherian.) The ring R is of finite type over Z.*
(4) *Add more here as needed.*[1]

Proof. Omitted. □

5. Noetherian schemes

Recall that a ring R is *Noetherian* if it satsifies the ascending chain condition of ideals. Equivalently every ideal of R is finitely generated.

Definition 5.1. Let X be a scheme.

(1) We say X is *locally Noetherian* if every $x \in X$ has an affine open neighbourhood $\mathrm{Spec}(R) = U \subset X$ such that the ring R is Noetherian.
(2) We say X is *Noetherian* if X is Noetherian and quasi-compact.

Here is the standard result characterizing locally Noetherian schemes.

Lemma 5.2. *Let X be a scheme. The following are equivalent:*

(1) *The scheme X is locally Noetherian.*
(2) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is Noetherian.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Noetherian.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is locally Noetherian.*

Moreover, if X is locally Noetherian then every open subscheme is locally Noetherian.

Proof. To show this it suffices to show that being Noetherian is a local property of rings, see Lemma 4.3. Any localization of a Noetherian ring is Noetherian, see Algebra, Lemma 29.1. By Algebra, Lemma 22.2 we see the second property to Definition 4.1. □

Lemma 5.3. *Any immersion $Z \to X$ with X locally Noetherian is quasi-compact.*

Proof. A closed immersion is clearly quasi-compact. A composition of quasi-compact morphisms is quasi-compact, see Topology, Lemma 9.2. Hence it suffices to show that an open immersion into a locally Noetherian scheme is quasi-compact. Using Schemes, Lemma 19.2 we reduce to the case where X is affine. Any open subset of the spectrum of a Noetherian ring is quasi-compact (for example combine Algebra, Lemma 29.5 and Topology, Lemmas 6.2 and 9.9). □

[1]But we only list those properties here which we have not already dealt with separately somewhere else.

Lemma 5.4. *A locally Noetherian scheme is quasi-separated.*

Proof. By Schemes, Lemma 21.7 we have to show that the intersection $U \cap V$ of two affine opens of X is quasi-compact. This follows from Lemma 5.3 above on considering the open immersion $U \cap V \to U$ for example. (But really it is just because any open of the spectrum of a Noetherian ring is quasi-compact.) □

Lemma 5.5. *A (locally) Noetherian scheme has a (locally) Noetherian underlying topological space, see Topology, Definition 6.1.*

Proof. This is because a Noetherian scheme is a finite union of spectra of Noetherian rings and Algebra, Lemma 29.5 and Topology, Lemma 6.4. □

Lemma 5.6. *Any morphism of schemes $f : X \to Y$ with X Noetherian is quasi-compact.*

Proof. Use Lemma 5.5 and use that any subset of a Noetherian topological space is quasi-compact (see Topology, Lemmas Lemmas 6.2 and 9.9). □

Lemma 5.7. *Any locally closed subscheme of a (locally) Noetherian scheme is (locally) Noetherian.*

Proof. Omitted. Hint: Any quotient, and any localization of a Noetherian ring is Noetherian. For the Noetherian case use again that any subset of a Noetherian space is a Noetherian space (with induced topology). □

Here is a fun lemma. It says that every locally Noetherian scheme has plenty of closed points (at least one in every closed subset).

Lemma 5.8. *Any locally Noetherian scheme has a closed point. Any closed subset of a locally Noetherian scheme has a closed point. Equivalently, any point of a locally Noetherian scheme specializes to a closed point.*

Proof. The second assertion follows from the first (using Schemes, Lemma 12.4 and Lemma 5.7). Consider any nonempty affine open $U \subset X$. Let $x \in U$ be a closed point. If x is a closed point of X then we are done. If not, let $y \in \overline{\{x\}}$ be a specialization of x. Note that $y \in X \setminus U$. Consider the local ring $R = \mathcal{O}_{X,y}$. This is a Noetherian local ring. Denote $V \subset \text{Spec}(R)$ the inverse image of U in $\text{Spec}(R)$ by the canonical morphism $\text{Spec}(R) \to X$ (see Schemes, Section 13.) By construction V is a singleton with unique point corresponding to x (use Schemes, Lemma 13.2). Say $V = \{\mathfrak{q}\}$. Consider the Noetherian local domain $R/\mathfrak{q}$. By Algebra, Lemma 59.1 we see that $\dim(R/\mathfrak{q}) = 1$. In other words, we see that y is an immediate specialization of x (see Topology, Definition 16.1). In other words, any point $y \neq x$ such that $x \rightsquigarrow y$ is an immediate specialization of x. Clearly each of these points is a closed point, and we win. □

Lemma 5.9. *Let X be a locally Noetherian scheme. Let $x' \rightsquigarrow x$ be a specialization of points of X. Then*

(1) *there exists a discrete valuation ring R and a morphism $f : \text{Spec}(R) \to X$ such that the generic point η of $\text{Spec}(R)$ maps to x' and the special point maps to x, and*
(2) *given a finitely generated field extension $\kappa(x') \subset K$ we may arrange it so that the extension $\kappa(x') \subset \kappa(\eta)$ induced by f is isomorphic to the given one.*

Proof. Let $x' \rightsquigarrow x$ be a specialization in X, and let $\kappa(x') \subset K$ be a finitely generated extension of fields. By Schemes, Lemma 13.2 and the discussion following Schemes, Lemma 13.3 this leads to ring maps $\mathcal{O}_{X,x} \to \kappa(x') \to K$. Let $R \subset K$ be any discrete valuation ring whose field of fractions is K and which dominates the image of $\mathcal{O}_{X,x} \to K$, see Algebra, Lemma 111.11. The ring map $\mathcal{O}_{X,x} \to R$ induces the morphism $f : \mathrm{Spec}(R) \to X$, see Schemes, Lemma 13.1. This morphism has all the desired properties by construction. □

6. Jacobson schemes

Recall that a space is said to be *Jacobson* if the closed points are dense in every closed subset, see Topology, Section 13.

Definition 6.1. A scheme S is said to be *Jacobson* if its underlying topological space is Jacobson.

Recall that a ring R is Jacobson if every radical ideal of R is the intersection of maximal ideals, see Algebra, Definition 32.1.

Lemma 6.2. *An affine scheme* $\mathrm{Spec}(R)$ *is Jacobson if and only if the ring* R *is Jacobson.*

Proof. This is Algebra, Lemma 32.4. □

Here is the standard result characterizing Jacobson schemes. Intuitively it claims that Jacobson $\Leftrightarrow$ locally Jacobson.

Lemma 6.3. *Let X be a scheme. The following are equivalent:*

(1) *The scheme X is Jacobson.*
(2) *The scheme X is "locally Jacobson" in the sense of Definition 4.2.*
(3) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is Jacobson.*
(4) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Jacobson.*
(5) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is Jacobson.*

Moreover, if X is Jacobson then every open subscheme is Jacobson.

Proof. The final assertion of the lemma holds by Topology, Lemma 13.5. The equivalence of (5) and (1) is Topology, Lemma 13.4. Hence, using Lemma 6.2, we see that (1) $\Leftrightarrow$ (2). To finish proving the lemma it suffices to show that "Jacobson" is a local property of rings, see Lemma 4.3. Any localization of a Jacobson ring at an element is Jacobson, see Algebra, Lemma 32.14. Suppose R is a ring, $f_1, \ldots, f_n \in R$ generate the unit ideal and each R_{f_i} is Jacobson. Then we see that $\mathrm{Spec}(R) = \bigcup D(f_i)$ is a union of open subsets which are all Jacobson, and hence $\mathrm{Spec}(R)$ is Jacobson by Topology, Lemma 13.4 again. This proves the second property of Definition 4.1. □

Many schemes used commonly in algebraic geometry are Jacobson, see Morphisms, Lemma 16.10. We mention here the following interesting case.

Lemma 6.4. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. In this case the scheme $S = \mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$ is Jacobson.*

Proof. Since $\text{Spec}(R)$ is a Noetherian scheme, hence S is a Noetherian scheme (Lemma 5.7). Hence S is a sober, Noetherian topological space (use Schemes, Lemma 11.1). Assume S is not Jacobson to get a contradiction. By Topology, Lemma 13.3 there exists some non-closed point $\xi \in S$ such that $\{\xi\}$ is locally closed. This corresponds to a prime $\mathfrak{p} \subset R$ such that (1) there exists a prime $\mathfrak{q}$, $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{m}$ with both inclusions strict, and (2) $\{\mathfrak{p}\}$ is open in $\text{Spec}(R/\mathfrak{p})$. This is impossible by Algebra, Lemma 59.1. □

7. Normal schemes

Recall that a ring R is said to be normal if all its local rings are normal domains, see Algebra, Definition 34.10. A normal domain is a domain which is integrally closed in its field of fractions, see Algebra, Definition 34.1. Thus it makes sense to define a normal scheme as follows.

Definition 7.1. A scheme X is *normal* if and only if for all $x \in X$ the local ring $\mathcal{O}_{X,x}$ is a normal domain.

This seems to be the definition used in EGA, see [DG67, 0, 4.1.4]. Suppose $X = \text{Spec}(A)$, and A is reduced. Then saying that X is normal is not equivalent to saying that A is integrally closed in its total ring of fractions. However, if A is Noetherian then this is the case (see Algebra, Lemma 34.14).

Lemma 7.2. *Let X be a scheme. The following are equivalent:*

(1) *The scheme X is normal.*
(2) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is normal.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is normal.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is normal.*

Moreover, if X is normal then every open subscheme is normal.

Proof. This is clear from the definitions. □

Lemma 7.3. *A normal scheme is reduced.*

Proof. Immediate from the defintions. □

Lemma 7.4. *Let X be an integral scheme. Then X is normal if and only if for every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is a normal domain.*

Proof. This follows from Algebra, Lemma 34.9. □

Lemma 7.5. *Let X be a scheme with a finite number of irreducible components. The following are equivalent:*

(1) *X is normal, and*
(2) *X is a finite disjoint union of normal integral schemes.*

Proof. It is immediate from the definitions that (2) implies (1). Let X be a normal scheme with a finite number of irreducible components. If X is affine then X satisfies (2) by Algebra, Lemma 34.14. For a general X, let $X = \bigcup X_i$ be an affine open covering. Note that also each X_i has but a finite number of irreducible components, and the lemma holds for each X_i. Let $T \subset X$ be an irreducible component. By the affine case each intersection $T \cap X_i$ is open in X_i and an integral normal scheme. Hence $T \subset X$ is open, and an integral normal scheme. This proves that X is the disjoint union of its irreducible components, which are integral normal schemes. There are only finitely many by assumption. □

Lemma 7.6. *Let X be a Noetherian scheme. The following are equivalent:*

1. *X is normal, and*
2. *X is a finite disjoint union of normal integral schemes.*

Proof. This is a special case of Lemma 7.5 because a Noetherian scheme has a Noetherian underlying topological space (Lemma 5.5 and Topology, Lemma 6.2. □

Lemma 7.7. *Let X be a locally Noetherian normal scheme. The following are equivalent:*

1. *X is normal, and*
2. *X is a disjoint union of integral normal schemes.*

Proof. Omitted. Hint: This is purely topological from Lemma 7.6. □

Remark 7.8. Let X be a normal scheme. If X is locally Noetherian then we see that X is integral if and only if X is connected, see Lemma 7.7. But there exists a connected affine scheme X such that $\mathcal{O}_{X,x}$ is a domain for all $x \in X$, but X is not irreducible, see Example 3.5. This example is even a normal scheme (proof omitted), so beware!

Lemma 7.9. *Let X be an integral normal scheme. Then $\Gamma(X, \mathcal{O}_X)$ is a normal domain.*

Proof. Set $R = \Gamma(X, \mathcal{O}_X)$. It is clear that R is a domain. Suppose $f = a/b$ is an element of its fraction field which is integral over R. Say we have $f^d + \sum_{i=1,\ldots,d} a_i f^i = 0$ with $a_i \in R$. Let $U \subset X$ be affine open. Since $b \in R$ is not zero and since X is integral we see that also $b|_U \in \mathcal{O}_X(U)$ is not zero. Hence a/b is an element of the fraction field of $\mathcal{O}_X(U)$ which is integral over $\mathcal{O}_X(U)$ (because we can use the same polynomial $f^d + \sum_{i=1,\ldots,d} a_i|_U f^i = 0$ on U). Since $\mathcal{O}_X(U)$ is a normal domain (Lemma 7.2), we see that $f_U = (a|_U)/(b|_U) \in \mathcal{O}_X(U)$. It is easy to see that $f_U|_V = f_V$ whenever $V \subset U \subset X$ are affine open. Hence the local sections f_U glue to a global section f as desired. □

8. Cohen-Macaulay schemes

Recall, see Algebra, Definition 97.1, that a local Noetherian ring $(R, \mathfrak{m})$ is said to be Cohen-Macaulay if $\text{depth}_\mathfrak{m}(R) = \dim(R)$. Recall that a Noetherian ring R is said to be Cohen-Macaulay if every local ring $R_\mathfrak{p}$ of R is Cohen-Macaulay, see Algebra, Definition 97.6.

Definition 8.1. Let X be a scheme. We say X is *Cohen-Macaulay* if for every $x \in X$ there exists an affine open neighbourhood $U \subset X$ of x such that the ring $\mathcal{O}_X(U)$ is Noetherian and Cohen-Macaulay.

Lemma 8.2. *Let X be a scheme. The following are equivalent:*

1. *X is Cohen-Macaulay,*
2. *X is locally Noetherian and all of its local rings are Cohen-Macaulay, and*
3. *X is locally Noetherian and for any closed point $x \in X$ the local ring $\mathcal{O}_{X,x}$ is Cohen-Macaulay.*

Proof. Algebra, Lemma 97.5 says that the localization of a Cohen-Macaulay local ring is Cohen-Macaulay. The lemma follows by combining this with Lemma 5.2, with the existence of closed points on locally Noetherian schemes (Lemma 5.8), and the definitions. □

Lemma 8.3. *Let X be a scheme. The following are equivalent:*

1. *The scheme X is Cohen-Macaulay.*
2. *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is Noetherian and Cohen-Macaulay.*
3. *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Noetherian and Cohen-Macaulay.*
4. *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is Cohen-Macaulay.*

Moreover, if X is Cohen-Macaulay then every open subscheme is Cohen-Macaulay.

Proof. Combine Lemmas 5.2 and 8.2. □

More information on Cohen-Macaulay schemes and depth can be found in Coherent, Section 13.

9. Regular schemes

Recall, see Algebra, Definition 58.9, that a local Noetherian ring $(R, \mathfrak{m})$ is said to be *regular* if $\mathfrak{m}$ can be generated by $\dim(R)$ elements. Recall that a Noetherian ring R is said to be *regular* if every local ring $R_{\mathfrak{p}}$ of R is regular, see Algebra, Definition 103.6.

Definition 9.1. Let X be a scheme. We say X is *regular*, or *nonsingular* if for every $x \in X$ there exists an affine open neighbourhood $U \subset X$ of x such that the ring $\mathcal{O}_X(U)$ is Noetherian and regular.

Lemma 9.2. *Let X be a scheme. The following are equivalent:*

1. *X is regular,*
2. *X is locally Noetherian and all of its local rings are regular, and*
3. *X is locally Noetherian and for any closed point $x \in X$ the local ring $\mathcal{O}_{X,x}$ is regular.*

Proof. By the discussion in Algebra preceding Algebra, Definition 103.6 we know that the localization of a regular local ring is regular. The lemma follows by combining this with Lemma 5.2, with the existence of closed points on locally Noetherian schemes (Lemma 5.8), and the definitions. □

Lemma 9.3. *Let X be a scheme. The following are equivalent:*

1. *The scheme X is regular.*
2. *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is Noetherian and regular.*
3. *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Noetherian and regular.*
4. *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is regular.*

Moreover, if X is regular then every open subscheme is regular.

Proof. Combine Lemmas 5.2 and 9.2. □

Lemma 9.4. *A regular scheme is normal.*

Proof. See Algebra, Lemma 141.5. ☐

10. Dimension

The dimension of a scheme is just the dimension of its underlying topological space.

Definition 10.1. Let X be a scheme.

(1) The *dimension* of X is just the dimension of X as a topological spaces, see Topology, Definition 7.1.
(2) For $x \in X$ we denote $\dim_x(X)$ the dimension of the underlying topological space of X at x as in Topology, Definition 7.1. We say $\dim_x(X)$ is the *dimension of X at x*.

As a scheme has a sober underlying topological space (Schemes, Lemma 11.1) we may compute the dimension of X as the supremum of the lengths n of chains

$$T_0 \subset T_1 \subset \ldots \subset T_n$$

of irreducible closed subsets of X, or as the supremum of the lengths n of chains of specializations

$$\xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots \rightsquigarrow \xi_0$$

of points of X.

Lemma 10.2. *Let X be a scheme. The following are equal*

(1) *The dimension of X.*
(2) *The supremum of the dimensions of the local rings of X.*
(3) *The supremum of* $\dim_x(X)$ *for $x \in X$.*

Proof. Note that given a chain of specializations

$$\xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots \rightsquigarrow \xi_0$$

of points of X all of the points ξ_i correspond to prime ideals of the local ring of X at ξ_0 by Schemes, Lemma 13.2. Hence we see that the dimension of X is the supremum of the dimensions of its local rings. In particular $\dim_x(X) \geq \dim(\mathcal{O}_{X,x})$ as $\dim_x(X)$ is the minimum of the dimensions of open neighbourhoods of x. Thus $\sup_{x \in X} \dim_x(X) \geq \dim(X)$. On the other hand, it is clear that $\sup_{x \in X} \dim_x(X) \leq \dim(X)$ as $\dim(U) \leq \dim(X)$ for any open subset of X. ☐

11. Catenary schemes

Recall that a topological space X is called *catenary* if for every pair of irreducible closed subsets $T \subset T'$ there exist a maximal chain of irreducible closed subsets

$$T = T_0 \subset T_1 \subset \ldots \subset T_e = T'$$

and every such chain has the same length. See Topology, Definition 8.1.

Definition 11.1. Let S be a scheme. We say S is *catenary* if the underlying topological space of S is catenary.

Recall that a ring A is called *catenary* if for any pair of prime ideals $\mathfrak{p} \subset \mathfrak{q}$ there exists a maximal chain of primes

$$\mathfrak{p} = \mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_e = \mathfrak{q}$$

and all of these have the same length. See Algebra, Definition 98.1.

Lemma 11.2. *Let S be a scheme. The following are equivalent*

1. *S is catenary,*
2. *there exists an open covering of S all of whose members are catenary schemes,*
3. *for every affine open $\operatorname{Spec}(R) = U \subset S$ the ring R is catenary, and*
4. *there exists an affine open covering $S = \bigcup U_i$ such that each U_i is the spectrum of a catenary ring.*

Moreover, in this case any locally closed subscheme of S is catenary as well.

Proof. Combine Topology, Lemma 8.2, and Algebra, Lemma 98.2. □

Lemma 11.3. *Let S be a locally Noetherian scheme. The following are equivalent:*

1. *S is catenary, and*
2. *locally in the Zariski topology there exists a dimension function on S (see Topology, Definition 16.1).*

Proof. This follows from Topology, Lemmas 8.2, 16.2, and 16.4, Schemes, Lemma 11.1 and finally Lemma 5.5. □

Lemma 11.4. *Let X be a scheme. Let $Y \subset X$ be an irreducible closed subset. Let $\xi \in Y$ be the generic point. Then*

$$\mathit{codim}(Y, X) = \dim(\mathcal{O}_{X,\xi})$$

where the codimension is as defined in Topology, Definition 8.3.

Proof. By Topology, Lemma 8.4 we may replace X by an affine open neighbourhood of ξ. In this case the result follows easily from Algebra, Lemma 24.2. □

In particular the dimension of a scheme is the supremum of the dimensions of all of its local rings. It turns out that we can use this lemma to characterize a catenary scheme as a scheme all of whose local rings are catenary.

Lemma 11.5. *Let X be a scheme. The following are equivalent*

1. *X is catenary, and*
2. *for any $x \in X$ the local ring $\mathcal{O}_{X,x}$ is catenary.*

Proof. Assume X is catenary. Let $x \in X$. By Lemma 11.2 we may replace X by an affine open neighbourhood of x, and then $\Gamma(X, \mathcal{O}_X)$ is a catenary ring. By Algebra, Lemma 98.3 any localization of a catenary ring is catenary. Whence $\mathcal{O}_{X,x}$ is catenary.

Conversely assume all local rings of X are catenary. Let $Y \subset Y'$ be an inclusion of irreducible closed subsets of X. Let $\xi \in Y$ be the generic point. Let $\mathfrak{p} \subset \mathcal{O}_{X,\xi}$ be the prime corresponding to the generic point of Y', see Schemes, Lemma 13.2. By that same lemma the irreducible closed subsets of X in between Y and Y' correspond to primes $\mathfrak{q} \subset \mathcal{O}_{X,\xi}$ with $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{m}_\xi$. Hence we see all maximal chains of these are finite and have the same length as $\mathcal{O}_{X,\xi}$ is a catenary ring. □

12. Serre's conditions

Here are two technical notions that are often useful. See also Coherent, Section 13.

Definition 12.1. Let X be a locally Noetherian scheme. Let $k \geq 0$.

(1) We say X is *regular in codimension* k, or we say X has property (R_k) if for every $x \in X$ we have

$$\dim(\mathcal{O}_{X,x}) \leq k \Rightarrow \mathcal{O}_{X,x} \text{ is regular}$$

(2) We say X has property (S_k) if for every $x \in X$ we have $\text{depth}(\mathcal{O}_{X,x}) \geq \min(k, \dim(\mathcal{O}_{X,x}))$.

The phrase "regular in codimension k" makes sense since we have seen in Section 11 that if $Y \subset X$ is irreducible closed with generic point x, then $\dim(\mathcal{O}_{X,x}) = \text{codim}(Y, X)$. For example condition (R_0) means that for every generic point $\eta \in X$ of an irreducible component of X the local ring $\mathcal{O}_{X,\eta}$ is a field. But for general Noetherian schemes it can happen that the regular locus of X is badly behaved, so care has to be taken.

Lemma 12.2. *Let X be a locally Noetherian scheme. Then X is Cohen-Macaulay if and only if X has (S_k) for all $k \geq 0$.*

Proof. By Lemma 8.2 we reduce to looking at local rings. Hence the lemma is true because a Noetherian local ring is Cohen-Macaulay if and only if it has depth equal to its dimension. □

Lemma 12.3. *Let X be a locally Noetherian scheme. Then X is reduced if and only if X has properties (S_1) and (R_0).*

Proof. This is Algebra, Lemma 141.3. □

Lemma 12.4. *Let X be a locally Noetherian scheme. Then X is normal if and only if X has properties (S_2) and (R_1).*

Proof. This is Algebra, Lemma 141.4. □

13. Japanese and Nagata schemes

The notions considered in this section are not prominently defined in EGA. A "universally Japanese scheme" is mentioned and defined in [DG67, IV Corollary 5.11.4]. A "Japanese scheme" is mentioned in [DG67, IV Remark 10.4.14 (ii)] but no definition is given. A Nagata scheme (as given below) occurs in a few places in the literature (see for example [Liu02, Definition 8.2.30] and [Gre76, Page 142]).

We briefly recall that a domain R is called *Japanese* if the integral closure of R in any finite extension of its fraction field is finite over R. A ring R is called *universally Japanese* if for any finite type ring map $R \to S$ with S a domain S is Japanese. A ring R is called *Nagata* if it is Noetherian and $R/\mathfrak{p}$ is Japanese for every prime $\mathfrak{p}$ of R.

Definition 13.1. Let X be a scheme.

(1) Assume X integral. We say X is *Japanese* if for every $x \in X$ there exists an affine open neighbourhood $x \in U \subset X$ such that the ring $\mathcal{O}_X(U)$ is Japanese (see Algebra, Definition 145.1).
(2) We say X is *universally Japanese* if for every $x \in X$ there exists an affine open neighbourhood $x \in U \subset X$ such that the ring $\mathcal{O}_X(U)$ is universally Japanese (see Algebra, Definition 145.15).
(3) We say X is *Nagata* if for every $x \in X$ there exists an affine open neighbourhood $x \in U \subset X$ such that the ring $\mathcal{O}_X(U)$ is Nagata (see Algebra, Definition 145.15).

Being Nagata is the same thing as being locally Noetherian and universally Japanese, see Lemma 13.8.

Remark 13.2. In [Hoo72] a (locally Noetherian) scheme X is called Japanese if for every $x \in X$ and every associated prime $\mathfrak{p}$ of $\mathcal{O}_{X,x}$ the ring $\mathcal{O}_{X,x}/\mathfrak{p}$ is Japanese. We do not use this definition since it is not clear that this gives the same notion as above for Noetherian integral schemes. In other words, we do not know whether a Noetherian domain all of whose local rings are Japanese is Japanese. If you do please email stacks.project@gmail.com. On the other hand, we could circumvent this problem by calling a scheme X Japanese if for every affine open $\text{Spec}(A) \subset X$ the ring $A/\mathfrak{p}$ is Japanese for every associated prime $\mathfrak{p}$ of A.

Lemma 13.3. *A Nagata scheme is locally Noetherian.*

Proof. This is true because a Nagata ring is Noetherian by definition. □

Lemma 13.4. *Let X be an integral scheme. The following are equivalent:*

1. *The scheme X is Japanese.*
2. *For every affine open $U \subset X$ the domain $\mathcal{O}_X(U)$ is Japanese.*
3. *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Japanese.*
4. *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is Japanese.*

Moreover, if X is Japanese then every open subscheme is Japanese.

Proof. This follows from Lemma 4.3 and Algebra, Lemmas 145.3 and 145.4. □

Lemma 13.5. *Let X be a scheme. The following are equivalent:*

1. *The scheme X is universally Japanese.*
2. *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is universally Japanese.*
3. *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is universally Japanese.*
4. *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is universally Japanese.*

Moreover, if X is universally Japanese then every open subscheme is universally Japanese.

Proof. This follows from Lemma 4.3 and Algebra, Lemmas 145.18 and 145.21. □

Lemma 13.6. *Let X be a scheme. The following are equivalent:*

1. *The scheme X is Nagata.*
2. *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is Nagata.*
3. *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Nagata.*
4. *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is Nagata.*

Moreover, if X is Nagata then every open subscheme is Nagata.

Proof. This follows from Lemma 4.3 and Algebra, Lemmas 145.20 and 145.21. □

Lemma 13.7. *Let X be a locally Noetherian scheme. Then X is Nagata if and only if every integral closed subscheme $Z \subset X$ is Japanese.*

Proof. Assume X is Nagata. Let $Z \subset X$ be an integral closed subscheme. Let $z \in Z$. Let $\text{Spec}(A) = U \subset X$ be an affine open containing z such that A is Nagata. Then $Z \cap U \cong \text{Spec}(A/\mathfrak{p})$ for some prime $\mathfrak{p}$, see Schemes, Lemma 10.1 (and Definition 3.1). By Algebra, Definition 145.15 we see that $A/\mathfrak{p}$ is Japanese. Hence Z is Japanese by definition.

Assume every integral closed subscheme of X is Japanese. Let $\text{Spec}(A) = U \subset X$ be any affine open. As X is locally Noetherian we see that A is Noetherian (Lemma 5.2). Let $\mathfrak{p} \subset A$ be a prime ideal. We have to show that $A/\mathfrak{p}$ is Japanese. Let $T \subset U$ be the closed subset $V(\mathfrak{p}) \subset \text{Spec}(A)$. Let $\overline{T} \subset X$ be the closure. Then $\overline{T}$ is irreducible as the closure of an irreducible subset. Hence the reduced closed subscheme defined by $\overline{T}$ is an integral closed subscheme (called $\overline{T}$ again), see Schemes, Lemma 12.4. In other words, $\text{Spec}(A/\mathfrak{p})$ is an affine open of an integral closed subscheme of X. This subscheme is Japanese by assumption and by Lemma 13.4 we see that $A/\mathfrak{p}$ is Japanese. ☐

Lemma 13.8. *Let X be a scheme. The following are equivalent:*

1. *X is Nagata, and*
2. *X is locally Noetherian and universally Japanese.*

Proof. This is Algebra, Proposition 145.30. ☐

This discussion will be continued in Morphisms, Section 18.

14. The singular locus

Here is the definition.

Definition 14.1. Let X be a locally Noetherian scheme. The *regular locus* $\text{Reg}(X)$ of X is the set of $x \in X$ such that $\mathcal{O}_{X,x}$ is a regular local ring. The *singular locus* $\text{Sing}(X)$ is the complement $X \setminus \text{Reg}(X)$, i.e., the set of points $x \in X$ such that $\mathcal{O}_{X,x}$ is not a regular local ring.

The regular locus of a locally Noetherian scheme is stable under generalizations, see the discussion preceding Algebra, Definition 103.6. However, for general locally Noetherian schemes the regular locus need not be open. In More on Algebra, Section 36 the reader can find some criteria for when this is the case. We will discuss this further in Morphisms, Section 19.

15. Quasi-affine schemes

Definition 15.1. A scheme X is called *quasi-affine* if it is quasi-compact and isomorphic to an open subscheme of an affine scheme.

Lemma 15.2. *Let X be a scheme. Let $f \in \Gamma(X, \mathcal{O}_X)$. Denote X_f the maximal open subscheme of X where f is invertible, see Schemes, Lemma 6.2 or Modules, Lemma 21.7. If X is quasi-compact and quasi-separated, the canonical map*

$$\Gamma(X, \mathcal{O}_X)_f \longrightarrow \Gamma(X_f, \mathcal{O}_X)$$

is an isomorphism. Moreover, if $\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules the map

$$\Gamma(X, \mathcal{F})_f \longrightarrow \Gamma(X_f, \mathcal{F})$$

is an isomorphism.

Proof. Write $R = \Gamma(X, \mathcal{O}_X)$. Consider the canonical morphism

$$\varphi : X \longrightarrow \text{Spec}(R)$$

of schemes, see Schemes, Lemma 6.4. Then the inverse image of the standard open $D(f)$ on the right hand side is X_f on the left hand side. Moreover, since X is assumed quasi-compact and quasi-separated the morphism φ is quasi-compact and quasi-separated, see Schemes, Lemma 19.2 and 21.14. Hence by Schemes, Lemma 24.1 we see that $\varphi_*\mathcal{F}$ is quasi-coherent. Hence we see that $\varphi_*\mathcal{F} = \widetilde{M}$ with $M = \Gamma(X, \mathcal{F})$ as an R-module. Thus we see that

$$\Gamma(X_f, \mathcal{F}) = \Gamma(D(f), \varphi_*\mathcal{F}) = \Gamma(D(f), \widetilde{M}) = M_f$$

which is exactly the content of the lemma. The case of $\mathcal{F} = \mathcal{O}_X$ will given the first displayed isomorphism of the lemma. □

Lemma 15.3. *Let X be a scheme. Let $f \in \Gamma(X, \mathcal{O}_X)$. Assume X is quasi-compact and quasi-separated and assume that X_f is affine. Then the canonical morphism*

$$j : X \longrightarrow \text{Spec}(\Gamma(X, \mathcal{O}_X))$$

from Schemes, Lemma 6.4 induces an isomorphism of $X_f = j^{-1}(D(f))$ onto the standard affine open $D(f) \subset \text{Spec}(\Gamma(X, \mathcal{O}_X))$.

Proof. This is clear as j induces an isomorphism of rings $\Gamma(X, \mathcal{O}_X)_f \to \mathcal{O}_X(X_f)$ by Lemma 15.2 above. □

Lemma 15.4. *Let X be a scheme. Then X is quasi-affine if and only if the canonical morphism*

$$X \longrightarrow \text{Spec}(\Gamma(X, \mathcal{O}_X))$$

from Schemes, Lemma 6.4 is a quasi-compact open immersion.

Proof. If the displayed morphism is a quasi-compact open immersion then X is ismorphic to a quasi-compact open subscheme of $\text{Spec}(\Gamma(X, \mathcal{O}_X))$ and clearly X is quasi-affine.

Assume X is quasi-affine, say $X \subset \text{Spec}(R)$ is quasi-compact open. This in particular implies that X is separated, see Schemes, Lemma 23.8. Let $A = \Gamma(X, \mathcal{O}_X)$. Consider the ring map $R \to A$ coming from $R = \Gamma(\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)})$ and the restriction mapping of the sheaf $\mathcal{O}_{\text{Spec}(R)}$. By Schemes, Lemma 6.4 we obtain a factorization:

$$X \longrightarrow \text{Spec}(A) \longrightarrow \text{Spec}(R)$$

of the inclusion morphism. Let $x \in X$. Choose $r \in R$ such that $x \in D(r)$ and $D(r) \subset X$. Denote $f \in A$ the image of r in A. The open X_f of Lemma 15.2 above is equal to $D(r) \subset X$ and hence $A_f \cong R_r$ by the conclusion of that lemma. Hence $D(r) \to \text{Spec}(A)$ is an isomorphism onto the standard affine open $D(f)$ of $\text{Spec}(A)$. Since X can be covered by such affine opens $D(f)$ we win. □

16. Characterizing modules of finite type and finite presentation

Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following lemma implies that $\mathcal{F}$ is of finite type (see Modules, Definition 9.1) if and only if $\mathcal{F}$ is on each open affine $\text{Spec}(A) = U \subset X$ of the form $\widetilde{M}$ for some finite type A-module M. Similarly, $\mathcal{F}$ is of finite presentation (see Modules, Definition 11.1) if and only if $\mathcal{F}$ is on each open affine $\text{Spec}(A) = U \subset X$ of the form $\widetilde{M}$ for some finitely presented A-module M.

Lemma 16.1. *Let $X = \text{Spec}(R)$ be an affine scheme. The quasi-coherent sheaf of $\mathcal{O}_X$-modules $\widetilde{M}$ is a finite type $\mathcal{O}_X$-module if and only if M is a finite R-module.*

Proof. Assume $\widetilde{M}$ is a finite type $\mathcal{O}_X$-module. This means there exists an open covering of X such that $\widetilde{M}$ restricted to the members of this covering is globally generated by finitely many sections. Thus there also exists a standard open covering $X = \bigcup_{i=1,\ldots,n} D(f_i)$ such that $\widetilde{M}|_{D(f_i)}$ is generated by finitely many sections. Thus M_{f_i} is finitely generated for each i. Hence we conclude by Algebra, Lemma 22.2. □

Lemma 16.2. *Let $X = \text{Spec}(R)$ be an affine scheme. The quasi-coherent sheaf of $\mathcal{O}_X$-modules $\widetilde{M}$ is an $\mathcal{O}_X$-module of finite presentation if and only if M is an R-module of finite presentation.*

Proof. Assume $\widetilde{M}$ is an $\mathcal{O}_X$-module of finite presentation. By Lemma 16.1 we see that M is a finite R-module. Choose a surjection $R^n \to M$ with kernel K. By Schemes, Lemma 5.4 there is a short exact sequence

$$0 \to \widetilde{K} \to \bigoplus \mathcal{O}_X^{\oplus n} \to \widetilde{M} \to 0$$

By Modules, Lemma 11.3 we see that $\widetilde{K}$ is a finite type $\mathcal{O}_X$-module. Hence by Lemma 16.1 again we see that K is a finite R-module. Hence M is an R-module of finite presentation. □

17. Flat modules

On any ringed space $(X, \mathcal{O}_X)$ we know what it means for an $\mathcal{O}_X$-module to be flat (at a point), see Modules, Definition 16.1 (Definition 16.3). On an affine scheme this matches the notion defined in the algebra chapter.

Lemma 17.1. *Let $X = \text{Spec}(R)$ be an affine scheme. Let $\mathcal{F} = \widetilde{M}$ for some R-module M. The quasi-coherent sheaf $\mathcal{F}$ is a flat $\mathcal{O}_X$-module of if and only if M is a flat R-module.*

Proof. Flatness of $\mathcal{F}$ may be checked on the stalks, see Modules, Lemma 16.2. The same is true in the case of modules over a ring, see Algebra, Lemma 36.19. And since $\mathcal{F}_x = M_{\mathfrak{p}}$ if x corresponds to $\mathfrak{p}$ the lemma is true. □

18. Locally free modules

On any ringed space we know what it means for an $\mathcal{O}_X$-module to be (finite) locally free. On an affine scheme this matches the notion defined in the algebra chapter.

Lemma 18.1. *Let $X = \text{Spec}(R)$ be an affine scheme. Let $\mathcal{F} = \widetilde{M}$ for some R-module M. The quasi-coherent sheaf $\mathcal{F}$ is a (finite) locally free $\mathcal{O}_X$-module of if and only if M is a (finite) locally free R-module.*

Proof. Follows from the definitions, see Modules, Definition 14.1 and Algebra, Definition 73.1. □

We can characterize finite locally free modules in many different ways.

Lemma 18.2. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent:*

(1) *$\mathcal{F}$ is a flat $\mathcal{O}_X$-module of finite presentation,*
(2) *$\mathcal{F}$ is $\mathcal{O}_X$-module of finite presentation and for all $x \in X$ the stalk $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module,*
(3) *$\mathcal{F}$ is a locally free, finite type $\mathcal{O}_X$-module,*
(4) *$\mathcal{F}$ is a finite locally free $\mathcal{O}_X$-module, and*
(5) *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, for every $x \in X$ the the stalk $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module, and the function*

$$\rho_{\mathcal{F}} : X \to \mathbf{Z}, \quad x \longmapsto \dim_{\kappa(x)} \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$

is locally constant in the Zariski topology on X.

Proof. This lemma immediately reduces to the affine case. In this case the lemma is a reformulation of Algebra, Lemma 73.2. The translation uses Lemmas 16.1, 16.2, 17.1, and 18.1. □

19. Locally projective modules

A consequence of the work done in the algebra chapter is that it makes sense to define a locally projective module as follows.

Definition 19.1. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say $\mathcal{F}$ is *locally projective* if for every affine open $U \subset X$ the $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ is projective.

Lemma 19.2. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent*

(1) *$\mathcal{F}$ is locally projective, and*
(2) *there exists an affine open covering $X = \bigcup U_i$ such that the $\mathcal{O}_X(U_i)$-module $\mathcal{F}(U_i)$ is projective for every i.*

In particular, if $X = \text{Spec}(A)$ and $\mathcal{F} = \widetilde{M}$ then $\mathcal{F}$ is locally projective if and only if M is a projective A-module.

Proof. First, note that if M is a projective A-module and $A \to B$ is a ring map, then $M \otimes_A B$ is a projective B-module, see Algebra, Lemma 89.1. Hence if U is an affine open such that $\mathcal{F}(U)$ is a projective $\mathcal{O}_X(U)$-module, then the standard open $D(f)$ is an affine open such that $\mathcal{F}(D(f))$ is a projective $\mathcal{O}_X(D(f))$-module for all $f \in \mathcal{O}_X(U)$. Assume (2) holds. Let $U \subset X$ be an arbitrary affine open. We can find an open covering $U = \bigcup_{j=1,\ldots,m} D(f_j)$ by finitely many standard opens $D(f_j)$ such that for each j the open $D(f_j)$ is a standard open of some U_i, see Schemes, Lemma 11.5. Hence, if we set $A = \mathcal{O}_X(U)$ and if M is an A-module such that $\mathcal{F}|_U$ corresponds to M, then we see that M_{f_j} is a projective A_{f_j}-module. It follows that $A \to B = \prod A_{f_j}$ is a faithfully flat ring map such that $M \times_A B$ is a projective B-module. Hence M is projective by Algebra, Theorem 90.5. □

Lemma 19.3. *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. If $\mathcal{G}$ is locally projective on Y, then $f^*\mathcal{G}$ is locally projective on X.*

Proof. See Algebra, Lemma 89.1. □

20. Extending quasi-coherent sheaves

It is sometimes useful to be able to show that a given quasi-coherent sheaf on an open subscheme extends to the whole scheme.

Lemma 20.1. *Let $j : U \to X$ be a quasi-compact open immersion of schemes.*

1. *Any quasi-coherent sheaf on U extends to a quasi-coherent sheaf on X.*
2. *Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\mathcal{G} \subset \mathcal{F}|_U$ be a quasi-coherent subsheaf. There exists a quasi-coherent subsheaf $\mathcal{H}$ of $\mathcal{F}$ such that $\mathcal{H}|_U = \mathcal{G}$ as subsheaves of $\mathcal{F}|_U$.*
3. *Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\mathcal{G}$ be a quasi-coherent sheaf on U. Let $\varphi : \mathcal{G} \to \mathcal{F}|_U$ be a morphism of $\mathcal{O}_U$-modules. There exists a quasi-coherent sheaf $\mathcal{H}$ of $\mathcal{O}_X$-modules and a map $\psi : \mathcal{H} \to \mathcal{F}$ such that $\mathcal{H}|_U = \mathcal{G}$ and that $\psi|_U = \varphi$.*

Proof. An immersion is separated (see Schemes, Lemma 23.7) and j is quasi-compact by assumption. Hence for any quasi-coherent sheaf $\mathcal{G}$ on U the sheaf $j_*\mathcal{G}$ is an extension to X. See Schemes, Lemma 24.1 and Sheaves, Section 31.

Assume $\mathcal{F}$, $\mathcal{G}$ are as in (2). Then $j_*\mathcal{G}$ is a quasi-coherent sheaf on X (see above). It is a subsheaf of $j_*j^*\mathcal{F}$. Hence the kernel

$$\mathcal{H} = \ker(\mathcal{F} \oplus j_*\mathcal{G} \longrightarrow j_*j^*\mathcal{F})$$

is quasi-coherent as well, see Schemes, Section 24. It is formal to check that $\mathcal{H} \subset \mathcal{F}$ and that $\mathcal{H}|_U = \mathcal{G}$ (using the material in Sheaves, Section 31 again).

The same proof as above works. Just take $\mathcal{H} = \ker(\mathcal{F} \oplus j_*\mathcal{G} \to j_*j^*\mathcal{F})$ with its obvious map to $\mathcal{F}$ and its obvious identification with $\mathcal{G}$ over U. $\square$

Lemma 20.2. *Let X be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G} \subset \mathcal{F}|_U$ be a quasi-coherent $\mathcal{O}_U$-submodule which is of finite type. Then there exists a quasi-coherent submodule $\mathcal{G}' \subset \mathcal{F}$ which is of finite type such that $\mathcal{G}'|_U = \mathcal{G}$.*

Proof. Let n be the minimal number of affine opens $U_i \subset X$, $i = 1, \ldots, n$ such that $X = U \cup \bigcup U_i$. (Here we use that X is quasi-compact.) Suppose we can prove the lemma for the case $n = 1$. Then we can successively extend $\mathcal{G}$ to a $\mathcal{G}_1$ over $U \cup U_1$ to a $\mathcal{G}_2$ over $U \cup U_1 \cup U_2$ to a $\mathcal{G}_3$ over $U \cup U_1 \cup U_2 \cup U_3$, and so on. Thus we reduce to the case $n = 1$.

Thus we may assume that $X = U \cup V$ with V affine. Since X is quasi-separated and U, V are quasi-compact open, we see that $U \cap V$ is a quasi-compact open. It suffices to prove the lemma for the system $(V, U \cap V, \mathcal{F}|_V, \mathcal{G}|_{U \cap V})$ since we can glue the resulting sheaf $\mathcal{G}'$ over V to the given sheaf $\mathcal{G}$ over U along the common value over $U \cap V$. Thus we reduce to the case where X is affine.

Assume $X = \text{Spec}(R)$. Write $\mathcal{F} = \widetilde{M}$ for some R-module M. By Lemma 20.1 above we may find a quasi-coherent subsheaf $\mathcal{H} \subset \mathcal{F}$ which restricts to $\mathcal{G}$ over U. Write $\mathcal{H} = \widetilde{N}$ for some R-module N. For every $u \in U$ there exists an $f \in R$ such that $u \in D(f) \subset U$ and such that N_f is finitely generated, see Lemma 16.1. Since U is quasi-compact we can cover it by finitely many $D(f_i)$ such that N_{f_i} is generated by finitely many elements, say $x_{i,1}/f_i^N, \ldots, x_{i,r_i}/f_i^N$. Let $N' \subset N$ be the submodule generated by the elements $x_{i,j}$. Then the subsheaf $\mathcal{G} := \widetilde{N'} \subset \mathcal{H} \subset \mathcal{F}$ works. $\square$

Lemma 20.3. *Let X be a quasi-compact and quasi-separated scheme. Any quasi-coherent sheaf of $\mathcal{O}_X$-modules is the directed colimit of its quasi-coherent $\mathcal{O}_X$-submodules which are of finite type.*

Proof. The colimit is direct because if $\mathcal{G}_1$, $\mathcal{G}_2$ are quasi-coherent subsheaves of finite type, then $\mathcal{G}_1 + \mathcal{G}_2 \subset \mathcal{F}$ is a quasi-coherent subsheaf of finite type. Let $U \subset X$ be any affine open, and let $s \in \Gamma(U, \mathcal{F})$ be any section. Let $\mathcal{G} \subset \mathcal{F}|_U$ be the subsheaf generated by s. Then clearly $\mathcal{G}$ is quasi-coherent and has finite type as an $\mathcal{O}_U$-module. By Lemma 20.2 we see that $\mathcal{G}$ is the restriction of a quasi-coherent subsheaf $\mathcal{G}' \subset \mathcal{F}$ which has finite type. Since X has a basis for the topology consisting of affine opens we conclude that every local section of $\mathcal{F}$ is locally contained in a quasi-coherent submodule of finite type. Thus we win. $\square$

Lemma 20.4. *Let X be a scheme. Assume X is quasi-compact and quasi-separated. For any quasi-compact open $U \subset X$ there exists a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ which is of finite type such that the corresponding closed subscheme $Z \subset X$ has the property $X = U \coprod Z$ (set theoretically).*

Proof. Let $T = X \setminus U$. By Schemes, Lemma 12.4 there exists a unique quasi-coherent sheaf of ideals $\mathcal{J}$ cutting out the reduced induced closed subscheme structure on T. Note that $\mathcal{J}|_U = \mathcal{O}_U$ which is an $\mathcal{O}_U$-modules of finite type. By Lemma 20.2 there exists a quasi-coherent subsheaf $\mathcal{I} \subset \mathcal{J}$ which is of finite type and has the property that $\mathcal{I}|_U = \mathcal{J}|_U$. It is easy to see that $\mathcal{I}$ has the required properties. $\square$

Lemma 20.5. *(Variant of Lemma 20.2 dealing with modules of finite presentation.) Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $U \subset X$ be a quasi-compact open. Let $\mathcal{G}$ be an $\mathcal{O}_U$-module which of finite presentation. Let $\varphi : \mathcal{G} \to \mathcal{F}|_U$ be a morphism of $\mathcal{O}_U$-modules. Then there exists an $\mathcal{O}_X$-module $\mathcal{G}'$ of finite presentation, and a morphism of $\mathcal{O}_X$-modules $\varphi' : \mathcal{G}' \to \mathcal{F}$ such that $\mathcal{G}'|_U = \mathcal{G}$ and such that $\varphi'|_U = \varphi$.*

Proof. The beginning of the proof is a repeat of the beginning of the proof of Lemma 20.2. We write it out carefully anyway.

Let n be the minimal number of affine opens $U_i \subset X$, $i = 1, \ldots, n$ such that $X = U \cup \bigcup U_i$. (Here we use that X is quasi-compact.) Suppose we can prove the lemma for the case $n = 1$. Then we can successively extend the pair $(\mathcal{G}, \varphi)$ to a pair $(\mathcal{G}_1, \varphi_1)$ over $U \cup U_1$ to a pair $(\mathcal{G}_2, \varphi_2)$ over $U \cup U_1 \cup U_2$ to a pair $(\mathcal{G}_3, \varphi_3)$ over $U \cup U_1 \cup U_2 \cup U_3$, and so on. Thus we reduce to the case $n = 1$.

Thus we may assume that $X = U \cup V$ with V affine. Since X is quasi-separated and U quasi-compact, we see that $U \cap V \subset V$ is quasi-compact. Suppose we prove the lemma for the system $(V, U \cap V, \mathcal{F}|_V, \mathcal{G}|_{U\cap V}, \varphi|_{U\cap V})$ thereby producing $(\mathcal{G}', \varphi')$ over V. Then we can glue $\mathcal{G}'$ over V to the given sheaf $\mathcal{G}$ over U along the common value over $U \cap V$, and similarly we can glue the map φ' to the map φ along the common value over $U \cap V$. Thus we reduce to the case where X is affine.

Assume $X = \text{Spec}(R)$. By Lemma 20.1 above we may find a quasi-coherent sheaf $\mathcal{H}$ with a map $\psi : \mathcal{H} \to \mathcal{F}$ over X which restricts to $\mathcal{G}$ and φ over U. By Lemma 20.2 we can find a finite type quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{H}' \subset \mathcal{H}$ such that $\mathcal{H}'|_U = \mathcal{G}$. Thus after replacing $\mathcal{H}$ by $\mathcal{H}'$ and ψ by the restriction of ψ to $\mathcal{H}'$ we may assume that $\mathcal{H}$ is of finite type. By Lemma 16.2 we conclude that $\mathcal{H} = \widetilde{N}$

with N a finitely generated R-module. Hence there exists a surjection as in the following short exact sequence of quasi-coherent $\mathcal{O}_X$-modules

$$0 \to \mathcal{K} \to \mathcal{O}_X^{\oplus n} \to \mathcal{H} \to 0$$

where $\mathcal{K}$ is defined as the kernel. Since $\mathcal{G}$ is of finite presentation and $\mathcal{H}|_U = \mathcal{G}$ by Modules, Lemma 11.3 the restriction $\mathcal{K}|_U$ is an $\mathcal{O}_U$-module of finite type. Hence by Lemma 20.2 again we see that there exists a finite type quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{K}' \subset \mathcal{K}$ such that $\mathcal{K}'|_U = \mathcal{K}|_U$. The solution to the problem posed in the lemma is to set

$$\mathcal{G}' = \mathcal{O}_X^{\oplus n}/\mathcal{K}'$$

which is clearly of finite presentation and restricts to give $\mathcal{G}$ on U with φ' equal to the composition

$$\mathcal{G}' = \mathcal{O}_X^{\oplus n}/\mathcal{K}' \to \mathcal{O}_X^{\oplus n}/\mathcal{K} = \mathcal{H} \xrightarrow{\psi} \mathcal{F}.$$

This finishes the proof of the lemma. □

The following lemma says that every quasi-coherent sheaf on a quasi-compact and quasi-separated scheme is a filtered colimit of $\mathcal{O}$-modules of finite presentation. Actually, we reformulate this in (perhaps more familiar) terms of directed colimits over posets in the next lemma.

Lemma 20.6. *Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. There exist*

1. *a filtered index category $\mathcal{I}$ (see Categories, Definition 17.1),*
2. *a diagram $\mathcal{I} \to Mod(\mathcal{O}_X)$ (see Categories, Section 13), $i \mapsto \mathcal{F}_i$,*
3. *morphisms of $\mathcal{O}_X$-modules $\varphi_i : \mathcal{F}_i \to \mathcal{F}$*

such that each $\mathcal{F}_i$ is of finite presentation and such that the morphisms φ_i induce an isomorphism

$$\operatorname{colim}_i \mathcal{F}_i = \mathcal{F}.$$

Proof. Choose a set I and for each $i \in I$ an $\mathcal{O}_X$-module of finite presentation and a homomorphism of $\mathcal{O}_X$-modules $\varphi_i : \mathcal{F}_i \to \mathcal{F}$ with the following property: For any $\psi : \mathcal{G} \to \mathcal{F}$ with $\mathcal{G}$ of finite presentation there is an $i \in I$ such that there exists an isomorphism $\alpha : \mathcal{F}_i \to \mathcal{G}$ with $\varphi_i = \psi \circ \alpha$. It is clear from Modules, Lemma 9.8 that such a set exists (see also its proof). We denote $\mathcal{I}$ the category with $\text{Ob}(\mathcal{I}) = I$ and given $i, i' \in I$ we set

$$\text{Mor}_{\mathcal{I}}(i, i') = \{\alpha : \mathcal{F}_i \to \mathcal{F}_{i'} \mid \alpha \circ \varphi_{i'} = \varphi_i\}.$$

We claim that $\mathcal{I}$ filtered category and that $\mathcal{F} = \operatorname{colim}_i \mathcal{F}_i$.

Let $i, i' \in I$. Then we can consider the morphism

$$\mathcal{F}_i \oplus \mathcal{F}_{i'} \longrightarrow \mathcal{F}$$

which is the direct sum of φ_i and $\varphi_{i'}$. Since a direct sum of finitely presented $\mathcal{O}_X$-modules is finitely presented we see that there exists some $i'' \in I$ such that $\varphi_{i''} : \mathcal{F}_{i''} \to \mathcal{F}$ is isomorphic to the displayed arrow towards $\mathcal{F}$ above. Since there are commutative diagrams

$$\begin{array}{ccc} \mathcal{F}_i & \longrightarrow & \mathcal{F} \\ \downarrow & & \| \\ \mathcal{F}_i \oplus \mathcal{F}_{i'} & \longrightarrow & \mathcal{F} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{F}_{i'} & \longrightarrow & \mathcal{F} \\ \downarrow & & \| \\ \mathcal{F}_i \oplus \mathcal{F}_{i'} & \longrightarrow & \mathcal{F} \end{array}$$

we see that there are morphisms $i \to i''$ and $i' \to i''$ in $\mathcal{I}$. Next, suppose that we have $i, i' \in I$ and morphisms $\alpha, \beta : i \to i'$ (corresponding to $\mathcal{O}_X$-module maps $\alpha, \beta : \mathcal{F}_i \to \mathcal{F}_{i'}$). In this case consider the coequalizer

$$\mathcal{G} = \text{Coker}(\mathcal{F}_i \xrightarrow{\alpha - \beta} \mathcal{F}_{i'})$$

Note that $\mathcal{G}$ is an $\mathcal{O}_X$-module of finite presentation. Since by definition of morphisms in the category $\mathcal{I}$ we have $\varphi_{i'} \circ \alpha = \varphi_{i'} \circ \beta$ we see that we get an induced map $\psi : \mathcal{G} \to \mathcal{F}$. Hence again the pair $(\mathcal{G}, \psi)$ is isomorphic to the pair $(\mathcal{F}_{i''}, \varphi_{i''})$ for some i''. Hence we see that there exists a morphism $i' \to i''$ in $\mathcal{I}$ which equalizes α and β. Thus we have shown that the category $\mathcal{I}$ is filtered.

We still have to show that the colimit of the diagram is $\mathcal{F}$. By definition of the colimit, and by our definition of the category $\mathcal{I}$ there is a canonical map

$$\varphi : \text{colim}_i \mathcal{F}_i \longrightarrow \mathcal{F}.$$

Pick $x \in X$. Let us show that φ_x is an isomorphism. Recall that

$$(\text{colim}_i \mathcal{F}_i)_x = \text{colim}_i \mathcal{F}_{i,x},$$

see Sheaves, Section 29. First we show that the map φ_x is injective. Suppose that $s \in \mathcal{F}_{i,x}$ is an element such that s maps to zero in $\mathcal{F}_x$. Then there exists a quasi-compact open U such that s comes from $s \in \mathcal{F}_i(U)$ and such that $\varphi_i(s) = 0$ in $\mathcal{F}(U)$. By Lemma 20.2 we can find a finite type quasi-coherent subsheaf $\mathcal{K} \subset \text{Ker}(\varphi_i)$ which restricts to the quasi-coherent $\mathcal{O}_U$-submodule of $\mathcal{F}_i$ generated by s: $\mathcal{K}|_U = \mathcal{O}_U \cdot s \subset \mathcal{F}_i|_U$. Clearly, $\mathcal{F}_i/\mathcal{K}$ is of finite presentation and the map φ_i factors through the quotient map $\mathcal{F}_i \to \mathcal{F}_i/\mathcal{K}$. Hence we can find an $i' \in I$ and a morphism $\alpha : \mathcal{F}_i \to \mathcal{F}_{i'}$ in $\mathcal{I}$ which can be identified with the quotient map $\mathcal{F}_i \to \mathcal{F}_i/\mathcal{K}$. Then it follows that the section s maps to zero in $\mathcal{F}_{i'}(U)$ and in particular in $(\text{colim}_i \mathcal{F}_i)_x = \text{colim}_i \mathcal{F}_{i,x}$. The injectivity follows. Finally, we show that the map φ_x is surjective. Pick $s \in \mathcal{F}_x$. Choose a quasi-compact open neighbourhood $U \subset X$ of x such that s corresponds to a section $s \in \mathcal{F}(U)$. Consider the map $s : \mathcal{O}_U \to \mathcal{F}$ (multiplication by s). By Lemma 20.5 there exists an $\mathcal{O}_X$-module $\mathcal{G}$ of finite presentation and an $\mathcal{O}_X$-module map $\mathcal{G} \to \mathcal{F}$ such that $\mathcal{G}|_U \to \mathcal{F}|_U$ is identified with $s : \mathcal{O}_U \to \mathcal{F}$. Again by definition of $\mathcal{I}$ there exists an $i \in I$ such that $\mathcal{G} \to \mathcal{F}$ is isomorphic to $\varphi_i : \mathcal{F}_i \to \mathcal{F}$. Clearly there exists a section $s' \in \mathcal{F}_i(U)$ mapping to $s \in \mathcal{F}(U)$. This proves surjectivity and the proof of the lemma is complete. □

Lemma 20.7. *Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. There exist*

1. *a directed partially ordered set I (see Categories, Definition 19.2),*
2. *a system $(\mathcal{F}_i, \varphi_{ii'})$ over I in $Mod(\mathcal{O}_X)$ (see Categories, Definition 19.1)*
3. *morphisms of $\mathcal{O}_X$-modules $\varphi_i : \mathcal{F}_i \to \mathcal{F}$*

such that each $\mathcal{F}_i$ is of finite presentation and such that the morphisms φ_i induce an isomorphism

$$\text{colim}_i \mathcal{F}_i = \mathcal{F}.$$

Proof. This is a direct consequence of Lemma 20.6 and Categories, Lemma 19.3 (combined with the fact that colimits exist in the category of sheaves of $\mathcal{O}_X$-modules, see Sheaves, Section 29). □

Lemma 20.8. *Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is the directed colimit of its finite type quasi-coherent submodules.*

Proof. If $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ are finite type quasi-coherent $\mathcal{O}_X$-submodules then the image of $\mathcal{G} \oplus \mathcal{H} \to \mathcal{F}$ is another finite type quasi-coherent $\mathcal{O}_X$-submodule which contains both of them. In this way we see that the system is directed. To show that $\mathcal{F}$ is the colimit of this system, write $\mathcal{F} = \text{colim}_i \mathcal{F}_i$ as a directed colimit of finitely presented quasi-coherent sheaves as in Lemma 20.7. Then the images $\mathcal{G}_i = \text{Im}(\mathcal{F}_i \to \mathcal{F})$ are finite type quasi-coherent subsheaves of $\mathcal{F}$. Since $\mathcal{F}$ is the colimit of these the result follows. □

Let X be a scheme. In the following lemma we use the notion of a *quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ of finite presentation*. This means that for every affine open $\text{Spec}(R) \subset X$ we have $\mathcal{A} = \widetilde{A}$ where A is a (commutative) R-algebra which is of finite presentation as an R-algebra.

Lemma 20.9. *Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. There exist*

(1) *a directed partially ordered set I (see Categories, Definition 19.2),*
(2) *a system $(\mathcal{A}_i, \varphi_{ii'})$ over I in the category of $\mathcal{O}_X$-algebras,*
(3) *morphisms of $\mathcal{O}_X$-algebras $\varphi_i : \mathcal{A}_i \to \mathcal{A}$*

such that each $\mathcal{A}_i$ is a quasi-coherent $\mathcal{O}_X$-algebra of finite presentation and such that the morphisms φ_i induce an isomorphism

$$\text{colim}_i \mathcal{A}_i = \mathcal{A}.$$

Proof. First we write $\mathcal{A} = \text{colim}_i \mathcal{F}_i$ as a directed colimit of finitely presented quasi-coherent sheaves as in Lemma 20.7. For each i let $\mathcal{B}_i = \text{Sym}(\mathcal{F}_i)$ be the symmetric algebra on $\mathcal{F}_i$ over $\mathcal{O}_X$. Write $\text{I}_i = \ker(\mathcal{B}_i \to \mathcal{A})$. Write $\mathcal{I}_i = \text{colim}_j \mathcal{F}_{i,j}$ where $\mathcal{F}_{i,j}$ is a finite type quasi-coherent submodule of $\mathcal{I}_i$, see Lemma 20.8. Set $\mathcal{I}_{i,j} \subset \mathcal{I}_i$ equal to the $\mathcal{B}_i$-ideal generated by $\mathcal{F}_{i,j}$. Set $\mathcal{A}_{i,j} = \mathcal{B}_i/\mathcal{I}_{i,j}$. Then $\mathcal{A}_{i,j}$ is a quasi-coherent finitely presented $\mathcal{O}_X$-algebra. Define $(i,j) \leq (i',j')$ if $i \leq i'$ and the map $\mathcal{B}_i \to \mathcal{B}_{i'}$ maps the ideal $\mathcal{I}_{i,j}$ into the ideal $\mathcal{I}_{i',j'}$. Then it is clear that $\mathcal{A} = \text{colim}_{i,j} \mathcal{A}_{i,j}$. □

Let X be a scheme. In the following lemma we use the notion of a *quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ of finite type*. This means that for every affine open $\text{Spec}(R) \subset X$ we have $\mathcal{A} = \widetilde{A}$ where A is a (commutative) R-algebra which is of finite type as an R-algebra.

Lemma 20.10. *Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. Then $\mathcal{A}$ is the directed colimit of its finite type quasi-coherent $\mathcal{O}_X$-subalgebras.*

Proof. Omitted. Hint: Compare with the proof of Lemma 20.8. □

21. Gabber's result

In this section we prove a result of Gabber which guarantees that on every scheme there exists a cardinal κ such that every quasi-coherent module $\mathcal{F}$ is the union of its quasi-coherent κ-generated subsheaves. It follows that the category of quasi-coherent sheaves on a scheme is a Grothendieck abelian category having limits and enough injectives[2].

Definition 21.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let κ be an infinite cardinal. We say a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is *κ-generated* if there exists an open covering $X = \bigcup U_i$ such that $\mathcal{F}|_{U_i}$ is generated by a subset $R_i \subset \mathcal{F}(U_i)$ whose cardinality is at most κ.

Note that a direct sum of at most κ κ-generated modules is again κ-generated because $\kappa \otimes \kappa = \kappa$, see Sets, Section 6. In particular this holds for the direct sum of two κ-generated modules. Moreover, a quotient of a κ-generated sheaf is κ-generated. (But the same needn't be true for submodules.)

Lemma 21.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let κ be a cardinal. There exists a set T and a family $(\mathcal{F}_t)_{t \in T}$ of κ-generated $\mathcal{O}_X$-modules such that every κ-generated $\mathcal{O}_X$-module is isomorphic to one of the $\mathcal{F}_t$.*

Proof. There is a set of coverings of X (provided we disallow repeats). Suppose $X = \bigcup U_i$ is a covering and suppose $\mathcal{F}_i$ is an $\mathcal{O}_{U_i}$-module. Then there is a set of isomorphism classes of $\mathcal{O}_X$-modules $\mathcal{F}$ with the property that $\mathcal{F}|_{U_i} \cong \mathcal{F}_i$ since there is a set of glueing maps. This reduces us to proving there is a set of (isomorphism classes of) quotients $\oplus_{k \in \kappa} \mathcal{O}_X \to \mathcal{F}$ for any ringed space X. This is clear. $\square$

Here is the result the title of this section refers to.

Lemma 21.3. *Let X be a scheme. There exists a cardinal κ such that every quasi-coherent module $\mathcal{F}$ is the directed colimit of its quasi-coherent κ-generated quasi-coherent subsheaves.*

Proof. Choose an affine open covering $X = \bigcup_{i \in I} U_i$. For each pair i, j choose an affine open covering $U_i \cap U_j = \bigcup_{k \in I_{ij}} U_{ijk}$. Write $U_i = \mathrm{Spec}(A_i)$ and $U_{ijk} = \mathrm{Spec}(A_{ijk})$. Let κ be any infinite cardinal $\geq$ than the cardinality of any of the sets I, I_{ij}.

Let $\mathcal{F}$ be a quasi-coherent sheaf. Set $M_i = \mathcal{F}(U_i)$ and $M_{ijk} = \mathcal{F}(U_{ijk})$. Note that

$$M_i \otimes_{A_i} A_{ijk} = M_{ijk} = M_j \otimes_{A_j} A_{ijk}.$$

see Schemes, Lemma 7.3. Using the axiom of choice we choose a map

$$(i, j, k, m) \mapsto S(i, j, k, m)$$

which associates to every $i, j \in I$, $k \in I_{ij}$ and $m \in M_i$ a finite subset $S(i, j, k, m) \subset M_j$ such that we have

$$m \otimes 1 = \sum\nolimits_{m' \in S(i,j,k,m)} m' \otimes a_{m'}$$

in M_{ijk} for some $a_{m'} \in A_{ijk}$. Moreover, let's agree that $S(i, i, k, m) = \{m\}$ for all $i, j = i, k, m$ as above. Fix such a map.

Given a family $\mathcal{S} = (S_i)_{i \in I}$ of subsets $S_i \subset M_i$ of cardinality at most κ we set $\mathcal{S}' = (S'_i)$ where

$$S'_j = \bigcup\nolimits_{(i,j,k,m) \text{ such that } m \in S_i} S(i, j, k, m)$$

[2] Nicely explained in a blog post by Akhil Mathew.

Note that $S_i \subset S'_i$. Note that S'_i has cardinality at most κ because it is a union over a set of cardinality at most κ of finite sets. Set $\mathcal{S}^{(0)} = \mathcal{S}$, $\mathcal{S}^{(1)} = \mathcal{S}'$ and by induction $\mathcal{S}^{(n+1)} = (\mathcal{S}^{(n)})'$. Then set $\mathcal{S}^{(\infty)} = \bigcup_{n \geq 0} \mathcal{S}^{(n)}$. Writing $\mathcal{S}^{(\infty)} = (S_i^{(\infty)})$ we see that for any element $m \in S_i^{(\infty)}$ the image of m in M_{ijk} can be written as a finite sum $\sum m' \otimes a_{m'}$ with $m' \in S_j^{(\infty)}$. In this way we see that setting

$$N_i = A_i\text{-submodule of } M_i \text{ generated by } S_i^{(\infty)}$$

we have

$$N_i \otimes_{A_i} A_{ijk} = N_j \otimes_{A_j} A_{ijk}.$$

as submodules of M_{ijk}. Thus there exists a quasi-coherent subsheaf $\mathcal{G} \subset \mathcal{F}$ with $\mathcal{G}(U_i) = N_i$. Moreover, by construction the sheaf $\mathcal{G}$ is κ-generated.

Let $\{\mathcal{G}_t\}_{t \in T}$ be the set of κ-generated quasi-coherent subsheaves. If $t, t' \in T$ then $\mathcal{G}_t + \mathcal{G}_{t'}$ is also a κ-generated quasi-coherent subsheaf as it is the image of the map $\mathcal{G}_t \oplus \mathcal{G}_{t'} \to \mathcal{F}$. Hence the system (ordered by inclusion) is directed. The arguments above show that every section of $\mathcal{F}$ over U_i is in one of the $\mathcal{G}_t$ (because we can start with $\mathcal{S}$ such that the given section is an element of S_i). Hence $\text{colim}_t \mathcal{G}_t \to \mathcal{F}$ is both injective and surjective as desired. □

Proposition 21.4. *Let X be a scheme. The inclusion functor $QCoh(\mathcal{O}_X) \to Mod(\mathcal{O}_X)$ has a right adjoint*

$$Q^3 : Mod(\mathcal{O}_X) \longrightarrow QCoh(\mathcal{O}_X)$$

such that for every quasi-coherent sheaf $\mathcal{F}$ the adjunction mapping $Q(\mathcal{F}) \to \mathcal{F}$ is an isomorphism. Moreover, the category $QCoh(\mathcal{O}_X)$ has limits and enough injectives.

Proof. The two assertions about $Q(\mathcal{F}) \to \mathcal{F}$ and limits in $QCoh(\mathcal{O}_X)$ are formal consequences of the existence of Q, the fact that the inclusion is fully faithful, and the fact that $Mod(\mathcal{O}_X)$ has limits (see Modules, Section 3). The existence of injectives follows from the existence of injectives in $Mod(\mathcal{O}_X)$ (see Injectives, Lemma 9.1) and Homology, Lemma 22.3. Thus it suffices to construct Q.

Pick a cardinal κ as in Lemma 21.3. Pick a collection $(\mathcal{F}_t)_{t \in T}$ of κ-generated quasi-coherent sheaves as in Lemma 21.2. Given an object $\mathcal{G}$ of $QCoh(\mathcal{O}_X)$ we set

$$Q(\mathcal{G}) = \text{colim}_{(t,\alpha)} \mathcal{F}_t$$

The colimit is over the category of pairs (t, α) where $t \in T$ and $\alpha : \mathcal{F}_t \to \mathcal{G}$ is a morphism of $\mathcal{O}_X$-modules. A morphism $(t, \alpha) \to (t', \alpha')$ is given by a morphism $\beta : \mathcal{F}_t \to \mathcal{F}_{t'}$ such that $\alpha' \circ \beta = \alpha$. By Schemes, Section 24 the colimit is quasi-coherent. Note that there is a canonical map $Q(\mathcal{G}) \to \mathcal{G}$ by definition of the colimit. The formula

$$\text{Hom}(\mathcal{H}, Q(\mathcal{G})) = \text{Hom}(\mathcal{H}, \mathcal{G})$$

holds for κ-generated quasi-coherent modules $\mathcal{H}$ by choice of the system $(\mathcal{F}_t)_{t \in T}$. It follows formally from Lemma 21.3 that this equality continuous to hold for any quasi-coherent module $\mathcal{H}$ on X. This finishes the proof. □

[3] This functor is sometimes called the *coherator*.

22. Sections of quasi-coherent sheaves

Here is a computation of sections of a quasi-coherent sheaf on a quasi-compact open of an affine spectrum.

Lemma 22.1. *Let A be a ring. Let $I \subset A$ be a finitely generated ideal. Let M be an A-module. Then there is a canonical map*

$$\mathrm{colim}_n \mathrm{Hom}_A(I^n, M) \longrightarrow \Gamma(\mathrm{Spec}(A) \setminus V(I), \widetilde{M}).$$

This map is always injective. If for all $x \in M$ we have $Ix = 0 \Rightarrow x = 0$ then this map is an isomorphism. In general, set $M_n = \{x \in M \mid I^n x = 0\}$, then there is an isomorphism

$$\mathrm{colim}_n \mathrm{Hom}_A(I^n, M/M_n) \longrightarrow \Gamma(\mathrm{Spec}(A) \setminus V(I), \widetilde{M}).$$

Proof. Since $I^n \subset I^{n+1}$ and $M_n \subset M_{n+1}$ we can use composition via these maps to get canonical maps of A-modules

$$\mathrm{Hom}_A(I^n, M) \longrightarrow \mathrm{Hom}_A(I^{n+1}, M)$$

and

$$\mathrm{Hom}_A(I^n, M/M_n) \longrightarrow \mathrm{Hom}_A(I^{n+1}, M/M_{n+1})$$

which we will use as the transition maps in the systems. Given an A-module map $\varphi : I^n \to M$, then we get a map of sheaves $\widetilde{\varphi} : \widetilde{I} \to \widetilde{M}$ which we can restrict to the open $\mathrm{Spec}(A) \setminus V(I)$. Since $\widetilde{I}$ restricted to this open gives the structure sheaf we get an element of $\Gamma(\mathrm{Spec}(A) \setminus V(I), \widetilde{M})$. We omit the verification that this is compatible with the transition maps in the system $\mathrm{Hom}_A(I^n, M)$. This gives the first arrow. To get the second arrow we note that $\widetilde{M}$ and $\widetilde{M/M_n}$ agree over the open $\mathrm{Spec}(A) \setminus V(I)$ since the sheaf $\widetilde{M_n}$ is clearly supported on $V(I)$. Hence we can use the same mechanism as before.

Next, we work out how to define this arrow in terms of algebra. Say $I = (f_1, \ldots, f_t)$. Then $\mathrm{Spec}(A) \setminus V(I) = \bigcup_{i=1,\ldots,t} D(f_i)$. Hence

$$0 \to \Gamma(\mathrm{Spec}(A) \setminus V(I), \widetilde{M}) \to \bigoplus\nolimits_i M_{f_i} \to \bigoplus\nolimits_{i,j} M_{f_i f_j}$$

is exact. Suppose that $\varphi : I^n \to M$ is an A-module map. Consider the vector of elements $\varphi(f_i^n)/f_i^n \in M_{f_i}$. It is easy to see that this vector maps to zero in the second direct sum of the exact sequence above. Whence an element of $\Gamma(\mathrm{Spec}(A) \setminus V(I), \widetilde{M})$. We omit the verification that this description agrees with the one given above.

Let us show that the first arrow is injective using this description. Namely, if φ maps to zero, then for each i the element $\varphi(f_i^n)/f_i^n$ is zero in M_{f_i}. In other words we see that for each i we have $f_i^m \varphi(f_i^n) = 0$ for some $m \geq 0$. We may choose a single m which works for all i. Then we see that $\varphi(f_i^{n+m}) = 0$ for all i. It is easy to see that this means that $\varphi|_{I^{t(n+m-1)+1}} = 0$ in other words that φ maps to zero in the $t(n+m-1)+1$st term of the colimit. Hence injectivity follows.

Note that each $M_n = 0$ in case we have $Ix = 0 \Rightarrow x = 0$ for $x \in M$. Thus to finish the proof of the lemma it suffices to show that the second arrow is an isomorphism.

Let us attempt to construct an inverse of the second map of the lemma. Let $s \in \Gamma(\mathrm{Spec}(A) \setminus V(I), \widetilde{M})$. This corresponds to a vector x_i/f_i^n with $x_i \in M$ of the first direct sum of the exact sequence above. Hence for each i, j there exists $m \geq 0$ such that $f_i^m f_j^m (f_j^n x_i - f_i^n x_j) = 0$ in M. We may choose a single m which works for all pairs i, j. After replacing x_i by $f_i^m x_i$ and n by $n + m$ we see that we get $f_j^n x_i = f_i^n x_j$ in M for all i, j. Let us introduce

$$K_n = \{x \in M \mid f_1^n x = \ldots = f_t^n x = 0\}$$

We claim there is an A-module map

$$\varphi : I^{t(n-1)+1} \longrightarrow M/K_n$$

which maps the monomial $f_1^{e_1} \ldots f_t^{e_t}$ with $\sum e_i = t(n-1)+1$ to the class modulo K_n of the expression $f_1^{e_1} \ldots f_i^{e_i - n} \ldots f_t^{e_t} x_i$ where i is chosen such that $e_i \geq n$ (note that there is at least one such i). To see that this is indeed the case suppose that

$$\sum\nolimits_{E=(e_1,\ldots,e_t), |E| = t(n-1)+1} a_E f_1^{e_1} \ldots f_t^{e_t} = 0$$

is a relation between the monomials with coefficients a_E in A. Then we would map this to

$$z = \sum\nolimits_{E=(e_1,\ldots,e_t), |E| = t(n-1)+1} a_E f_1^{e_1} \ldots f_{i(E)}^{e_{i(E)} - n} \ldots f_t^{e_t} x_i$$

where for each multiindex E we have chosen a particular $i(E)$ such that $e_{i(E)} \geq n$. Note that if we multiply this by f_j^n for any j, then we get zero, since by the relations $f_j^n x_i = f_i^n x_j$ above we get

$$\begin{aligned} f_j^n z &= \sum\nolimits_{E=(e_1,\ldots,e_t), |E| = t(n-1)+1} a_E f_1^{e_1} \ldots f_j^{e_j + n} \ldots f_{i(E)}^{e_{i(E)} - n} \ldots f_t^{e_t} x_i \\ &= \sum\nolimits_{E=(e_1,\ldots,e_t), |E| = t(n-1)+1} a_E f_1^{e_1} \ldots f_t^{e_t} x_j = 0. \end{aligned}$$

Hence $z \in K_n$ and we see that every relation gets mapped to zero in M/K_n. This proves the claim.

Note that $K_n \subset M_{t(n-1)+1}$. Hence the map φ in particular gives rise to a A-module map $I^{t(n-1)+1} \to M/M_{t(n-1)+1}$. This proves the second arrow of the lemma is surjective. We omit the proof of injectivity. $\square$

Example 22.2. Let k be a field. Consider the ring

$$A = k[f, g, x, y, \{a_n, b_n\}_{n \geq 1}]/(fy - gx, \{a_n f^n + b_n g^n\}_{n \geq 1}).$$

Then $x/f \in A_f$ and $y/g \in A_g$ map to the same element of A_{fg}. Hence these define a section s of the structure sheaf of $\text{Spec}(A)$ over $D(f) \cup D(g) = \text{Spec}(A) \setminus V(I)$. Here $I = (f, g) \subset A$. However, there is no $n \geq 0$ such that s comes from an A-module map $\varphi : I^n \to A$ as in the source of the first displayed arrow of Lemma 22.1. Namely, given such a module map set $x_n = \varphi(f^n)$ and $y_n = \varphi(g^n)$. Then $f^m x_n = f^{n+m-1} x$ and $g^m y_n = g^{n+m-1} y$ for some $m \geq 0$ (see proof of the lemma). But then we would have $0 = \varphi(0) = \varphi(a_{n+m} f^{n+m} + b_{n+m} g^{n+m}) = a_{n+m} f^{n+m-1} x + b_{n+m} g^{n+m-1} y$ which is not the case in the ring A.

Lemma 22.3. *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Consider the sheaf of $\mathcal{O}_X$-modules $\mathcal{F}'$ which associates to every open $U \subset X$*

$$\mathcal{F}'(U) = \{s \in \mathcal{F}(U) \mid \mathcal{I}s = 0\}$$

Assume $\mathcal{I}$ is of finite type. Then

(1) *$\mathcal{F}'$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules,*

(2) *on any affine open* $U \subset X$ *we have* $\mathcal{F}'(U) = \{s \in \mathcal{F}(U) \mid \mathcal{I}(U)s = 0\}$, *and*
(3) $\mathcal{F}'_x = \{s \in \mathcal{F}_x \mid \mathcal{I}_x s = 0\}$.

Proof. It is clear that the rule defining $\mathcal{F}'$ gives a subsheaf of $\mathcal{F}$ (the sheaf condition is easy to verify). Hence we may work locally on X to verify the other statements. In other words we may assume that $X = \mathrm{Spec}(A)$, $\mathcal{F} = \widetilde{M}$ and $\mathcal{I} = \widetilde{I}$. It is clear that in this case $\mathcal{F}'(U) = \{x \in M \mid Ix = 0\} =: M'$ because $\widetilde{I}$ is generated by its global sections I which proves (2). To show $\mathcal{F}'$ is quasi-coherent it suffices to show that for every $f \in A$ we have $\{x \in M_f \mid I_f x = 0\} = (M')_f$. Write $I = (g_1, \ldots, g_t)$, which is possible because $\mathcal{I}$ is of finite type, see Lemma 16.1. If $x = y/f^n$ and $I_f x = 0$, then that means that for every i there exists an $m \geq 0$ such that $f^m g_i x = 0$. We may choose one m which works for all i (and this is where we use that I is finitely generated). Then we see that $f^m x \in M'$ and $x/f^n = f^m x/f^{n+m}$ in $(M')_f$ as desired. The proof of (3) is similar and omitted. □

Definition 22.4. Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The subsheaf $\mathcal{F}' \subset \mathcal{F}$ defined in Lemma 22.3 above is called the *subsheaf of sections annihilated by* $\mathcal{I}$.

Lemma 22.5. *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $Z \subset X$ be the closed subscheme defined by $\mathcal{I}$ and set $U = X \setminus Z$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume that X is quasi-compact and quasi-separated and that $\mathcal{I}$ is of finite type. Let $\mathcal{F}_n \subset \mathcal{F}$ be subsheaf of sections annihilated by $\mathcal{I}^n$. The canonical map*

$$\mathrm{colim}_n \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{F})$$

is injective and the canonical map

$$\mathrm{colim}_n \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}/\mathcal{F}_n) \longrightarrow \Gamma(U, \mathcal{F})$$

is an isomorphism.

Proof. Let $\mathrm{Spec}(A) = W \subset X$ be an affine open. Write $\mathcal{F}|_W = \widetilde{M}$ for some A-module M and $\mathcal{I}|_W = \widetilde{I}$ for some ideal $I \subset A$. We omit the verification that $\mathcal{F}_n = \widetilde{M_n}$ where $M_n \subset M$ is defined as in Lemma 22.1. This proves (1). It also follows from Lemma 22.1 that we have an injection

$$\mathrm{colim}_n \mathrm{Hom}_{\mathcal{O}_W}(\mathcal{I}^n|_W, \mathcal{F}|_W) \longrightarrow \Gamma(U \cap W, \mathcal{F})$$

and a bijection

$$\mathrm{colim}_n \mathrm{Hom}_{\mathcal{O}_W}(\mathcal{I}^n|_W, (\mathcal{F}/\mathcal{F}_n)|_W) \longrightarrow \Gamma(U \cap W, \mathcal{F})$$

for any such affine open W.

To see (2) we choose a finite affine open covering $X = \bigcup_{j=1,\ldots,m} W_j$. The injectivity of the first arrow of (2) follows immediately from the above and the finiteness of the covering. Moreover for each pair j, j' we choose a finite affine open covering

$$W_j \cap W_{j'} = \bigcup\nolimits_{k=1,\ldots,m_{jj'}} W_{jj'k}.$$

Let $s \in \Gamma(U, \mathcal{F})$. As seen above for each j there exists an n_j and a map $\varphi_j : \mathcal{I}^{n_j}|_{W_j} \to (\mathcal{F}/\mathcal{F}_{n_j})|_{W_j}$ which corresponds to $s|_{W_j}$. By the same token for each triple (j, j', k) there exists an integer $n_{jj'k}$ such that the restriction of φ_j and $\varphi_{j'}$ as maps $\mathcal{I}^{n_{jj'k}} \to \mathcal{F}/\mathcal{F}_{n_{jj'k}}$ agree over $W_{jj'l}$. Let $n = \max\{n_j, n_{jj'k}\}$ and we see that the φ_j glue as maps $\mathcal{I}^n \to \mathcal{F}/\mathcal{F}_n$ over X. This proves surjectivity of the map. We omit the proof of injectivity. ☐

23. Ample invertible sheaves

Recall from Modules, Lemma 21.7 that given an invertible sheaf $\mathcal{L}$ on a locally ringed space X, and given a global section s of $\mathcal{L}$ the set $X_s = \{x \in X \mid s \not\in \mathfrak{m}_x\mathcal{L}_x\}$ is open. A general remark is that $X_s \cap X_{s'} = X_{ss'}$, where ss' denote the section $s \otimes s' \in \Gamma(X, \mathcal{L} \otimes \mathcal{L}')$.

Definition 23.1. Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We say $\mathcal{L}$ is *ample* if

(1) X is quasi-compact, and
(2) for every $x \in X$ there exists an $n \geq 1$ and $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $x \in X_s$ and X_s is affine.

Lemma 23.2. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $n \geq 1$. Then $\mathcal{L}$ is ample if and only if $\mathcal{L}^{\otimes n}$ is ample.*

Proof. This follows from the fact that $X_{s^n} = X_s$. ☐

Lemma 23.3. *Let X be a scheme. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. For any closed subscheme $Z \subset X$ the restriction of $\mathcal{L}$ to Z is ample.*

Proof. This is clear since a closed subset of a quasi-compact space is quasi-compact and a closed subscheme of an affine scheme is affine (see Schemes, Lemma 8.2). ☐

Lemma 23.4. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. For any affine $U \subset X$ the intersection $U \cap X_s$ is affine.*

Proof. This translates into the following algebra problem. Let R be a ring. Let N be an invertible R-module (i.e., locally free of rank 1). Let $s \in N$ be an element. Then $U = \{\mathfrak{p} \mid s \not\in \mathfrak{p}N\}$ is an affine open subset of $\text{Spec}(R)$. This you can see as follows. Think of s as an R-module map $R \to N$. This gives rise to R-module maps $N^{\otimes k} \to N^{\otimes k+1}$. Consider

$$R' = \text{colim}_n N^{\otimes n}$$

with transition maps as above. Define an R-algebra structure on R' by the rule $x \cdot y = x \otimes y \in N^{\otimes n+m}$ if $x \in N^{\otimes n}$ and $y \in N^{\otimes m}$. We claim that $\text{Spec}(R') \to \text{Spec}(R)$ is an open immersion with image U.

To prove this is a local question on $\text{Spec}(R)$. Let $\mathfrak{p} \in \text{Spec}(R)$. Pick $f \in R$, $f \not\in \mathfrak{p}$ such that $N_f \cong R_f$ as a module. Replacing R by R_f, N by N_f and R' by $R'_f = \text{colim}\, N_f^{\otimes n}$ we may assume that $N \cong R$. Say $N = R$. In this case s is an element of R and it is easy to see that $R' \cong R_s$. Thus the lemma follows. ☐

Recall that given a scheme X and an invertible sheaf $\mathcal{L}$ on X we get a graded ring $\Gamma_*(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$, see Modules, Definition 21.4. Also, given a sheaf of $\mathcal{O}_X$-modules we have the graded $\Gamma_*(X, \mathcal{L})$-module $\Gamma_*(X, \mathcal{F}) = \Gamma_*(X, \mathcal{L}, \mathcal{F})$.

Lemma 23.5. *Let X be a scheme. Let $\mathcal{L}$ be an invertible sheaf on X. Let $s \in \Gamma(X, \mathcal{L})$. If X is quasi-compact and quasi-separated, the canonical map*

$$\Gamma_*(X, \mathcal{L})_{(s)} \longrightarrow \Gamma(X_s, \mathcal{O})$$

which maps a/s^n to $a \otimes s^{-n}$ is an isomorphism. Moreover, if $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module then the map

$$\Gamma_*(X, \mathcal{L}, \mathcal{F})_{(s)} \longrightarrow \Gamma(X_s, \mathcal{F})$$

is an isomorphism.

Proof. Consider the scheme

$$\pi : L^* = \underline{\mathrm{Spec}}_X \left(\bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{L}^{\otimes n}\right) \longrightarrow X$$

see Constructions, Section 4. Since the inverse image $\pi^{-1}(U)$ of every affine open $U \subset X$ is affine (see Constructions, Lemma 4.6), it follows that L^* quasi-compact and separated, since X is assumed quasi-compact and separated (use Schemes, Lemma 21.7). Note that s gives rise to an element $f \in \Gamma(L^*, \mathcal{O})$, via $\pi_*\mathcal{O}_{L^*} = \bigoplus_{n \in \mathbf{Z}} \mathcal{L}^{\otimes n}$. Note that $(L^*)_f = \pi^{-1}(X_s)$. Hence we have

$$\begin{aligned} \left(\bigoplus\nolimits_{n \in \mathbf{Z}} \Gamma(X, \mathcal{L}^{\otimes n})\right)_s &= \Gamma(L^*, \mathcal{O}_{L^*})_f \\ &= \Gamma((L^*)_f, \mathcal{O}_{L^*}) \\ &= \bigoplus\nolimits_{n \in \mathbf{Z}} \Gamma(X_s, \mathcal{L}^{\otimes n}) \end{aligned}$$

where the middle "=" is Lemma 15.2. The first statement of the lemma follows from this equality by looking at degree zero terms. The second statement also follows from Lemma 15.2 applied to the quasi-coherent sheaf of $\mathcal{O}_{L^*}$-modules $\pi^*\mathcal{F}$ using that

$$\pi_*\pi^*\mathcal{F} = \mathcal{F} \otimes_{\mathcal{O}_X} \left(\bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{L}^{\otimes n}\right) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$$

which is proved by computing both sides on affine opens of X. □

Lemma 23.6. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume the open sets X_s, where $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ and $n \geq 1$, form a basis for the topology on X. Then among those opens, the open sets X_s which are affine form a basis for the topology on X.*

Proof. Let $x \in X$. Choose an affine open neighbourhood $\mathrm{Spec}(R) = U \subset X$ of x. By assumption, there exists a $n \geq 1$ and a $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $X_s \subset U$. By Lemma 23.4 above the intersection $X_s = U \cap X_s$ is affine. Since U can be chosen arbitrarily small we win. □

Lemma 23.7. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume for every point x of X there exists $n \geq 1$ and $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $x \in X_s$ and X_s is affine. Then X is quasi-separated.*

Proof. By assumption we can find a covering of X by affine opens of the form X_s. By Schemes, Lemma 21.7 it suffices to show that $X_s \cap X_{s'}$ is quasi-compact whenever X_s is affine. This is true by Lemma 23.4. □

Lemma 23.8. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Set $S = \Gamma_*(X, \mathcal{L})$ as a graded ring. If every point of X is contained in one of the open subschemes X_s, for some $s \in S_+$ homogeneous, then there is a canonical morphism of schemes*

$$f : X \longrightarrow Y = \mathit{Proj}(S),$$

to the homogeneous spectrum of S (see Constructions, Section 8). This morphism has the following properties

(1) *$f^{-1}(D_+(s)) = X_s$ for any $s \in S_+$ homogeneous,*
(2) *there are $\mathcal{O}_Y$-module maps $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ compatible with multiplication maps, see Constructions, Equation (10.1.1),*
(3) *the compositions $S_n \to \Gamma(Y, \mathcal{O}_Y(n)) \to \Gamma(X, \mathcal{L}^{\otimes n})$ are equal to the identity maps, and*
(4) *for every $x \in X$ there is an integer $d \geq 1$ and an open neighbourhood $U \subset X$ of x such that $f^*\mathcal{O}_Y(dn)|_U \to \mathcal{L}^{\otimes dn}|_U$ is an isomorphism for all $n \in \mathbf{Z}$.*

Proof. Denote $\psi : S \to \Gamma_*(X, \mathcal{L})$ the identity map. We are going to use the triple $(U(\psi), r_{\mathcal{L},\psi}, \theta)$ of Constructions, Lemma 14.1. By assumption the open subscheme $U(\psi)$ of equals X. Hence $r_{\mathcal{L},\psi} : U(\psi) \to Y$ is defined on all of X. We set $f = r_{\mathcal{L},\psi}$. The maps in part (2) are the components of θ. Part (3) follows from condition (2) in the lemma cited above. Part (1) follows from (3) combined with condition (1) in the lemma cited above. Part (4) follows from the last statement in Constructions, Lemma 14.1 since the map α mentioned there is an isomorphism. □

Lemma 23.9. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Set $S = \Gamma_*(X, \mathcal{L})$. Assume (a) every point of X is contained in one of the open subschemes X_s, for some $s \in S_+$ homogeneous, and (b) X is quasi-compact. Then the canonical morphism of schemes $f : X \longrightarrow Proj(S)$ of Lemma 23.8 above is quasi-compact.*

Proof. It suffices to show that $f^{-1}(D_+(s))$ is quasi-compact for any $s \in S_+$ homogeneous. Write $X = \bigcup_{i=1,\ldots,n} X_i$ as a finite union of affine opens. By Lemma 23.4 each intersection $X_s \cap X_i$ is affine. Hence $X_s = \bigcup_{i=1,\ldots,n} X_s \cap X_i$ is quasi-compact. □

Lemma 23.10. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Set $S = \Gamma_*(X, \mathcal{L})$. Assume $\mathcal{L}$ is ample. Then the canonical morphism of schemes $f : X \longrightarrow Proj(S)$ of Lemma 23.8 is an open immersion.*

Proof. By Lemma 23.7 we see that X is quasi-separated. Choose finitely many $s_1, \ldots, s_n \in S_+$ homogeneous such that X_{s_i} are affine, and $X = \bigcup X_{s_i}$. Say s_i has degree d_i. The inverse image of $D_+(s_i)$ under f is X_{s_i}, see Lemma 23.8. By Lemma 23.5 the ring map

$$(S^{(d_i)})_{(s_i)} = \Gamma(D_+(s_i), \mathcal{O}_{\text{Proj}(S)}) \longrightarrow \Gamma(X_{s_i}, \mathcal{O}_X)$$

is an isomorphism. Hence f induces an isomorphism $X_{s_i} \to D_+(s_i)$. Thus f is an isomorphism of X onto the open subscheme $\bigcup_{i=1,\ldots,n} D_+(s_i)$ of $\text{Proj}(S)$. □

Lemma 23.11. *Let X be a scheme. Let S be a graded ring. Assume X is quasi-compact, and assume there exists an open immersion*

$$j : X \longrightarrow Y = Proj(S).$$

Then $j^*\mathcal{O}_Y(d)$ is an invertible ample sheaf for some $d > 0$.

Proof. This is Constructions, Lemma 10.6. □

Proposition 23.12. *Let X be a quasi-compact scheme. Let $\mathcal{L}$ be an invertible sheaf on X. Set $S = \Gamma_*(X, \mathcal{L})$. The following are equivalent:*

(1) *$\mathcal{L}$ is ample,*
(2) *the open sets X_s, with $s \in S_+$ homogeneous, cover X and the associated morphism $X \to Proj(S)$ is an open immersion,*

(3) *the open sets* X_s, *with* $s \in S_+$ *homogeneous, form a basis for the topology of* X,
(4) *the open sets* X_s, *with* $s \in S_+$ *homogeneous, which are affine form a basis for the topology of* X,
(5) *for every quasi-coherent sheaf* $\mathcal{F}$ *on* X *the sum of the images of the canonical maps*

$$\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \otimes_{\mathbf{Z}} \mathcal{L}^{\otimes -n} \longrightarrow \mathcal{F}$$

with $n \geq 1$ *equals* $\mathcal{F}$,
(6) *same property as (5) with* $\mathcal{F}$ *ranging over all quasi-coherent sheaves of ideals,*
(7) X *is quasi-separated and for every quasi-coherent sheaf* $\mathcal{F}$ *of finite type on* X *there exists an integer* n_0 *such that* $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ *is globally generated for all* $n \geq n_0$,
(8) X *is quasi-separated and for every quasi-coherent sheaf* $\mathcal{F}$ *of finite type on* X *there exist integers* $n > 0$, $k \geq 0$ *such that* $\mathcal{F}$ *is a quotient of a direct sum of* k *copies of* $\mathcal{L}^{\otimes -n}$, *and*
(9) *same as in (8) with* $\mathcal{F}$ *ranging over all sheaves of ideals of finite type on* X.

Proof. Lemma 23.10 is (1) $\Rightarrow$ (2). Lemmas 23.2 and 23.11 provide the implication (1) $\Leftarrow$ (2). The implications (2) $\Rightarrow$ (4) $\Rightarrow$ (3) are clear from Constructions, Section 8. Lemma 23.6 is (3) $\Rightarrow$ (1). Thus we see that the first 4 conditions are all equivalent.

Assume the equivalent conditions (1) – (4). Note that in particular X is separated (as an open subscheme of the separated scheme $\text{Proj}(S)$). Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Choose $s \in S_+$ homogeneous such that X_s is affine. We claim that any section $m \in \Gamma(X_s, \mathcal{F})$ is in the image of one of the maps displayed in (5) above. This will imply (5) since these affines X_s cover X. Namely, by Lemma 23.5 we may write m as the image of $m' \otimes s^{-n}$ for some $n \geq 1$, some $m' \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$. This proves the claim.

Clearly (5) $\Rightarrow$ (6). Let us assume (6) and prove $\mathcal{L}$ is ample. Pick $x \in X$. Let $U \subset X$ be an affine open which contains x. Set $Z = X \setminus U$. We may think of Z as a reduced closed subscheme, see Schemes, Section 12. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals corresponding to the closed subscheme Z. By assumption (6), there exists an $n \geq 1$ and a section $s \in \Gamma(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n})$ such that s does not vanish at x (more precisely such that $s \notin \mathfrak{m}_x \mathcal{I}_x \otimes \mathcal{L}_x^{\otimes n}$). We may think of s as a section of $\mathcal{L}^{\otimes n}$. Since it clearly vanishes along Z we see that $X_s \subset U$. Hence X_s is affine, see Lemma 23.4. This proves that $\mathcal{L}$ is ample. At this point we have proved that (1) – (6) are equivalent.

Assume the equivalent conditions (1) – (6). In the following we will use the fact that the tensor product of two sheaves of modules which are globally generated is globally generated without further mention (see Modules, Lemma 4.3). By (1) we can find elements $s_i \in S_{d_i}$ with $d_i \geq 1$ such that $X = \bigcup_{i=1,\ldots,n} X_{s_i}$. Set $d = d_1 \ldots d_n$. It follows that $\mathcal{L}^{\otimes d}$ is globally generated by

$$s_1^{d/d_1}, \ldots, s_n^{d/d_n}.$$

This means that if $\mathcal{L}^{\otimes j}$ is globally generated then so is $\mathcal{L}^{\otimes j + dn}$ for all $n \geq 0$. Fix a $j \in \{0, \ldots, d-1\}$. For any point $x \in X$ there exists an $n \geq 1$ and a global section s of $\mathcal{L}^{j+dn}$ which does not vanish at x, as follows from (5) applied to $\mathcal{F} = \mathcal{L}^{\otimes j}$ and ample invertible sheaf $\mathcal{L}^{\otimes d}$. Since X is quasi-compact there we may find a finite list of integers n_i and global sections s_i of $\mathcal{L}^{\otimes j+dn_i}$ which do not vanish at any point of X. Since $\mathcal{L}^{\otimes d}$ is globally generated this means that $\mathcal{L}^{\otimes j+dn}$ is globally generated where $n = \max\{n_i\}$. Since we proved this for every congruence class mod d we conclude that there exists an $n_0 = n_0(\mathcal{L})$ such that $\mathcal{L}^{\otimes n}$ is globally generated for all $n \geq n_0$. At this point we see that if $\mathcal{F}$ is globally generated then so is $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ for all $n \geq n_0$.

We continue to assume the equivalent conditions (1) – (6). Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules of finite type. Denote $\mathcal{F}_n \subset \mathcal{F}$ the image of the canonical map of (5). By construction $\mathcal{F}_n \otimes \mathcal{L}^{\otimes n}$ is globally generated. By (5) we see $\mathcal{F}$ is the sum of the subsheaves $\mathcal{F}_n$, $n \geq 1$. By Modules, Lemma 9.7 we see that $\mathcal{F} = \sum_{n=1,\ldots,N} \mathcal{F}_n$ for some $N \geq 1$. It follows that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated whenever $n \geq N + n_0(\mathcal{L})$ with $n_0(\mathcal{L})$ as above. We conclude that (1) – (6) implies (7).

Assume (7). Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules of finite type. By (7) there exists an integer $n \geq 1$ such that the canonical map

$$\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \otimes_{\mathbf{Z}} \mathcal{L}^{\otimes -n} \longrightarrow \mathcal{F}$$

is surjective. Let I be the set of finite subsets of $\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$ partially ordered by inclusion. Then I is a directed partially ordered set. For $i = \{s_1, \ldots, s_{r(i)}\}$ let $\mathcal{F}_i \subset \mathcal{F}$ be the image of the map

$$\bigoplus\nolimits_{j=1,\ldots,r(i)} \mathcal{L}^{\otimes -n} \longrightarrow \mathcal{F}$$

which is multiplication by s_j on the jth factor. The surjectivity above implies that $\mathcal{F} = \operatorname{colim}_{i \in I} \mathcal{F}_i$. Hence Modules, Lemma 9.7 applies and we conclude that $\mathcal{F} = \mathcal{F}_i$ for some i. Hence we have proved (8). In other words, (7) $\Rightarrow$ (8).

The implication (8) $\Rightarrow$ (9) is trivial.

Finally, assume (9). Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. By Lemma 20.3 (this is where we use the condition that X be quasi-separated) we see that $\mathcal{I} = \operatorname{colim}_\alpha I_\alpha$ with each I_α quasi-coherent of finite type. Since by assumption each of the I_α is a quotient of negative tensor powers of $\mathcal{L}$ we conclude the same for $\mathcal{I}$ (but of course without the finiteness or boundedness of the powers). Hence we conclude that (9) implies (6). This ends the proof of the proposition. □

24. Affine and quasi-affine schemes

Lemma 24.1. *Let X be a scheme. Then X is quasi-affine if and only if $\mathcal{O}_X$ is ample.*

Proof. Suppose that X is quasi-affine. Consider the open immersion

$$j : X \longrightarrow \operatorname{Spec}(\Gamma(X, \mathcal{O}_X))$$

from Lemma 15.4. Note that $\operatorname{Spec}(A) = \operatorname{Proj}(A[T])$, see Constructions, Example 8.14. Hence we can apply Lemma 23.11 to deduce that $\mathcal{O}_X$ is ample.

Suppose that $\mathcal{O}_X$ is ample. Note that $\Gamma_*(X, \mathcal{O}_X) \cong \Gamma(X, \mathcal{O}_X)[T]$ as graded rings. Hence the result follows from Lemmas 23.10 and 15.4 taking into account that $\operatorname{Spec}(A) = \operatorname{Proj}(A[T])$ for any ring A as seen above. □

Lemma 24.2. *Let X be a scheme. Suppose that there exist finitely many elements $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$ such that*

1. *each X_{f_i} is an affine open of X, and*
2. *the ideal generated by $f_1, \ldots, f_n$ in $\Gamma(X, \mathcal{O}_X)$ is equal to the unit ideal.*

Then X is affine.

Proof. Assume we have $f_1, \ldots, f_n$ as in the lemma. We may write $1 = \sum g_i f_i$ for some $g_j \in \Gamma(X, \mathcal{O}_X)$ and hence it is clear that $X = \bigcup X_{f_i}$. (The f_i's cannot all vanish at a point.) Since each X_{f_i} is quasi-compact (being affine) it follows that X is quasi-compact. Hence we see that X is quasi-affine by Lemma 24.1 above. Consider the open immersion

$$j : X \to \mathrm{Spec}(\Gamma(X, \mathcal{O}_X)),$$

see Lemma 15.4. The inverse image of the standard open $D(f_i)$ on the right hand side is equal to X_{f_i} on the left hand side and the morphism j induces an isomorphism $X_{f_i} \cong D(f_i)$, see Lemma 15.3. Since the f_i generate the unit ideal we see that $\mathrm{Spec}(\Gamma(X, \mathcal{O}_X)) = \bigcup_{i=1,\ldots,n} D(f_i)$. Thus j is an isomorphism. □

25. Quasi-coherent sheaves and ample invertible sheaves

Situation 25.1. Let X be a scheme. Let $\mathcal{L}$ be an invertible sheaf on X. Assume $\mathcal{L}$ is ample. Set $S = \Gamma_*(X, \mathcal{L})$ as a graded ring. Set $Y = \mathrm{Proj}(S)$. Let $f : X \to Y$ be the canonical morphism of Lemma 23.8. It comes equipped with a $\mathbf{Z}$-graded $\mathcal{O}_X$-algebra map $\bigoplus f^*\mathcal{O}_Y(n) \to \bigoplus \mathcal{L}^{\otimes n}$.

The following lemma is really a special case of the next lemma but it seems like a good idea to point out its validity first.

Lemma 25.2. *In Situation 25.1. The canonical morphism $f : X \to Y$ maps X into the open subscheme $W = W_1 \subset Y$ where $\mathcal{O}_Y(1)$ is invertible and where all multiplication maps $\mathcal{O}_Y(n) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(m) \to \mathcal{O}_Y(n+m)$ are isomorphisms (see Constructions, Lemma 10.4). Moreover, the maps $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ are all isomorphisms.*

Proof. By Proposition 23.12 there exists an integer n_0 such that $\mathcal{L}^{\otimes n}$ is globally generated for all $n \geq n_0$. Let $x \in X$ be a point. By the above we can find $a \in S_{n_0}$ and $b \in S_{n_0+1}$ such that a and b do not vanish at x. Hence $f(x) \in D_+(a) \cap D_+(b) = D_+(ab)$. By Constructions, Lemma 10.4 we see that $f(x) \in W_1$ as desired. By Constructions, Lemma 14.1 which was used in the construction of the map f the maps $f^*\mathcal{O}_Y(n_0) \to \mathcal{L}^{\otimes n_0}$ and $f^*\mathcal{O}_Y(n_0+1) \to \mathcal{L}^{\otimes n_0+1}$ are isomorphisms in a neighbourhood of x. By compatibility with the algebra structure and the fact that f maps into W we conclude all the maps $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ are isomorphisms in a neighbourhood of x. Hence we win. □

Recall from Modules, Definition 21.4 that given a locally ringed space X, an invertible sheaf $\mathcal{L}$, and a $\mathcal{O}_X$-module $\mathcal{F}$ we have the graded $\Gamma_*(X, \mathcal{L})$-module

$$\Gamma(X, \mathcal{L}, \mathcal{F}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}).$$

The following lemma says that, in Situation 25.1, we can recover a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ from this graded module. Take a look also at Constructions, Lemma 13.7 where we prove this lemma in the special case $X = \mathbf{P}^n_R$.

Lemma 25.3. *In Situation 25.1. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Set $M = \Gamma_*(X, \mathcal{L}, \mathcal{F})$ as a graded S-module. There are isomorphisms*

$$f^*\widetilde{M} \longrightarrow \mathcal{F}$$

functorial in $\mathcal{F}$ such that $M_0 \to \Gamma(Proj(S), \widetilde{M}) \to \Gamma(X, \mathcal{F})$ is the identity map.

Proof. Let $s \in S_+$ be homogeneous such that X_s is affine open in X. Recall that $\widetilde{M}|_{D_+(s)}$ corresponds to the $S_{(s)}$-module $M_{(s)}$, see Constructions, Lemma 8.4. Recall that $f^{-1}(D_+(s)) = X_s$. As X carries an ample invertible sheaf it is quasi-compact and quasi-separated, see Section 23. By Lemma 23.5 there is a canonical isomorphism $M_{(s)} = \Gamma_*(X, \mathcal{L}, \mathcal{F})_{(s)} \to \Gamma(X_s, \mathcal{F})$. Since $\mathcal{F}$ is quasi-coherent this leads to a canonical isomorphism

$$f^*\widetilde{M}|_{X_s} \to \mathcal{F}|_{X_s}$$

Since $\mathcal{L}$ is ample on X we know that X is covered by the affine opens of the form X_s. Hence it suffices to prove that the displayed maps glue on overlaps. Proof of this is omitted. □

Remark 25.4. With assumptions and notation of Lemma 25.3. Denote the displayed map of the lemma by $\theta_\mathcal{F}$. Note that the isomorphism $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ of Lemma 25.2 is just $\theta_{\mathcal{L}^{\otimes n}}$. Consider the multiplication maps

$$\widetilde{M} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(n) \longrightarrow \widetilde{M(n)}$$

see Constructions, Equation (10.1.5). Pull this back to X and consider

$$\begin{array}{ccc} f^*\widetilde{M} \otimes_{\mathcal{O}_X} f^*\mathcal{O}_Y(n) & \longrightarrow & f^*\widetilde{M(n)} \\ \downarrow{\scriptstyle \theta_\mathcal{F} \otimes \theta_{\mathcal{L}^{\otimes n}}} & & \downarrow{\scriptstyle \theta_{\mathcal{F} \otimes \mathcal{L}^{\otimes n}}} \\ \mathcal{F} \otimes \mathcal{L}^{\otimes n} & \xrightarrow{\text{id}} & \mathcal{F} \otimes \mathcal{L}^{\otimes n} \end{array}$$

Here we have used the obvious identification $M(n) = \Gamma_*(X, \mathcal{L}, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$. This diagram commutes. Proof omitted.

26. Finding suitable affine opens

In this section we collect some results on the existence of affine opens in more and less general situations.

Lemma 26.1. *Let X be a quasi-separated scheme. Let $Z_1, \ldots, Z_n$ be pairwise distinct irreducible components of X, see Topology, Section 5. Let $\eta_i \in Z_i$ be their generic points, see Schemes, Lemma 11.1. There exist affine open neighbourhoods $\eta_i \in U_i$ such that $U_i \cap U_j = \emptyset$ for all $i \neq j$. In particular, $U = U_1 \cup \ldots \cup U_n$ is an affine open containing all of the points $\eta_1, \ldots, \eta_n$.*

Proof. Let V_i be any affine open containing η_i and disjoint from the closed set $Z_1 \cup \ldots \hat{Z}_i \ldots \cup Z_n$. Since X is quasi-separated for each i the union $W_i = \bigcup_{j, j \neq i} V_i \cap V_j$ is a quasi-compact open of V_i not containing η_i. We can find open neighbourhoods $U_i \subset V_i$ containing η_i and disjoint from W_i by Algebra, Lemma 24.4. Finally, U is affine since it is the spectrum of the ring $R_1 \times \ldots \times R_n$ where $R_i = \mathcal{O}_X(U_i)$, see Schemes, Lemma 6.8. □

Remark 26.2. Lemma 26.1 above is false if X is not quasi-separated. Here is an example. Take $R = \mathbf{Q}[x, y_1, y_2, \ldots]/((x-i)y_i)$. Consider the minimal prime ideal $\mathfrak{p} = (y_1, y_2, \ldots)$ of R. Glue two copies of $\text{Spec}(R)$ along the (not quasi-compact) open $\text{Spec}(R)\setminus V(\mathfrak{p})$ to get a scheme X (glueing as in Schemes, Example 14.3). Then the two maximal points of X corresponding to $\mathfrak{p}$ are not contained in a common affine open. The reason is that any open of $\text{Spec}(R)$ containing $\mathfrak{p}$ contains infinitely many of the "lines" $x = i$, $y_j = 0$, $j \neq i$ with parameter y_i. Details omitted.

Notwithstanding the example above, for "most" finite sets of irreducible closed subsets one can apply Lemma 26.1 above, at least if X is quasi-compact. This is true because X contains a dense open which is separated.

Lemma 26.3. *Let X be a quasi-compact scheme. There exists a dense open $V \subset X$ which is separated.*

Proof. Say $X = \bigcup_{i=1,\ldots,n} U_i$ is a union of n affine open subschemes. We will prove the lemma by induction on n. It is trivial for $n = 1$. Let $V' \subset \bigcup_{i=1,\ldots,n-1} U_i$ be a separated dense open subscheme, which exists by induction hypothesis. Consider

$$V = V' \coprod (U_n \setminus \overline{V'}).$$

It is clear that V is separated and a dense open subscheme of X. □

Here is a slight refinement of Lemma 26.1 above.

Lemma 26.4. *Let X be a quasi-separated scheme. Let $Z_1, \ldots, Z_n$ be pairwise distinct irreducible components of X. Let $\eta_i \in Z_i$ be their generic points. Let $x \in X$ be arbitrary. There exists an affine open $U \subset X$ containing x and all the η_i.*

Proof. Suppose that $x \in Z_1 \cap \ldots \cap Z_r$ and $x \notin Z_{r+1}, \ldots, Z_n$. Then we may choose an affine open $W \subset X$ such that $x \in W$ and $W \cap Z_i = \emptyset$ for $i = r+1, \ldots, n$. Note that clearly $\eta_i \in W$ for $i = 1, \ldots, r$. By Lemma 26.1 we may choose affine opens $U_i \subset X$ which are pairwise disjoint such that $\eta_i \in U_i$ for $i = r+1, \ldots, n$. Since X is quasi-separated the opens $W \cap U_i$ are quasi-compact and do not contain η_i for $i = r+1, \ldots, n$. Hence by Algebra, Lemma 24.4 we may shrink U_i such that $W \cap U_i = \emptyset$ for $i = r+1, \ldots, n$. Then the union $U = W \cup \bigcup_{i=r+1,\ldots,n} U_i$ is disjoint and hence (by Schemes, Lemma 6.8) a suitable affine open. □

Lemma 26.5. *Let X be a scheme. Assume either*

1. *The scheme X is quasi-affine.*
2. *The scheme X is isomorphic to a locally closed subscheme of an affine scheme.*
3. *There exists an ample invertible sheaf on X.*
4. *The scheme X is isomorphic to a locally closed subscheme of Proj(S) for some graded ring S.*

Then for any finite subset $E \subset X$ there exists an affine open $U \subset X$ with $E \subset U$.

Proof. By Properties, Definition 15.1 a quasi-affine scheme is a quasi-compact open subscheme of an affine scheme. Any affine scheme $\text{Spec}(R)$ is isomorphic to $\text{Proj}(R[X])$ where $R[X]$ is graded by setting $\deg(X) = 1$. By Properties, Proposition 23.12 if X has an ample invertible sheaf then X is isomorphic to an open subscheme of $\text{Proj}(S)$ for some graded ring S. Hence, it suffices to prove the lemma in case (4). (We urge the reader to prove case (2) directly for themselves.)

Thus assume $X \subset \text{Proj}(S)$ is a locally closed subscheme where S is some graded ring. Let $T = \overline{X} \setminus X$. Recall that the standard opens $D_+(f)$ form a basis of the topology on $\text{Proj}(S)$. Since E is finite we may choose finitely many homogeneous elements $f_i \in S_+$ such that

$$E \subset D_+(f_1) \cup \ldots \cup D_+(f_n) \subset \text{Proj}(S) \setminus T$$

Suppose that $E = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ as a subset of $\text{Proj}(S)$. Consider the ideal $I = (f_1, \ldots, f_n) \subset S$. Since $I \not\subset \mathfrak{p}_j$ for all $j = 1, \ldots, m$ we see from Algebra, Lemma 54.6 that there exists a homogeneous element $f \in I$, $f \notin \mathfrak{p}_j$ for all $j = 1, \ldots, m$. Then $E \subset D_+(f) \subset D_+(f_1) \cup \ldots \cup D_+(f_n)$. Since $D_+(f)$ does not meet T we see that $X \cap D_+(f)$ is a closed subscheme of the affine scheme $D_+(f)$, hence is an affine open of X as desired. □

27. Other chapters

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