SMOOTHING RING MAPS

Contents

1. Introduction

The main result of this chapter is the following:

> A regular map of Noetherian rings is a filtered colimit of smooth ones.

This theorem is due to Popescu, see [Pop90]. A readable exposition of Popescu's proof was given by Richard Swan, see [Swa98] who used notes by André and a paper of Ogoma, see [Ogo94].

Our exposition follows Swan's, but we first prove an intermediate result which let's us work in a slightly simpler situation. Here is an overview. We first solve the following "lifting problem": A flat infinitesimal deformation of a filtered colimit of smooth algebras is a filtered colimit of smooth algebras. This result essentially says that it suffices to prove the main theorem for maps between reduced Noetherian rings. Next we prove two very clever lemmas called the "lifting lemma" and the "desingularization lemma". We show that these lemmas combined reduce the main theorem to proving a Noetherian, geometrically regular k-algebra Λ is a filtered limit of smooth k-algebras. Next, we discuss the necessary local tricks that go into the Popescu-Ogoma-Swan-André proof. Finally, in the last three sections we give the proof.

We end this introduction with some pointers to references. Let A be a henselian Noetherian local ring. We say A has the *approximation property* if for any $f_1, \ldots, f_m \in$

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$A[x_1, \ldots, x_n]$ the system of equations $f_1 = 0, \ldots, f_m = 0$ has a solution in the completion of A if and only if it has a solution in A. This definition is due to Artin. Artin first proved the approximation property for analytic systems of equations, see [Art68]. In [Art69] Artin proved the approximation property for local rings essentially of finite type over an excellent discrete valuation ring. Artin conjectured (page 26 of [Art69]) that every excellent henselian local ring should have the approximation property.

At some point in time it became a conjecture that that every regular homomorphism of Noetherian rings is a filtered colimit of smooth algebras (see for example [Ray72], [Pop81], [Art82], [AD83]). We're not sure who this conjecture[1] is due to. The relationship with the approximation property is that if $A \to A^\wedge$ is a colimit of smooth algebras, then the approximation property holds (insert future reference here). Moreover, the main theorem applies to the map $A \to A^\wedge$ if A is an excellent local ring, as one of the conditions of an excellent local ring is that the formal fibres are geometrically regular. Note that excellent local rings were defined by Grothendieck and their definition appeared in print in 1965.

In [Art82] it was shown that $R \to R^\wedge$ is a filtered colimit of smooth algebras for any local ring R essentially of finite type over a field. In [AR88] it was shown that $R \to R^\wedge$ is a filtered colimit of smooth algebras for any local ring R essentially of finite type over an excellent discrete valution ring. Finally, the main theorem was shown in [Pop85], [Pop86], [Pop90], [Ogo94], and [Swa98] as discussed above.

Conversely, using some of the results above, in [Rot90] it was shown that any local ring with the approximation property is excellent.

The paper [Spi99] provides an alternative approach to the main theorem, but it seems hard to read (for example [Spi99, Lemma 5.2] appears to be an incorrectly reformulated version of [Elk73, Lemma 3]). There is also a Bourbaki lecture about this material, see [Tei95].

2. Colimits

In Categories, Section 17 we discuss filtered colimits. In particular, note that Categories, Lemma 19.3 tells us that colimits over filtered index categories are the same thing as colimits over directed partially ordered sets.

Lemma 2.1. *Let $R \to \Lambda$ be a ring map. Let $\mathcal{E}$ be a set of R-algebras such that each $A \in \mathcal{E}$ is of finite presentation over R. Then the following two statements are equivalent*

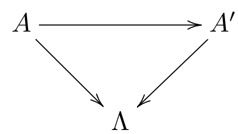
1. *Λ is a filtered colimit of elements of $\mathcal{E}$, and*
2. *for any R algebra map $A \to \Lambda$ with A of finite presentation over R we can find a factorization $A \to B \to \Lambda$ with $B \in \mathcal{E}$.*

Proof. Suppose that $\mathcal{I} \to \mathcal{E}$, $i \mapsto A_i$ is a diagram such that $\Lambda = \text{colim}_i A_i$. Let $A \to \Lambda$ with A of finite presentation over R. Pick a presentation $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Say $A \to \Lambda$ maps x_s to $\lambda_s \in \Lambda$. We can find an $i \in \text{Ob}(\mathcal{I})$ and elements $a_s \in A_i$ whose image in Λ is λ_s. Increasing i if necessary

[1] The question/conjecture as formulated in [Art82], [AD83], and [Pop81] is stronger and was shown to be equivalent to the original version in [CP84].

we may also assume that $f_t(a_1, \ldots, a_n) = 0$ in A_i. Hence we can factor $A \to \Lambda$ through A_i by mapping x_s to a_s.

Conversely, suppose that (2) holds. Consider the category $\mathcal{I}$ whose objects are R-algebra maps $A \to \Lambda$ with $A \in \mathcal{E}$ and whose morphisms are commutative diagrams



of R-algebras. We claim that $\mathcal{I}$ is a filtered index category and that $\Lambda = \operatorname{colim}_{\mathcal{I}} A$. To see that $\mathcal{I}$ is filtered, let $A \to \Lambda$ and $A' \to \Lambda$ be two objects. Then we can factor $A \otimes_R A' \to \Lambda$ through an object of $\mathcal{I}$ by assumption (2) and the fact that the elements of $\mathcal{E}$ are of finite presentation over R. Suppose that $\varphi, \psi : A \to A'$ are two morphisms of $\mathcal{I}$. Let $x_1, \ldots, x_n$ be generators of A as an R-algebra. By assumption (2) we can factor the R-algebra map $A'/(\varphi(x_i) - \psi(x_i)) \to \Lambda$ through an object of $\mathcal{I}$. This proves that $\mathcal{I}$ is filtered. We omit the proof that $\Lambda = \operatorname{colim}_{\mathcal{I}} A$. □

3. Singular ideals

Let $R \to A$ be a ring map. The singular ideal of A over R is the radical ideal in A cutting out the singular locus of the morphism $\operatorname{Spec}(A) \to \operatorname{Spec}(R)$. Here is a formal definition.

Definition 3.1. Let $R \to A$ be a ring map. The *singular ideal of A over R*, denoted $H_{A/R}$ is the unique radical ideal $H_{A/R} \subset A$ with

$$V(H_{A/R}) = \{\mathfrak{q} \in \operatorname{Spec}(A) \mid R \to A \text{ not smooth at } \mathfrak{q}\}$$

This makes sense because the set of primes where $R \to A$ is smooth is open, see Algebra, Definition 127.11. In order to find an explicit set of generators for the singular ideal we first prove the following lemma.

Lemma 3.2. *Let R be a ring. Let $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Let $\mathfrak{q} \subset A$. Assume $R \to A$ is smooth at $\mathfrak{q}$. Then there exists an $a \in A$, $a \notin \mathfrak{q}$, an integer c, $0 \leq c \leq \min(n, m)$, subsets $U \subset \{1, \ldots, n\}$, $V \subset \{1, \ldots, m\}$ of cardinality c such that*

$$a = a' \det(\partial f_j/\partial x_i)_{j \in V, i \in U}$$

for some $a' \in A$ and

$$a f_\ell \in (f_j, j \in V) + (f_1, \ldots, f_m)^2$$

for all $\ell \in \{1, \ldots, m\}$.

Proof. Set $I = (f_1, \ldots, f_m)$ so that the the naive cotangent complex of A over R is homotopy equivalent to $I/I^2 \to \bigoplus A\mathrm{d}x_i$, see Algebra, Lemma 124.2. We will use the formation of the naive cotangent complex commutes with localization, see Algebra, Section 124, especially Algebra, Lemma 124.10. By Algebra, Definitions 127.1 and 127.11 we see that $(I/I^2)_a \to \bigoplus A_a \mathrm{d}x_i$ is a split injection for some $a \in A$, $a \notin \mathfrak{p}$. After renumbering $x_1, \ldots, x_n$ and $f_1, \ldots, f_m$ we may assume that $f_1, \ldots, f_c$ form a basis for the vector space $I/I^2 \otimes_A \kappa(\mathfrak{q})$ and that $\mathrm{d}x_{c+1}, \ldots, \mathrm{d}x_n$ map to a basis of $\Omega_{A/R} \otimes_A \kappa(\mathfrak{q})$. Hence after replacing a by aa' for some $a' \in A$, $a' \notin \mathfrak{q}$ we may assume $f_1, \ldots, f_c$ form a basis for $(I/I^2)_a$ and that $\mathrm{d}x_{c+1}, \ldots, \mathrm{d}x_n$ map to a basis of $(\Omega_{A/R})_a$. In this situation a^N for some large integer N satisfies the conditions of the lemma (with $U = V = \{1, \ldots, c\}$). □

We will use the notion of a *strictly standard* element in a A over R. Our notion is slightly weaker than the one in Swan's paper [Swa98]. We also define an *elementary standard* element to be one of the type we found in the lemma above. We compare the different types of elements in Lemma 4.7.

Definition 3.3. Let $R \to A$ be a ring map of finite presentation. We say an element $a \in A$ is *elementary standard in A over R* if there exists a presentation $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and $0 \leq c \leq \min(n, m)$ such that

$$a = a' \det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c} \tag{3.3.1}$$

for some $a' \in A$ and

$$a f_{c+j} \in (f_1, \ldots, f_c) + (f_1, \ldots, f_m)^2 \tag{3.3.2}$$

for $j = 1, \ldots, m - c$. We say $a \in A$ is *strictly standard in A over R* if there exists a presentation $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and $0 \leq c \leq \min(n, m)$ such that

$$a = \sum\nolimits_{I \subset \{1,\ldots,n\},\ |I|=c} a_I \det(\partial f_j/\partial x_i)_{j=1,\ldots,c,\ i \in I} \tag{3.3.3}$$

for some $a_I \in A$ and

$$a f_{c+j} \in (f_1, \ldots, f_c) + (f_1, \ldots, f_m)^2 \tag{3.3.4}$$

for $j = 1, \ldots, m - c$.

The following lemma is useful to find implications of (3.3.3).

Lemma 3.4. *Let R be a ring. Let $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and write $I = (f_1, \ldots, f_n)$. Let $a \in A$. Then (3.3.3) implies there exists an A-linear map $\psi : \bigoplus_{i=1,\ldots,n} A dx_i \to A^{\oplus c}$ such that the composition*

$$A^{\oplus c} \xrightarrow{(f_1,\ldots,f_c)} I/I^2 \xrightarrow{f \mapsto df} \bigoplus\nolimits_{i=1,\ldots,n} A dx_i \xrightarrow{\psi} A^{\oplus c}$$

is multiplication by a. Conversely, if such a ψ exists, then a^c satisfies (3.3.3).

Proof. This is a special case of Algebra, Lemma 14.5. □

Lemma 3.5 (Elkik)**.** *Let $R \to A$ be a ring map of finite presentation. The singular ideal $H_{A/R}$ is the radical of the ideal generated by strictly standard elements in A over R and also the radical of the ideal generated by elementary standard elements in A over R.*

Proof. Assume a is strictly standard in A over R. We claim that A_a is smooth over R, which proves that $a \in H_{A/R}$. Namely, let $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, c, and $a' \in A$ be as in Definition 3.3. Write $I = (f_1, \ldots, f_m)$ so that the naive cotangent complex of A over R is given by $I/I^2 \to \bigoplus A\mathrm{d}x_i$. Assumption (3.3.4) implies that $(I/I^2)_a$ is generated by the classes of $f_1, \ldots, f_c$. Assumption (3.3.3) implies that the differential $(I/I^2)_a \to \bigoplus A_a \mathrm{d}x_i$ has a left inverse, see Lemma 3.4. Hence $R \to A_a$ is smooth by definition and Algebra, Lemma 124.10.

Let $H_e, H_s \subset A$ be the radical of the ideal generated by elementary, resp. strictly standard elements of A over R. By definition and what we just proved we have $H_e \subset H_s \subset H_{A/R}$. The inclusion $H_{A/R} \subset H_e$ follows from Lemma 3.2. □

Example 3.6. The set of points where a finitely presented ring map is smooth needn't be a quasi-compact open. For example, let $R = k[x, y_1, y_2, y_3, \ldots]/(xy_i)$ and $A = R/(x)$. Then the smooth locus of $R \to A$ is $\bigcup D(y_i)$ which is not quasi-compact.

Lemma 3.7. *Let $R \to A$ be a ring map of finite presentation. Let $R \to R'$ be a ring map. If $a \in A$ is elementary, resp. strictly standard in A over R, then $a \otimes 1$ is elementary, resp. strictly standard in $A \otimes_R R'$ over R'.*

Proof. If $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ is a presentation of A over R, then $A \otimes_R R' = R'[x_1, \ldots, x_n]/(f'_1, \ldots, f'_m)$ is a presentation of $A \otimes_R R'$ over R'. Here f'_j is the image of f_j in $R'[x_1, \ldots, x_n]$. Hence the result follows from the definitions. □

Lemma 3.8. *Let $R \to A \to \Lambda$ be ring maps with A of finite presentation over R. Assume that $H_{A/R}\Lambda = \Lambda$. Then there exists a factorization $A \to B \to \Lambda$ with B smooth over R.*

Proof. Choose $f_1, \ldots, f_r \in H_{A/R}$ and $\lambda_1, \ldots, \lambda_r \in \Lambda$ such that $\sum f_i \lambda_i = 1$ in Λ. Set $B = A[x_1, \ldots, x_r]/(f_1 x_1 + \ldots + f_r x_r - 1)$ and define $B \to \Lambda$ by mapping x_i to λ_i. Details omitted. □

4. Presentations of algebras

Some of the results in this section are due to Elkik. Note that the algebra C in the following lemma is a symmetric algebra over A. Moreover, if R is Noetherian, then C is of finite presentation over R.

Lemma 4.1. *Let R be a ring and let A be a finitely presented R-algebra. There exists finite type R-algebra map $A \to C$ which has a retraction with the following two properties*

(1) *for each $a \in A$ such that A_a is syntomic*[2] *over R the ring C_a is smooth over A_a and has a presentation $C_a = R[y_1, \ldots, y_m]/J$ such that J/J^2 is free over C_a, and*
(2) *for each $a \in A$ such that A_a is smooth over R the module $\Omega_{C_a/R}$ is free over C_a.*

Proof. Choose a presentation $A = R[x_1, \ldots, x_n]/I$ and write $I = (f_1, \ldots, f_m)$. Define the A-module K by the short exact sequence

$$0 \to K \to A^{\oplus m} \to I/I^2 \to 0$$

where the jth basis vector e_j in the middle is mapped to the class of f_j on the right. Set

$$C = \text{Sym}^*_A(I/I^2).$$

The retraction is just the projection onto the degree 0 part of C. We have a surjection $R[x_1, \ldots, x_n, y_1, \ldots, y_m] \to C$ which maps y_j to the class of f_j in I/I^2. The kernel J of this map is generated by the elements $f_1, \ldots, f_m$ and by elements $\sum h_j y_j$ with $h_j \in R[x_1, \ldots, x_n]$ such that $\sum h_j e_j$ defines an element of K. By Algebra, Lemma 124.3 applied to $R \to A \to C$ and the presentations above and More on Algebra, Lemma 9.10 there is a short exact sequence

$$I/I^2 \otimes_A C \to J/J^2 \to K \otimes_A C \to 0 \quad (4.1.1)$$

[2]Or just that $R \to A_a$ is a local complete intersection, see More on Algebra, Definition 24.2.

of C-modules. Let $h \in R[x_1, \ldots, x_n]$ be an element with image $a \in A$. We will use as presentations for the localized rings

$$A_a = R[x_0, x_1, \ldots, x_n]/I' \quad \text{and} \quad C_a = R[x_0, x_1, \ldots, x_n, y_1, \ldots, y_m]/J'$$

where $I' = (hx_0 - 1, I)$ and $J' = (hx_0 - 1, J)$. Hence $I'/(I')^2 = C_a \oplus I/I^2 \otimes_A C_a$ and $J'/(J')^2 = C_a \oplus (J/J^2)_a$ as C_a-modules. Thus we obtain

$$C_a \oplus I/I^2 \otimes_A C_a \to C_a \oplus (J/J^2)_a \to K \otimes_A C_a \to 0 \tag{4.1.2}$$

as the sequence of Algebra, Lemma 124.3 corresponding to $R \to A_a \to C_a$ and the presentations above.

Next, assume that $a \in A$ is such that A_a is syntomic over R. Then $(I/I^2)_a$ is finite projective over A_a, see Algebra, Lemma 126.17. Hence we see $K_a \oplus (I/I^2)_a \cong A_a^{\oplus m}$ is free. In particular K_a is finite projective too. By More on Algebra, Lemma 24.6 the sequence (4.1.2) is exact on the left. Hence

$$J'/(J')^2 \cong C_a \oplus I/I^2 \otimes_A C_a \oplus K \otimes_A C_a \cong C_a^{\oplus m+1}$$

This proves (1). Finally, suppose that in addition A_a is smooth over R. Then the same presentation shows that $\Omega_{C_a/R}$ is the cokernel of the map

$$J'/(J')^2 \longrightarrow \bigoplus\nolimits_i C_a \mathrm{d}x_i \oplus \bigoplus\nolimits_j C_a \mathrm{d}y_j$$

The summand C_a of $J'/(J')^2$ in the decomposition above corresponds to $hx_0 - 1$ and hence maps isomorphically to the summand $C_a \mathrm{d}x_0$. The summand $I/I^2 \otimes_A C_a$ of $J'/(J')^2$ maps injectively to $\bigoplus_{i=1,\ldots,n} C_a \mathrm{d}x_i$ with quotient $\Omega_{A_a/R} \otimes_{A_a} C_a$. The summand $K \otimes_A C_a$ maps injectively to $\bigoplus_{j \geq 1} C_a \mathrm{d}y_j$ with quotient isomorphic to $I/I^2 \otimes_A C_a$. Thus the cokernel of the last displayed map is the module $I/I^2 \otimes_A C_a \oplus \Omega_{A_a/R} \otimes_{A_a} C_a$. Since $(I/I^2)_a \oplus \Omega_{A_a/R}$ is free (from the definition of smooth ring maps) we see that (2) holds. □

The following proposition was proved for henselian pairs by Elkik in [Elk73]. In the form stated below it can be found in [Ara01], where they also prove that ring maps between smooth algebras can be lifted.

Proposition 4.2. *Let R be a ring and let $I \subset R$ be an ideal. Let $R/I \to \overline{A}$ be a smooth ring map. Then there exists a smooth ring map $R \to A$ such that A/IA is isomorphic to $\overline{A}$.*

Proof. Choose a presentation $\overline{A} = (R/I)[x_1, \ldots, x_n]/\overline{J}$. Set $\overline{C} = \mathrm{Sym}^*_{\overline{A}}(\overline{J}/\overline{J}^2)$. Note that $\overline{J}/\overline{J}^2$ is a finite projective $\overline{A}$-module (follows from the definition of smoothness). By Lemma 4.1 and its proof the ring map $\overline{A} \to \overline{C}$ is smooth and we can find a presentation $\overline{C} = R/I[y_1, \ldots, y_m]/\overline{K}$ with $\overline{K}/\overline{K}^2$ free over $\overline{C}$. By Algebra, Lemma 126.6 we can even assume that $\overline{C} = R/I[y_1, \ldots, y_m]/(\overline{f}_1, \ldots, \overline{f}_c)$ where $\overline{f}_1, \ldots, \overline{f}_c$ maps to a basis of $\overline{K}/\overline{K}^2$ over $\overline{C}$. Choose $f_1, \ldots, f_c \in R[y_1, \ldots, y_c]$ lifting $\overline{f}_1, \ldots, \overline{f}_c$ and set

$$C = R[y_1, \ldots, y_m]/(f_1, \ldots, f_c)$$

By construction $C/IC = \overline{C}$. Consider the naive cotangent complex

$$(f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2 \longrightarrow \bigoplus\nolimits_{i=1,\ldots,m} C \mathrm{d}y_i$$

associated to the presentation of C. For every prime $\mathfrak{q} \supset IC$ of C the images $\mathrm{d}f_j$ are linearly independent in $\bigoplus \kappa(\mathfrak{q})\mathrm{d}y_i$ because $\overline{C}$ is smooth over R/I. Hence we conclude that $((f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2)_{\mathfrak{q}}$ is free of rank c and maps to a direct summand of $\bigoplus C_{\mathfrak{q}}\mathrm{d}y_j$. Hence $R \to C$ is smooth at $\mathfrak{q}$, see Algebra, Lemma 127.12. Thus we can find a $g \in C$ mapping to an invertible element of C/IC such that $R \to C_g$ is smooth, see More on Algebra, Lemma 9.4. We conclude that there exists a finite projective $\overline{A}$-module $\overline{P}$ such that $\overline{C} = \mathrm{Sym}^*_{\overline{A}}(\overline{P})$ is isomorphic to C/IC for some smooth R-algebra C.

Choose an integer n and a direct sum decomposition $\overline{A}^{\oplus n} = \overline{P} \oplus \overline{Q}$. By More on Algebra, Lemma 9.9 we can find an étale ring map $C \to C'$ which induces an isomorphism $C/IC \to C'/IC'$ and a finite projective C'-module Q such that Q/IQ is isomorphic to $\overline{Q} \otimes_{\overline{A}} C/IC$. Then $D = \mathrm{Sym}^*_{C'}(Q)$ is a smooth C'-algebra (see More on Algebra, Lemma 9.11). Picture

$$\begin{array}{ccccccccc} R & \longrightarrow & & & C & \longrightarrow & C' & \longrightarrow & D \\ \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\ R/I & \longrightarrow & \overline{A} & \longrightarrow & C/IC & \xrightarrow{\cong} & C'/IC' & \longrightarrow & D/ID \end{array}$$

Observe that our choice of Q gives

$$\begin{aligned} D/ID &= \mathrm{Sym}^*_{C/IC}(\overline{Q} \otimes_{\overline{A}} C/IC) \\ &= \mathrm{Sym}^*_{\overline{A}}(\overline{Q}) \otimes_{\overline{A}} C/IC \\ &= \mathrm{Sym}^*_{\overline{A}}(\overline{Q}) \otimes_{\overline{A}} \mathrm{Sym}^*_{\overline{A}}(\overline{P}) \\ &= \mathrm{Sym}^*_{\overline{A}}(\overline{Q} \oplus \overline{P}) \\ &= \mathrm{Sym}^*_{\overline{A}}(\overline{A}^{\oplus n}) \\ &= \overline{A}[x_1, \ldots, x_n] \end{aligned}$$

Choose $f_1, \ldots, f_n \in D$ which map to $x_1, \ldots, x_n$ in $D/ID = \overline{A}[x_1, \ldots, x_n]$. Set $A = D/(f_1, \ldots, f_n)$. Note that $\overline{A} = A/IA$. By an argument similar to the argument in the first paragraph of the proof we see that $R \to A$ is smooth at all primes of IA. Hence, after replacing A by A_f for a suitable $f \in A$ (see More on Algebra, Lemma 9.4) we win. ☐

We know that any syntomic ring map $R \to A$ is locally a relative global complete intersection, see Algebra, Lemma 126.16. The next lemma says that a vector bundle over $\mathrm{Spec}(A)$ is a relative global complete intersection.

Lemma 4.3. *Let $R \to A$ be a syntomic ring map. Then there exists a smooth R-algebra map $A \to C$ with a retraction such that C is a global relative complete intersection over R, i.e.,*

$$C \cong R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$$

flat over R and all fibres of dimension $n - c$.

Proof. Apply Lemma 4.1 to get $A \to C$. By Algebra, Lemma 126.6 we can write $C = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ with f_i mapping to a basis of J/J^2. The ring map $R \to C$ is syntomic (hence flat) as it is a composition of a syntomic and a smooth ring map. The dimension of the fibres is $n - c$ by Algebra, Lemma 125.4 (the fibres are local complete intersections, so the lemma applies). □

Lemma 4.4. *Let $R \to A$ be a smooth ring map. Then there exists a smooth R-algebra map $A \to B$ with a retraction such that B is standard smooth over R, i.e.,*

$$B \cong R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$$

and $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ is invertible in B.

Proof. Apply Lemma 4.3 to get a smooth R-algebra map $A \to C$ with a retraction such that $C = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection over R. As C is smooth over R we have a short exact sequence

$$0 \to \bigoplus\nolimits_{j=1,\ldots,c} Cf_j \to \bigoplus\nolimits_{i=1,\ldots,n} C\mathrm{d}x_i \to \Omega_{C/R} \to 0$$

Since $\Omega_{C/R}$ is a projective C-module this sequence is split. Choose a left inverse t to the first map. Say $t(\mathrm{d}x_i) = \sum c_{ij} f_j$ so that $\sum_i \frac{\partial f_j}{\partial x_i} c_{i\ell} = \delta_{j\ell}$ (Kronecker delta). Let

$$B' = C[y_1, \ldots, y_c] = R[x_1, \ldots, x_n, y_1, \ldots, y_c]/(f_1, \ldots, f_c)$$

The R-algebra map $C \to B'$ has a retraction given by mapping y_j to zero. We claim that the map

$$R[z_1, \ldots, z_n] \longrightarrow B', \quad z_i \longmapsto x_i - \sum\nolimits_j c_{ij} y_j$$

is étale at every point in the image of $\mathrm{Spec}(C) \to \mathrm{Spec}(B')$. In $\Omega_{B'/R[z_1,\ldots,z_n]}$ we have

$$0 = \mathrm{d}f_j - \sum\nolimits_i \frac{\partial f_j}{\partial x_i}\mathrm{d}z_i \equiv \sum\nolimits_{i,\ell} \frac{\partial f_j}{\partial x_i} c_{i\ell}\mathrm{d}y_\ell \equiv \mathrm{d}y_j \bmod (y_1, \ldots, y_c)\Omega_{B'/R[z_1,\ldots,z_n]}$$

Since $0 = \mathrm{d}z_i = \mathrm{d}x_i$ modulo $\sum B'\mathrm{d}y_j + (y_1, \ldots, y_c)\Omega_{B'/R[z_1,\ldots,z_n]}$ we conclude that

$$\Omega_{B'/R[z_1,\ldots,z_n]}/(y_1, \ldots, y_c)\Omega_{B'/R[z_1,\ldots,z_n]} = 0.$$

As $\Omega_{B'/R[z_1,\ldots,z_n]}$ is a finite B'-module by Nakayama's lemma there exists a $g \in 1 + (y_1, \ldots, y_c)$ that $(\Omega_{B'/R[z_1,\ldots,z_n]})_g = 0$. This proves that $R[z_1, \ldots, z_n] \to B'_g$ is unramified, see Algebra, Definition 139.1. For any ring map $R \to k$ where k is a field we obtain an unramified ring map $k[z_1, \ldots, z_n] \to (B'_g) \otimes_R k$ between smooth k-algebras of dimension n. It follows that $k[z_1, \ldots, z_n] \to (B'_g) \otimes_R k$ is flat by Algebra, Lemmas 120.1 and 130.2. By the critère de platitude par fibre (Algebra, Lemma 120.8) we conclude that $R[z_1, \ldots, z_n] \to B'_g$ is flat. Finally, Algebra, Lemma 133.7 implies that $R[z_1, \ldots, z_n] \to B'_g$ is étale. Set $B = B'_g$. Note that $C \to B$ is smooth and has a retraction, so also $A \to B$ is smooth and has a retraction. Moreover, $R[z_1, \ldots, z_n] \to B$ is étale. By Algebra, Lemma 133.2 we can write

$$B = R[z_1, \ldots, z_n, w_1, \ldots, w_c]/(g_1, \ldots, g_c)$$

with $\det(\partial g_j/\partial w_i)$ invertible in B. This proves the lemma. □

Lemma 4.5. *Let $R \to \Lambda$ be a ring map. If Λ is a filtered colimit of smooth R-algebras, then Λ is a filtered colimit of standard smooth R-algebras.*

Proof. Let $A \to \Lambda$ be an R-algebra map with A of finite presentation over R. According to Lemma 2.1 we have to factor this map through a standard smooth algebra, and we know we can factor it as $A \to B \to \Lambda$ with B smooth over R. Choose an R-algebra map $B \to C$ with a retraction $C \to B$ such that C is standard smooth over R, see Lemma 4.4. Then the desired factorization is $A \to B \to C \to B \to \Lambda$. □

Lemma 4.6. *Let $R \to A$ be a standard smooth ring map. Let $E \subset A$ be a finite subset of order $|E| = n$. Then there exists a presentation $A = R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_c)$ with $c \geq n$, with $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ invertible in A, and such that E is the set of congruence classes of $x_1, \ldots, x_n$.*

Proof. Choose a presentation $A = R[y_1, \ldots, y_m]/(g_1, \ldots, g_d)$ such that the image of $\det(\partial g_j/\partial y_i)_{i,j=1,\ldots,d}$ is invertible in A. Choose an enumerations $E = \{a_1, \ldots, a_n\}$ and choose $h_i \in R[y_1, \ldots, y_m]$ whose image in A is a_i. Consider the presentation

$$A = R[x_1, \ldots, x_n, y_1, \ldots, y_m]/(x_1 - h_1, \ldots, x_n - h_n, g_1, \ldots, g_d)$$

and set $c = n + d$. □

Lemma 4.7. *Let $R \to A$ be a ring map of finite presentation. Let $a \in A$. Consider the following conditions on a:*

(1) *A_a is smooth over R,*
(2) *A_a is smooth over R and $\Omega_{A_a/R}$ is stably free,*
(3) *A_a is smooth over R and $\Omega_{A_a/R}$ is free,*
(4) *A_a is standard smooth over R,*
(5) *a is strictly standard in A over R,*
(6) *a is elementary standard in A over R.*

Then we have

(a) *(4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1),*
(b) *(6) $\Rightarrow$ (5),*
(c) *(6) $\Rightarrow$ (4),*
(d) *(5) $\Rightarrow$ (2),*
(e) *(2) $\Rightarrow$ the elements a^e, $e \geq e_0$ are strictly standard in A over R,*
(f) *(4) $\Rightarrow$ the elements a^e, $e \geq e_0$ are elementary standard in A over R.*

Proof. Part (a) is clear from the definitions and Algebra, Lemma 127.7. Part (b) is clear from Definition 3.3.

Proof of (c). Choose a presentation $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ such that (3.3.1) and (3.3.2) hold. Choose $h \in R[x_1, \ldots, x_n]$ mapping to a. Then

$$A_a = R[x_0, x_1, \ldots, x_n]/(x_0 h - 1, f_1, \ldots, f_n).$$

Write $J = (x_0 h - 1, f_1, \ldots, f_n)$. By (3.3.2) we see that the A_a-module J/J^2 is generated by $x_0 h - 1, f_1, \ldots, f_c$ over A_a. Hence, as in the proof of Algebra, Lemma 126.6, we can choose a $g \in 1 + J$ such that

$$A_a = R[x_0, \ldots, x_n, x_{n+1}]/(x_0 h - 1, f_1, \ldots, f_n, g x_{n+1} - 1).$$

At this point (3.3.1) implies that $R \to A_a$ is standard smooth (use the coordinates $x_0, x_1, \ldots, x_c, x_{n+1}$ to take derivatives).

Proof of (d). Choose a presentation $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ such that (3.3.3) and (3.3.4) hold. We already know that A_a is smooth over R, see Lemma 3.5. As above we get a presentation $A_a = R[x_0, x_1, \ldots, x_n]/J$ with J/J^2 free. Then $\Omega_{A_a/R} \oplus J/J^2 \cong A_a^{\oplus n+1}$ by the definition of smooth ring maps, hence we see that $\Omega_{A_a/R}$ is stably free.

Proof of (e). Choose a presentation $A = R[x_1, \ldots, x_n]/I$ with I finitely generated. By assumption we have a short exact sequence

$$0 \to (I/I^2)_a \to \bigoplus\nolimits_{i=1,\ldots,n} A_a \mathrm{d}x_i \to \Omega_{A_a/R} \to 0$$

which is split exact. Hence we see that $(I/I^2)_a \oplus \Omega_{A_a/R}$ is a free A_a-module. Since $\Omega_{A_a/R}$ is stably free we see that $(I/I^2)_a$ is stably free as well. Thus replacing the presentation chosen above by $A = R[x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+r}]/J$ with $J = (I, x_{n+1}, \ldots, x_{n+r})$ for some r we get that $(J/J^2)_a$ is (finite) free. Choose $f_1, \ldots, f_c \in J$ which map to a basis of $(J/J^2)_a$. Extend this to a list of generators $f_1, \ldots, f_m \in J$. Consider the presentation $A = R[x_1, \ldots, x_{n+r}]/(f_1, \ldots, f_m)$. Then (3.3.4) holds for a^e for all sufficiently large e by construction. Moreover, since $(J/J^2)_a \to \bigoplus_{i=1,\ldots,n} A_a \mathrm{d}x_i$ is a split injection we can find an A_a-linear left inverse. Writing this left inverse in terms of the basis $f_1, \ldots, f_c$ and clearing denominators we find a linear map $\psi_0 : A^{\oplus n} \to A^{\oplus c}$ such that

$$A^{\oplus c} \xrightarrow{(f_1,\ldots,f_c)} J/J^2 \xrightarrow{f \mapsto \mathrm{d}f} \bigoplus\nolimits_{i=1,\ldots,n} A\mathrm{d}x_i \xrightarrow{\psi_0} A^{\oplus c}$$

is multiplication by a^{e_0} for some $e_0 \geq 1$. By Lemma 3.4 we see (3.3.3) holds for all a^{ce_0} and hence for a^e for all e with $e \geq ce_0$.

Proof of (f). Choose a presentation $A_a = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ such that $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ is invertible in A_a. We may assume that for some $m < n$ the classes of the elements $x_1, \ldots, x_m$ correspond $a_i/1$ where $a_1, \ldots, a_m \in A$ are generators of A over R, see Lemma 4.6. After replacing x_i by $a^N x_i$ for $m < i \leq n$ we may assume the class of x_i is $a_i/1 \in A_a$ for some $a_i \in A$. Consider the ring map

$$\Psi : R[x_1, \ldots, x_n] \longrightarrow A, \quad x_i \longmapsto a_i.$$

This is a surjective ring map. By replacing f_j by $a^N f_j$ we may assume that $f_j \in R[x_1, \ldots, x_n]$ and that $\Psi(f_j) = 0$ (since after all $f_j(a_1/1, \ldots, a_n/1) = 0$ in A_a). Let $J = \mathrm{Ker}(\Psi)$. Then $A = R[x_1, \ldots, x_n]/J$ is a presentation and $f_1, \ldots, f_c \in J$ are elements such that $(J/J^2)_a$ is freely generated by $f_1, \ldots, f_c$ and such that $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ maps to an invertible element of A_a. It follows that (3.3.1) and (3.3.2) hold for a^e and all large enough e as desired. □

5. The lifting problem

The goal in this section is to prove (Proposition 5.3) that the collection of algebras which are filtered colimits of smooth algebras is closed under infinitesimal flat deformations. The proof is elementary and only uses the results on presentations of smooth algebras from Section 4.

Lemma 5.1. *Let $R \to \Lambda$ be a ring map. Let $I \subset R$ be an ideal. Assume that*

(1) *$I^2 = 0$, and*
(2) *$\Lambda/I\Lambda$ is a filtered colimit of smooth R/I-algebras.*

Let $\varphi : A \to \Lambda$ be an R-algebra map with A of finite presentation over R. Then there exists a factorization

$$A \to B/J \to \Lambda$$

where B is a smooth R-algebra and $J \subset IB$ is a finitely generated ideal.

Proof. Choose a factorization

$$A/IA \to \bar{B} \to \Lambda/I\Lambda$$

with $\bar{B}$ standard smooth over R/I; this is possible by assumption and Lemma 4.5. Write

$$\bar{B} = A/IA[t_1, \ldots, t_r]/(\bar{g}_1, \ldots, \bar{g}_s)$$

and say $\bar{B} \to \Lambda/I\Lambda$ maps t_i to the class of λ_i modulo $I\Lambda$. Choose $g_1, \ldots, g_s \in A[t_1, \ldots, t_r]$ lifting $\bar{g}_1, \ldots, \bar{g}_s$. Write $\varphi(g_i)(\lambda_1, \ldots, \lambda_r) = \sum \epsilon_{ij}\mu_{ij}$ for some $\epsilon_{ij} \in I$ and $\mu_{ij} \in \Lambda$. Define

$$A' = A[t_1, \ldots, t_r, \delta_{i,j}]/(g_i - \sum \epsilon_{ij}\delta_{ij})$$

and consider the map

$$A' \longrightarrow \Lambda, \quad a \longmapsto \varphi(a), \quad t_i \longmapsto \lambda_i, \quad \delta_{ij} \longmapsto \mu_{ij}$$

We have

$$A'/IA' = A/IA[t_1, \ldots, t_r]/(\bar{g}_1, \ldots, \bar{g}_s)[\delta_{ij}] \cong \bar{B}[\delta_{ij}]$$

This is a standard smooth algebra over R/I as $\bar{B}$ is standard smooth. Choose a presentation $A'/IA' = R/I[x_1, \ldots, x_n]/(\bar{f}_1, \ldots, \bar{f}_c)$ with $\det(\partial \bar{f}_j/\partial x_i)_{i,j=1,\ldots,c}$ invertible in A'/IA'. Choose lifts $f_1, \ldots, f_c \in R[x_1, \ldots, x_n]$ of $\bar{f}_1, \ldots, \bar{f}_c$. Then

$$B = R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, x_{n+1}\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c} - 1)$$

is smooth over R. Since smooth ring maps are formally smooth (Algebra, Proposition 128.13) there exists an R-algebra map $B \to A'$ which is an isomorphism modulo I. Then $B \to A'$ is surjective by Nakayama's lemma (Algebra, Lemma 18.1). Thus $A' = B/J$ with $J \subset IB$ finitely generated (see Algebra, Lemma 6.3). □

Lemma 5.2. *Let $R \to \Lambda$ be a ring map. Let $I \subset R$ be an ideal. Assume that*

1. $I^2 = 0$,
2. *$\Lambda/I\Lambda$ is a filtered colimit of smooth R/I-algebras, and*
3. *$R \to \Lambda$ is flat.*

Let $\varphi : B \to \Lambda$ be an R-algebra map with B smooth over R. Let $J \subset IB$ be a finitely generated ideal. Then there exists R-algebra maps

$$B \xrightarrow{\alpha} B' \xrightarrow{\beta} \Lambda$$

such that B' is smooth over R, such that $\alpha(J) = 0$ and such that $\beta \circ \alpha = \varphi \bmod I\Lambda$.

Proof. If we can prove the lemma in case $J = (h)$, then we can prove the lemma by induction on the number of generators of J. Namely, suppose that J can be generated by n elements $h_1, \ldots, h_n$ and the lemma holds for all cases where J is generated by $n-1$ elements. Then we apply the case $n = 1$ to produce $B \to B' \to \Lambda$ where the first map kills of h_n. Then we let J' be the ideal of B' generated by the images of $h_1, \ldots, h_{n-1}$ and we apply the case for $n-1$ to produce $B' \to B'' \to \Lambda$. It is easy to verify that $B \to B'' \to \Lambda$ does the job.

Assume $J = (h)$ and write $h = \sum \epsilon_i b_i$ for some $\epsilon_i \in I$ and $b_i \in B$. Note that $0 = \varphi(h) = \sum \epsilon_i \varphi(b_i)$. As Λ is flat over R, the equational criterion for flatness

(Algebra, Lemma 36.10) implies that we can find $\lambda_j \in \Lambda$, $j = 1, \ldots, m$ and $a_{ij} \in R$ such that $\varphi(b_i) = \sum_j a_{ij}\lambda_j$ and $\sum_i \epsilon_i a_{ij} = 0$. Set

$$C = B[x_1, \ldots, x_m]/(b_i - \sum a_{ij}x_j)$$

with $C \to \Lambda$ given by φ and $x_j \mapsto \lambda_j$. Choose a factorization

$$C \to B'/J' \to \Lambda$$

as in Lemma 5.1. Since B is smooth over R we can lift the map $B \to C \to B'/J'$ to a map $\psi : B \to B'$. We claim that $\psi(h) = 0$. Namely, the fact that ψ agrees with $B \to C \to B'/J'$ mod I implies that

$$\psi(b_i) = \sum a_{ij}\xi_j + \theta_i$$

for some $\xi_i \in B'$ and $\theta_i \in IB'$. Hence we see that

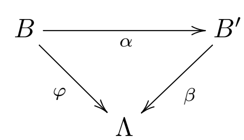
$$\psi(h) = \psi(\sum \epsilon_i b_i) = \sum \epsilon_i a_{ij}\xi_j + \sum \epsilon_i\theta_i = 0$$

because of the relations above and the fact that $I^2 = 0$. □

Proposition 5.3. *Let $R \to \Lambda$ be a ring map. Let $I \subset R$ be an ideal. Assume that*
1. *I is nilpotent,*
2. *$\Lambda/I\Lambda$ is a filtered colimit of smooth R/I-algebras, and*
3. *$R \to \Lambda$ is flat.*

Then Λ is a colimit of smooth R-algebras.

Proof. Since $I^n = 0$ for some n, it follows by induction on n that it suffices to consider the case where $I^2 = 0$. Let $\varphi : A \to \Lambda$ be an R-algebra map with A of finite presentation over R. We have to find a factorization $A \to B \to \Lambda$ with B smooth over R, see Lemma 2.1. By Lemma 5.1 we may assume that $A = B/J$ with B smooth over R and $J \subset IB$ a finitely generated ideal. By Lemma 5.2 we can find a (possibly noncommutative) diagram



of R-algebras which commutes modulo I and such that $\alpha(J) = 0$. The map

$$D : B \longrightarrow I\Lambda, \quad b \longmapsto \varphi(b) - \beta(\alpha(b))$$

is a derivation over R hence we can write it as $D = \xi \circ \mathrm{d}_{B/R}$ for some B-linear map $\xi : \Omega_{B/R} \to I\Lambda$. Since $\Omega_{B/R}$ is a finite projective B-module we can write $\xi = \sum_{i=1,\ldots,n} \epsilon_i \Xi_i$ for some $\epsilon_i \in I$ and B-linear maps $\Xi_i : \Omega_{B/R} \to \Lambda$. (Details omitted. Hint: write $\Omega_{B/R}$ as a direct sum of a finite free module to reduce to the finite free case.) We define

$$B'' = \mathrm{Sym}^*_{B'}\left(\bigoplus\nolimits_{i=1,\ldots,n} \Omega_{B/R} \otimes_{B,\alpha} B'\right)$$

and we define $\beta' : B'' \to \Lambda$ by β on B' and by

$$\beta'|_{i\text{th summand } \Omega_{B/R} \otimes_{B,\alpha} B'} = \Xi_i \otimes \beta$$

and $\alpha' : B \to B''$ by

$$\alpha'(b) = \alpha(b) \oplus \sum \epsilon_i \mathrm{d}_{B/R}(b) \otimes 1 \oplus 0 \oplus \ldots$$

At this point the diagram

$$\begin{array}{ccc} B & \xrightarrow{\alpha'} & B'' \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\beta'} & \\ & \Lambda & \end{array}$$

does commute. Moreover, it is direct from the definitions that $\alpha'(J) = 0$ as $I^2 = 0$. Hence the desired factorization. □

6. The lifting lemma

Here is a fiendishly clever lemma.

Lemma 6.1. *Let R be a Noetherian ring. Let Λ be an R-algebra. Let $\pi \in R$ and assume that $Ann_R(\pi) = Ann_R(\pi^2)$ and $Ann_\Lambda(\pi) = Ann_\Lambda(\pi^2)$. Suppose we have R-algebra maps $R/\pi^2R \to \bar{C} \to \Lambda/\pi^2\Lambda$ with $\bar{C}$ of finite presentation. Then there exists an R-algebra homomorphism $D \to \Lambda$ and a commutative diagram*

$$\begin{array}{ccccc} R/\pi^2R & \longrightarrow & \bar{C} & \longrightarrow & \Lambda/\pi^2\Lambda \\ \downarrow & & \downarrow & & \downarrow \\ R/\pi R & \longrightarrow & D/\pi D & \longrightarrow & \Lambda/\pi\Lambda \end{array}$$

with the following properties

(a) *D is of finite presentation,*
(b) *$R \to D$ is smooth at any prime $\mathfrak{q}$ with $\pi \notin \mathfrak{q}$,*
(c) *$R \to D$ is smooth at any prime $\mathfrak{q}$ with $\pi \in \mathfrak{q}$ lying over a prime of $\bar{C}$ where $R/\pi^2R \to \bar{C}$ is smooth, and*
(d) *$\bar{C}/\pi\bar{C} \to D/\pi D$ is smooth at any prime lying over a prime of $\bar{C}$ where $R/\pi^2R \to \bar{C}$ is smooth.*

Proof. We choose a presentation

$$\bar{C} = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$$

We also denote $I = (f_1, \ldots, f_m)$ and $\bar{I}$ the image of I in $R/\pi^2R[x_1, \ldots, x_n]$. Since R is Noetherian, so is $\bar{C}$. Hence the smooth locus of $R/\pi^2R \to \bar{C}$ is quasi-compact, see Topology, Lemma 6.2. Applying Lemma 3.2 we may choose a finite list of elements $a_1, \ldots, a_r \in R[x_1, \ldots, x_n]$ such that

(1) the union of the open subspaces $\text{Spec}(\bar{C}_{a_k}) \subset \text{Spec}(\bar{C})$ cover the smooth locus of $R/\pi^2R \to \bar{C}$, and
(2) for each $k = 1, \ldots, r$ there exists a finite subset $E_k \subset \{1, \ldots, m\}$ such that $(\bar{I}/\bar{I}^2)_{a_k}$ is freely generated by the classes of f_j, $j \in E_k$.

Set $I_k = (f_j, j \in E_k) \subset I$ and denote $\bar{I}_k$ the image of I_k in $R/\pi^2R[x_1, \ldots, x_n]$. By (2) and Nakayama's lemma we see that $(\bar{I}/\bar{I}_k)_{a_k}$ is annihilated by $1 + b'_k$ for some $b'_k \in \bar{I}_{a_k}$. Suppose b'_k is the image of $b_k/(a_k)^N$ for some $b_k \in I$ and some integer N. After replacing a_k by $a_k b_k$ we get

(3) $(\bar{I}_k)_{a_k} = (\bar{I})_{a_k}$.

Thus, after possibly replacing a_k by a high power, we may write

(4) $a_k f_\ell = \sum_{j \in E_k} h^j_{k,\ell} f_j + \pi^2 g_{k,\ell}$

for any $\ell \in \{1, \ldots, m\}$ and some $h^j_{i,\ell}, g_{i,\ell} \in R[x_1, \ldots, x_n]$. If $\ell \in E_k$ we choose $h^j_{k,\ell} = a_k \delta_{\ell,j}$ (Kronecker delta) and $g_{k,\ell} = 0$. Set

$$D = R[x_1, \ldots, x_n, z_1, \ldots, z_m]/(f_j - \pi z_j, p_{k,\ell}).$$

Here $j \in \{1, \ldots, m\}$, $k \in \{1, \ldots, r\}$, $\ell \in \{1, \ldots, m\}$, and

$$p_{k,\ell} = a_k z_\ell - \sum\nolimits_{j \in E_k} h^j_{k,\ell} z_j - \pi g_{k,\ell}.$$

Note that for $\ell \in E_k$ we have $p_{k,\ell} = 0$ by our choices above.

The map $R \to D$ is the given one. Say $\bar{C} \to \Lambda/\pi^2\Lambda$ maps x_i to the class of λ_i modulo π^2. For an element $f \in R[x_1, \ldots, x_n]$ we denote $f(\lambda) \in \Lambda$ the result of subsituting λ_i for x_i. Then we know that $f_j(\lambda) = \pi^2 \mu_j$ for some $\mu_j \in \Lambda$. Define $D \to \Lambda$ by the rules $x_i \mapsto \lambda_i$ and $z_j \mapsto \pi\mu_j$. This is well defined because

$$\begin{aligned} p_{k,\ell} &\mapsto a_k(\lambda)\pi\mu_\ell - \sum\nolimits_{j \in E_k} h^j_{k,\ell}(\lambda)\pi\mu_j - \pi g_{k,\ell}(\lambda) \\ &= \pi\left(a_k(\lambda)\mu_\ell - \sum\nolimits_{j \in E_k} h^j_{k,\ell}(\lambda)\mu_j - g_{k,\ell}(\lambda)\right) \end{aligned}$$

Substituting $x_i = \lambda_i$ in (4) above we see that the expression inside the brackets is annihilated by π^2, hence it is annihilated by π as we have assumed $\mathrm{Ann}_\Lambda(\pi) = \mathrm{Ann}_\Lambda(\pi^2)$. The map $\bar{C} \to D/\pi D$ is determined by $x_i \mapsto x_i$ (clearly well defined). Thus we are done if we can prove (b), (c), and (d).

Using (4) we obtain the following key equality

$$\begin{aligned} \pi p_{k,\ell} &= \pi a_k z_\ell - \sum\nolimits_{j \in E_k} \pi h^j_{k,\ell} z_j - \pi^2 g_{k,\ell} \\ &= -a_k(f_\ell - \pi z_\ell) + a_k f_\ell + \sum\nolimits_{j \in E_k} h^j_{k,\ell}(f_j - \pi z_j) - \sum\nolimits_{j \in E_k} h^j_{k,\ell} f_j - \pi^2 g_{k,\ell} \\ &= -a_k(f_\ell - \pi z_\ell) + \sum\nolimits_{j \in E_k} h^j_{k,\ell}(f_j - \pi z_j) \end{aligned}$$

The end result is an element of the ideal generated by $f_j - \pi z_j$. In particular, we see that $D[1/\pi]$ is isomorphic to $R[1/\pi][x_1, \ldots, x_n, z_1, \ldots, z_m]/(f_j - \pi z_j)$ which is isomorphic to $R[1/\pi][x_1, \ldots, x_n]$ hence smooth over R. This proves (b).

For fixed $k \in \{1, \ldots, r\}$ consider the ring

$$D_k = R[x_1, \ldots, x_n, z_1, \ldots, z_m]/(f_j - \pi z_j, j \in E_k, p_{k,\ell})$$

The number of equations is $m = |E_k| + (m - |E_k|)$ as $p_{k,\ell}$ is zero if $\ell \in E_k$. Also, note that

$$\begin{aligned} (D_k/\pi D_k)_{a_k} &= R/\pi R[x_1, \ldots, x_n, 1/a_k, z_1, \ldots, z_m]/(f_j, j \in E_k, p_{k,\ell}) \\ &= (\bar{C}/\pi\bar{C})_{a_k}[z_1, \ldots, z_m]/(a_k z_\ell - \sum\nolimits_{j \in E_k} h^j_{k,\ell} z_j) \\ &\cong (\bar{C}/\pi\bar{C})_{a_k}[z_j, j \in E_k] \end{aligned}$$

In particular $(D_k/\pi D_k)_{a_k}$ is smooth over $(\bar{C}/\pi\bar{C})_{a_k}$. By our choice of a_k we have that $(\bar{C}/\pi\bar{C})_{a_k}$ is smooth over $R/\pi R$ of relative dimension $n - |E_k|$, see (2). Hence for a prime $\mathfrak{q}_k \subset D_k$ containing π and lying over $\mathrm{Spec}(\bar{C}_{a_k})$ the fibre ring of $R \to D_k$ is smooth at $\mathfrak{q}_k$ of dimension n. Thus $R \to D_k$ is syntomic at $\mathfrak{q}_k$ by our count of the number of equations above, see Algebra, Lemma 126.11. Hence $R \to D_k$ is smooth at $\mathfrak{q}_k$, see Algebra, Lemma 127.16.

To finish the proof, let $\mathfrak{q} \subset D$ be a prime containing π lying over a prime where $R/\pi^2 R \to \bar{C}$ is smooth. Then $a_k \notin \mathfrak{q}$ for some k by (1). We will show that the surjection $D_k \to D$ induces an isomorphism on local rings at $\mathfrak{q}$. Since we know that the ring maps $\bar{C}/\pi\bar{C} \to D_k/\pi D_k$ and $R \to D_k$ are smooth at the corresponding prime $\mathfrak{q}_k$ by the preceding paragraph this will prove (c) and (d) and thus finish the proof.

First, note that for any ℓ the equation $\pi p_{k,\ell} = -a_k(f_\ell - \pi z_\ell) + \sum_{j\in E_k} h_{k,\ell}^j (f_j - \pi z_j)$ proved above shows that $f_\ell - \pi z_\ell$ maps to zero in $(D_k)_{a_k}$ and in particular in $(D_k)_{\mathfrak{q}_k}$. The relations (4) imply that $a_k f_\ell = \sum_{j\in E_k} h_{k,\ell}^j f_j$ in I/I^2. Since $(\bar{I}_k/\bar{I}_k^2)_{a_k}$ is free on f_j, $j \in E_k$ we see that

$$a_{k'} h_{k,\ell}^j - \sum\nolimits_{j'\in E_{k'}} h_{k',\ell}^{j'} h_{k,j'}^j$$

is zero in $\bar{C}_{a_k}$ for every k, k', ℓ and $j \in E_k$. Hence we can find a large integer N such that

$$a_k^N \left(a_{k'} h_{k,\ell}^j - \sum\nolimits_{j'\in E_{k'}} h_{k',\ell}^{j'} h_{k,j'}^j \right)$$

is in $I_k + \pi^2 R[x_1, \ldots, x_n]$. Computing modulo π we have

$$\begin{aligned} &a_k p_{k',\ell} - a_{k'} p_{k,\ell} + \sum h_{k',\ell}^{j'} p_{k,j'} \\ &= -a_k \sum h_{k',\ell}^{j'} z_{j'} + a_{k'} \sum h_{k,\ell}^j z_j + \sum h_{k',\ell}^{j'} a_k z_{j'} - \sum\sum h_{k',\ell}^{j'} h_{k,j'}^j z_j \\ &= \sum \left(a_{k'} h_{k,\ell}^j - \sum h_{k',\ell}^{j'} h_{k,j'}^j \right) z_j \end{aligned}$$

with Einstein summation convention. Combining with the above we see $a_k^{N+1} p_{k',\ell}$ is contained in the ideal generated by I_k and π in $R[x_1, \ldots, x_n, z_1, \ldots, z_m]$. Thus $p_{k',\ell}$ maps into $\pi(D_k)_{a_k}$. On the other hand, the equation

$$\pi p_{k',\ell} = -a_{k'}(f_\ell - \pi z_\ell) + \sum\nolimits_{j'\in E_{k'}} h_{k',\ell}^{j'} (f_{j'} - \pi z_{j'})$$

shows that $\pi p_{k',\ell}$ is zero in $(D_k)_{a_k}$. Since we have assumed that $\mathrm{Ann}_R(\pi) = \mathrm{Ann}_R(\pi^2)$ and since $(D_k)_{\mathfrak{q}_k}$ is smooth hence flat over R we see that $\mathrm{Ann}_{(D_k)_{\mathfrak{q}_k}}(\pi) = \mathrm{Ann}_{(D_k)_{\mathfrak{q}_k}}(\pi^2)$. We conclude that $p_{k',\ell}$ maps to zero as well, hence $D_\mathfrak{q} = (D_k)_{\mathfrak{q}_k}$ and we win. □

7. The desingularization lemma

Here is another fiendishly clever lemma.

Lemma 7.1. *Let R be a Noetherian ring. Let Λ be an R-algebra. Let $\pi \in R$ and assume that $Ann_\Lambda(\pi) = Ann_\Lambda(\pi^2)$. Let $A \to \Lambda$ be an R-algebra map with A of finite presentation. Assume*

1. *the image of π is strictly standard in A over R, and*
2. *there exists a section $\rho : A/\pi^4 A \to R/\pi^4 R$ which is compatible with the map to $\Lambda/\pi^4\Lambda$.*

Then we can find R-algebra maps $A \to B \to \Lambda$ with B of finite presentation such that $\mathfrak{a}B \subset H_{B/R}$ where $\mathfrak{a} = Ann_R(Ann_R(\pi^2)/Ann_R(\pi))$.

Proof. Choose a presentation

$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$$

and $0 \leq c \leq \min(n, m)$ such that (3.3.3) holds for π and such that

$$\pi f_{c+j} \in (f_1, \ldots, f_c) + (f_1, \ldots, f_m)^2 \quad (7.1.1)$$

for $j = 1, \ldots, m-c$. Say ρ maps x_i to the class of $r_i \in R$. Then we can replace x_i by $x_i - r_i$. Hence we may assume $\rho(x_i) = 0$ in $R/\pi^4 R$. This implies that $f_j(0) \in \pi^4 R$ and that $A \to \Lambda$ maps x_i to $\pi^4 \lambda_i$ for some $\lambda_i \in \Lambda$. Write

$$f_j = f_j(0) + \sum\nolimits_{i=1,\ldots,n} r_{ji} x_i + \text{h.o.t.}$$

This implies that the constant term of $\partial f_j/\partial x_i$ is r_{ji}. Apply ρ to (3.3.3) for π and we see that

$$\pi = \sum\nolimits_{I \subset \{1,\ldots,n\},\ |I|=c} r_I \det(r_{ji})_{j=1,\ldots,c,\ i \in I} \bmod \pi^4 R$$

for some $r_I \in R$. Thus we have

$$u\pi = \sum\nolimits_{I \subset \{1,\ldots,n\},\ |I|=c} r_I \det(r_{ji})_{j=1,\ldots,c,\ i \in I}$$

for some $u \in 1 + \pi^3 R$. By Algebra, Lemma 14.5 this implies there exists a $n \times c$ matrix (s_{ik}) such that

$$u\pi\delta_{jk} = \sum\nolimits_{i=1,\ldots,n} r_{ji} c_{ik} \quad \text{for all } j, k = 1, \ldots, c$$

(Kronecker delta). We introduce auxiliary variables $v_1, \ldots, v_c, w_1, \ldots, w_n$ and we set

$$h_i = x_i - \pi^2 \sum\nolimits_{j=1,\ldots c} s_{ij} v_j - \pi^3 w_i$$

In the following we will use that

$$R[x_1, \ldots, x_n, v_1, \ldots, v_c, w_1, \ldots, w_n]/(h_1, \ldots, h_n) = R[v_1, \ldots, v_c, w_1, \ldots, w_n]$$

without further mention. In $R[x_1, \ldots, x_n, v_1, \ldots, v_c, w_1, \ldots, w_n]/(h_1, \ldots, h_n)$ we have

$$\begin{aligned} f_j &= f_j(x_1 - h_1, \ldots, x_n - h_n) \\ &= \sum\nolimits_i \pi^2 r_{ji} s_{ik} v_k + \sum\nolimits_i \pi^3 r_{ji} w_i \bmod \pi^4 \\ &= \pi^3 v_j + \sum \pi^3 r_{ji} w_i \bmod \pi^4 \end{aligned}$$

for $1 \leq j \leq c$. Hence we can choose elements $g_j \in R[v_1, \ldots, v_c, w_1, \ldots, w_n]$ such that $g_j = v_j + \sum r_{ji} w_i \bmod \pi$ and such that $f_j = \pi^3 g_j$ in the R-algebra $R[x_1, \ldots, x_n, v_1, \ldots, v_c, w_1, \ldots, w_n]/(h_1, \ldots, h_n)$. We set

$$B = R[x_1, \ldots, x_n, v_1, \ldots, v_c, w_1, \ldots, w_n]/(f_1, \ldots, f_n, h_1, \ldots, h_n, g_1, \ldots, g_c).$$

The map $A \to B$ is clear. We define $B \to \Lambda$ by mapping $x_i \to \pi^4 \lambda_i$, $v_i \mapsto 0$, and $w_i \mapsto \pi \lambda_i$. Then it is clear that the elements f_j and h_i are mapped to zero in Λ. Moreover, it is clear that g_i is mapped to an element t of $\pi\Lambda$ such that $\pi^3 t = 0$ (as $f_i = \pi^3 g_i$ modulo the ideal generated by the h's). Hence our assumption that $\mathrm{Ann}_\Lambda(\pi) = \mathrm{Ann}_\Lambda(\pi^2)$ implies that $t = 0$. Thus we are done if we can prove the statement about smoothness.

Note that $B_\pi \cong A_\pi[v_1, \ldots, v_c]$ because the equations $g_i = 0$ are implied by $f_i = 0$. Hence B_π is smooth over R as A_π is smooth over R by the assumption that π is strictly standard in A over R, see Lemma 3.5.

Set $B' = R[v_1, \ldots, v_c, w_1, \ldots, w_n]/(g_1, \ldots, g_c)$. As $g_i = v_i + \sum r_{ji} w_i \bmod \pi$ we see that $B'/\pi B' = R/\pi R[w_1, \ldots, w_n]$. Hence $R \to B'$ is smooth of relative dimension n at every point of $V(\pi)$ by Algebra, Lemmas 126.11 and 127.16 (the first lemma shows it is syntomic at those primes, in particular flat, whereupon the second lemma shows it is smooth).

Let $\mathfrak{q} \subset B$ be a prime with $\pi \in \mathfrak{q}$ and for some $r \in \mathfrak{a}$, $r \notin \mathfrak{q}$. Denote $\mathfrak{q}' = B' \cap \mathfrak{q}$. We claim the surjection $B' \to B$ induces an isomorphism of local rings $(B')_{\mathfrak{q}'} \to B_{\mathfrak{q}}$. This will conclude the proof of the lemma. Note that $B_{\mathfrak{q}}$ is the quotient of $(B')_{\mathfrak{q}'}$ by the ideal generated by f_{c+j}, $j = 1, \ldots, m - c$. We observe two things: first the image of f_{c+j} in $(B')_{\mathfrak{q}'}$ is divisible by π^2 and second the image of πf_{c+j} in $(B')_{\mathfrak{q}'}$ can be written as $\sum b_{j_1 j_2} f_{c+j_1} f_{c+j_2}$ by (7.1.1). Thus we see that the image of each πf_{c+j} is contained in the ideal generated by the elements $\pi^2 f_{c+j'}$. Hence $\pi f_{c+j} = 0$ in $(B')_{\mathfrak{q}'}$ as this is a Noetherian local ring, see Algebra, Lemma 48.6. As $R \to (B')_{\mathfrak{q}'}$ is flat we see that

$$\big(\text{Ann}_R(\pi^2)/\text{Ann}_R(\pi)\big) \otimes_R (B')_{\mathfrak{q}'} = \text{Ann}_{(B')_{\mathfrak{q}'}}(\pi^2)/\text{Ann}_{(B')_{\mathfrak{q}'}}(\pi)$$

Because $r \in \mathfrak{a}$ is invertible in $(B')_{\mathfrak{q}'}$ we see that this module is zero. Hence we see that the image of f_{c+j} is zero in $(B')_{\mathfrak{q}'}$ as desired. □

Lemma 7.2. *Let R be a Noetherian ring. Let Λ be an R-algebra. Let $\pi \in R$ and assume that $Ann_R(\pi) = Ann_R(\pi^2)$ and $Ann_\Lambda(\pi) = Ann_\Lambda(\pi^2)$. Let $A \to \Lambda$ and $D \to \Lambda$ be R-algebra maps with A and D of finite presentation. Assume*

1. *π is strictly standard in A over R, and*
2. *there exists an R-algebra map $A/\pi^4 A \to D/\pi^4 D$ compatible with the maps to $\Lambda/\pi^4\Lambda$.*

Then we can find an R-algebra map $B \to \Lambda$ with B of finite presentation and R-algebra maps $A \to B$ and $D \to B$ compatible with the maps to Λ such that $H_{D/R}B \subset H_{B/D}$ and $H_{D/R}B \subset H_{B/R}$.

Proof. We apply Lemma 7.1 to

$$D \longrightarrow A \otimes_R D \longrightarrow \Lambda$$

and the image of π in D. By Lemma 3.7 we see that π is strictly standard in $A \otimes_R D$ over D. As our section $\rho : (A \otimes_R D)/\pi^4(A \otimes_R D) \to D/\pi^4 D$ we take the map induced by the map in (2). Thus Lemma 7.1 applies and we obtain a factorization $A \otimes_R D \to B \to \Lambda$ with B of finite presentation and $\mathfrak{a}B \subset H_{B/D}$ where

$$\mathfrak{a} = \text{Ann}_D(\text{Ann}_D(\pi^2)/\text{Ann}_D(\pi)).$$

For any prime $\mathfrak{q}$ of D such that $D_{\mathfrak{q}}$ is flat over R we have $\text{Ann}_{D_{\mathfrak{q}}}(\pi^2)/\text{Ann}_{D_{\mathfrak{q}}}(\pi) = 0$ because annihilators of elements commutes with flat base change and we assumed $\text{Ann}_R(\pi) = \text{Ann}_R(\pi^2)$. Because D is Noetherian we see that $\text{Ann}_D(\pi^2)/\text{Ann}_D(\pi)$ is a finite D-module, hence formation of its annihilator commutes with localization. Thus we see that $\mathfrak{a} \not\subset \mathfrak{q}$. Hence we see that $D \to B$ is smooth at any prime of B lying over $\mathfrak{q}$. Since any prime of D where $R \to D$ is smooth is one where $D_{\mathfrak{q}}$ is flat over R we conclude that $H_{D/R}B \subset H_{B/D}$. The final inclusion $H_{D/R}B \subset H_{B/R}$

follows because compositions of smooth ring maps are smooth (Algebra, Lemma 127.14). □

Lemma 7.3. *Let R be a Noetherian ring. Let Λ be an R-algebra. Let $\pi \in R$ and assume that $Ann_R(\pi) = Ann_R(\pi^2)$ and $Ann_\Lambda(\pi) = Ann_\Lambda(\pi^2)$. Let $A \to \Lambda$ be an R-algebra map with A of finite presentation and assume π is strictly standard in A over R. Let*

$$A/\pi^8 A \to \bar{C} \to \Lambda/\pi^8\Lambda$$

be a factorization with $\bar{C}$ of finite presentation. Then we can find a factorization $A \to B \to \Lambda$ with B of finite presentation such that $R_\pi \to B_\pi$ is smooth and such that

$$H_{\bar{C}/(R/\pi^8 R)} \cdot \Lambda/\pi^8\Lambda \subset \sqrt{H_{B/R}\Lambda} \bmod \pi^8\Lambda.$$

Proof. Apply Lemma 6.1 to get $R \to D \to \Lambda$ with a factorization $\bar{C}/\pi^4\bar{C} \to D/\pi^4 D \to \Lambda/\pi^4\Lambda$ such that $R \to D$ is smooth at any prime not containing π and at any prime lying over a prime of $\bar{C}/\pi^4\bar{C}$ where $R/\pi^8 R \to \bar{C}$ is smooth. By Lemma 7.2 we can find a finitely presented R-algebra B and factorizations $A \to B \to \Lambda$ and $D \to B \to \Lambda$ such that $H_{D/R}B \subset H_{B/R}$. We omit the verification that this is a solution to the problem posed by the lemma. □

8. Warmup: reduction to a base field

In this section we apply the lemmas in the previous sections to prove that it suffices to prove the main result when the base ring is a field, see Lemma 8.4.

Situation 8.1. Here $R \to \Lambda$ is a regular ring map of Noetherian rings.

Let $R \to \Lambda$ be as in Situation 8.1. We say *PT holds for* $R \to \Lambda$ if Λ is a filtered colimit of smooth R-algebras.

Lemma 8.2. *Let $R_i \to \Lambda_i$, $i = 1, 2$ be as in Situation 8.1. If PT holds for $R_i \to \Lambda_i$, $i = 1, 2$, then PT holds for $R_1 \times R_2 \to \Lambda_1 \times \Lambda_2$.*

Proof. Omitted. Hint: A product of colimits is a colimit. □

Lemma 8.3. *Let $R \to A \to \Lambda$ be ring maps with A of finite presentation over R. Let $S \subset R$ be a multiplicative set. Let $S^{-1}A \to B' \to S^{-1}\Lambda$ be a factorization with B' smooth over $S^{-1}R$. Then we can find a factorization $A \to B \to \Lambda$ such that some $s \in S$ maps to an elementary standard element in B over R.*

Proof. We first apply Lemma 4.4 to $S^{-1}R \to B'$. Thus we may assume B' is standard smooth over $S^{-1}R$. Write $A = R[x_1, \ldots, x_n]/(g_1, \ldots, g_t)$ and say $x_i \mapsto \lambda_i$ in Λ. We may write $B' = S^{-1}R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_c)$ for some $c \geq n$ where $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ is invertible in B' and such that $A \to B'$ is given by $x_i \mapsto x_i$, see Lemma 4.6. After multiplying x_i, $i > n$ by an element of S and correspondingly modifying the equations f_j we may assume $B' \to S^{-1}\Lambda$ maps x_i to $\lambda_i/1$ for some $\lambda_i \in \Lambda$ for $i > n$. Choose a relation

$$1 = a_0 \det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c} + \sum\nolimits_{j=1,\ldots,c} a_j f_j$$

for some $a_j \in S^{-1}R[x_1, \ldots, x_{n+m}]$. Since each element of S is invertible in B' we may (by clearing denominators) assume that $f_j, a_j \in R[x_1, \ldots, x_{n+m}]$ and that

$$s_0 = a_0 \det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c} + \sum\nolimits_{j=1,\ldots,c} a_j f_j$$

for some $s_0 \in S$. Since g_j maps to zero in $S^{-1}R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, x_c)$ we can find elements $s_j \in S$ such that $s_j g_j = 0$ in $R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_c)$. Since f_j maps to zero in $S^{-1}\Lambda$ we can find $s'_j \in S$ such that $s'_j f_j(\lambda_1, \ldots, \lambda_{n+m}) = 0$ in Λ. Consider the ring

$$B = R[x_1, \ldots, x_{n+m}]/(s'_1 f_1, \ldots, s'_c f_c, g_1, \ldots, g_t)$$

and the factorization $A \to B \to \Lambda$ with $B \to \Lambda$ given by $x_i \mapsto \lambda_i$. We claim that $s = s_0 s_1 \ldots s_t s'_1 \ldots s'_c$ is elementary standard in B over R which finishes the proof. Namely, $s_j g_j \in (f_1, \ldots, f_c)$ and hence $s g_j \in (s'_1 f_1, \ldots, s'_c f_c)$. Finally, we have

$$a_0 \det(\partial s'_j f_j / \partial x_i)_{i,j=1,\ldots,c} + \sum\nolimits_{j=1,\ldots,c} (s'_1 \ldots \hat{s'_j} \ldots s'_c) a_j s'_j f_j = s_0 s'_1 \ldots s'_c$$

which divides s as desired. □

Lemma 8.4. *If for every Situation 8.1 where R is a field PT holds, then PT holds in general.*

Proof. Assume PT holds for any Situation 8.1 where R is a field. Let $R \to \Lambda$ be as in Situation 8.1 arbitrary. Note that $R/I \to \Lambda/I\Lambda$ is another regular ring map of Noetherian rings, see More on Algebra, Lemma 31.3. Consider the set of ideals

$$\mathcal{I} = \{I \subset R \mid R/I \to \Lambda/I\Lambda \text{ does not have PT}\}$$

We have to show that $\mathcal{I}$ is empty. If this set is nonempty, then it contains a maxmimal element because R is Noetherian. Replacing R by R/I and Λ by Λ/I we obtain a situation where PT holds for $R/I \to \Lambda/I\Lambda$ for any nonzero ideal of R. In particular, we see by applying Proposition 5.3 that R is a reduced ring.

Let $A \to \Lambda$ be an R-algebra homomorphism with A of finite presentation. We have to find a factorization $A \to B \to \Lambda$ with B smooth over R, see Lemma 2.1.

Let $S \subset R$ be the set of nonzerodivisors and consider the total ring of fractions $Q = S^{-1}R$ of R. We know that $Q = K_1 \times \ldots \times K_n$ is a product of fields, see Algebra, Lemmas 23.2 and 29.6. By Lemma 8.2 and our assumption PT holds for the ring map $S^{-1}R \to S^{-1}\Lambda$. Hence we can find a factorization $S^{-1}A \to B' \to \Lambda$ with B' smooth over $S^{-1}R$.

We apply Lemma 8.3 and find a factorization $A \to B \to \Lambda$ such that some $\pi \in S$ is elementary standard in B over R. After replacing A by B we may assume that π is elementary standard, hence strictly standard in A. We know that $R/\pi^8 R \to \Lambda/\pi^8\Lambda$ satisfies PT. Hence we can find a factorization $R/\pi^8 R \to A/\pi^8 A \to \bar{C} \to \Lambda/\pi^8\Lambda$ with $R/\pi^8 R \to \bar{C}$ smooth. By Lemma 6.1 we can find an R-algebra map $D \to \Lambda$ with D smooth over R and a factorization $R/\pi^4 R \to A/\pi^4 A \to D/\pi^4 D \to \Lambda/\pi^4\Lambda$. By Lemma 7.2 we can find $A \to B \to \Lambda$ with B smooth over R which finishes the proof. □

9. Local tricks

Situation 9.1. We are given a Noetherian ring R and an R-algebra map $A \to \Lambda$ and a prime $\mathfrak{q} \subset \Lambda$. We assume A is of finite presentation over R. In this situation we denote $\mathfrak{h}_A = \sqrt{H_{A/R}\Lambda}$.

Let $R \to A \to \Lambda \supset \mathfrak{q}$ be as in Situation 9.1. We say $R \to A \to \Lambda \supset \mathfrak{q}$ *can be resolved* if there exists a factorization $A \to B \to \Lambda$ with B of finite presentation and $\mathfrak{h}_A \subset \mathfrak{h}_B \not\subset \mathfrak{q}$. In this case we will call the factorization $A \to B \to \Lambda$ a *resolution of* $R \to A \to \Lambda \supset \mathfrak{q}$.

Lemma 9.2. *Let* $R \to A \to \Lambda \supset \mathfrak{q}$ *be as in Situation 9.1. Let* $r \geq 1$ *and* $\pi_1, \ldots, \pi_r \in R$ *map to elements of* $\mathfrak{q}$. *Assume*

(1) *for* $i = 1, \ldots, r$ *we have*

$$Ann_{R/(\pi_1^8,\ldots,\pi_{i-1}^8)R}(\pi_i) = Ann_{R/(\pi_1^8,\ldots,\pi_{i-1}^8)R}(\pi_i^2)$$

and

$$Ann_{\Lambda/(\pi_1^8,\ldots,\pi_{i-1}^8)\Lambda}(\pi_i) = Ann_{\Lambda/(\pi_1^8,\ldots,\pi_{i-1}^8)\Lambda}(\pi_i^2)$$

(2) *for* $i = 1, \ldots, r$ *the element* π_i *maps to a strictly standard element in* A *over* R.

Then, if

$$R/(\pi_1^8, \ldots, \pi_r^8)R \to A/(\pi_1^8, \ldots, \pi_r^8)A \to \Lambda/(\pi_1^8, \ldots, \pi_r^8)\Lambda \supset \mathfrak{q}/(\pi_1^8, \ldots, \pi_r^8)\Lambda$$

can be resolved, so can $R \to A \to \Lambda \supset \mathfrak{q}$.

Proof. We are going to prove this by induction on r.

The case $r = 1$. Here the assumption is that there exists a factorization $A/\pi_1^8 \to \bar{C} \to \Lambda/\pi_1^8$ which resolves the situation modulo π_1^8. Conditions (1) and (2) are the assumptions needed to apply Lemma 7.3. Thus we can “lift” the resolution $\bar{C}$ to a resolution of $R \to A \to \Lambda \supset \mathfrak{q}$.

The case $r > 1$. In this case we apply the induction hypothesis for $r - 1$ to the situation $R/\pi_1^8 \to A/\pi_1^8 \to \Lambda/\pi_1^8 \supset \mathfrak{q}/\pi_1^8\Lambda$. Note that property (2) is preserved by Lemma 3.7. □

Lemma 9.3. *Let* $R \to A \to \Lambda \supset \mathfrak{q}$ *be as in Situation 9.1. Let* $\mathfrak{p} = R \cap \mathfrak{q}$. *Assume that* $\mathfrak{q}$ *is minimal over* $\mathfrak{h}_A$ *and that* $R_\mathfrak{p} \to A_\mathfrak{p} \to \Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}$ *can be resolved. Then there exists a factorization* $A \to C \to \Lambda$ *with* C *of finite presentation such that* $H_{C/R}\Lambda \not\subset \mathfrak{q}$.

Proof. Let $A_\mathfrak{p} \to C \to \Lambda_\mathfrak{q}$ be a resolution of $R_\mathfrak{p} \to A_\mathfrak{p} \to \Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}$. By our assumption that $\mathfrak{q}$ is minimal over $\mathfrak{h}_A$ this means that $H_{C/R_\mathfrak{p}}\Lambda_\mathfrak{q} = \Lambda_\mathfrak{q}$. By Lemma 3.8 we may assume that C is smooth over $\Lambda_\mathfrak{p}$. By Lemma 4.4 we may assume that C is standard smooth over $R_\mathfrak{p}$. Write $A = R[x_1, \ldots, x_n]/(g_1, \ldots, g_t)$ and say $A \to \Lambda$ is given by $x_i \mapsto \lambda_i$. Write $C = R_\mathfrak{p}[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_c)$ for some $c \geq n$ such that $A \to C$ maps x_i to x_i and such that $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ is invertible in C, see Lemma 4.6. After clearing denominators we may assume $f_1, \ldots, f_c$ are elements of $R[x_1, \ldots, x_{n+m}]$. Of course $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ is not invertible in $R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_c)$ but it becomes invertible after inverting some element $s_0 \in R$, $s_0 \notin \mathfrak{p}$. As g_j maps to zero under $R[x_1, \ldots, x_n] \to A \to C$ we can find $s_j \in R$, $s_j \notin \mathfrak{p}$ such that $s_j g_j$ is zero in $R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_c)$. Write $f_j = F_j(x_1, \ldots, x_{n+m}, 1)$ for some polynomial $F_j \in R[x_1, \ldots, x_n, X_{n+1}, \ldots, X_{n+m+1}]$ homogeneous in $X_{n+1}, \ldots, X_{n+m+1}$. Pick $\lambda_{n+i} \in \Lambda$, $i = 1, \ldots, m+1$ with $\lambda_{n+m+1} \notin \mathfrak{q}$

such that x_{n+i} maps to $\lambda_{n+i}/\lambda_{n+m+1}$ in $\Lambda_{\mathfrak q}$. Then

$$\begin{aligned} F_j(\lambda_1,\ldots,\lambda_{n+m+1}) &= (\lambda_{n+m+1})^{\deg(F_j)}F_j(\lambda_1,\ldots,\lambda_n,\frac{\lambda_{n+1}}{\lambda_{n+m+1}},\ldots,\frac{\lambda_{n+m}}{\lambda_{n+m+1}},1) \\ &= (\lambda_{n+m+1})^{\deg(F_j)}f_j(\lambda_1,\ldots,\lambda_n,\frac{\lambda_{n+1}}{\lambda_{n+m+1}},\ldots,\frac{\lambda_{n+m}}{\lambda_{n+m+1}}) \\ &= 0 \end{aligned}$$

in $\Lambda_{\mathfrak q}$. Thus we can find $\lambda_0 \in \Lambda$, $\lambda_0 \notin \mathfrak q$ such that $\lambda_0 F_j(\lambda_1,\ldots,\lambda_{n+m+1}) = 0$ in Λ. Now we set B equal to

$$R[x_0,\ldots,x_{n+m+1}]/(g_1,\ldots,g_t,x_0F_1(x_1,\ldots,x_{n+m+1}),\ldots,x_0F_c(x_1,\ldots,x_{n+m+1}))$$

which we map to Λ by mapping x_i to λ_i. Let b be the image of $x_0x_1s_0s_1\ldots s_t$ in B. Then B_b is isomorphic to

$$R_{s_0s_1}[x_0,x_1,\ldots,x_{n+m+1},1/x_0x_{n+m+1}]/(f_1,\ldots,f_c)$$

which is smooth over R by construction. Since b does not map to an element of $\mathfrak q$, we win. □

Lemma 9.4. *Let $R \to A \to \Lambda \supset \mathfrak q$ be as in Situation 9.1. Let $\mathfrak p = R \cap \mathfrak q$. Assume*

1. *$\mathfrak q$ is minimal over $\mathfrak h_A$,*
2. *$R_{\mathfrak p} \to A_{\mathfrak p} \to \Lambda_{\mathfrak q} \supset \mathfrak q\Lambda_{\mathfrak q}$ can be resolved, and*
3. *$\dim(\Lambda_{\mathfrak q}) = 0$.*

Then $R \to A \to \Lambda \supset \mathfrak q$ can be resolved.

Proof. By (3) the ring $\Lambda_{\mathfrak q}$ is Artinian local hence $\mathfrak q\Lambda_{\mathfrak q}$ is nilpotent. Thus $(\mathfrak h_A)^N\Lambda_{\mathfrak q} = 0$ for some $N > 0$. Thus there exists a $\lambda \in \Lambda$, $\lambda \notin \mathfrak q$ such that $\lambda(\mathfrak h_A)^N = 0$ in Λ. Say $H_{A/R} = (a_1,\ldots,a_r)$ so that $\lambda a_i^N = 0$ in Λ. By Lemma 9.3 we can find a factorization $A \to C \to \Lambda$ with C of finite presentation such that $\mathfrak h_C \not\subset \mathfrak q$. Write $C = A[x_1,\ldots,x_n]/(f_1,\ldots,f_m)$. Set

$$B = A[x_1,\ldots,x_n,y_1,\ldots,y_r,z,t_{ij}]/(f_j - \sum y_i t_{ij}, zy_i)$$

where t_{ij} is a set of rm variables. Note that there is a map $B \to C[y_i,z]/(y_iz)$ given by setting t_{ij} equal to zero. The map $B \to \Lambda$ is the composition $B \to C[y_i,z]/(y_iz) \to \Lambda$ where $C[y_i,z]/(y_iz) \to \Lambda$ is the given map $C \to \Lambda$, maps z to λ, and maps y_i to the image of a_i^N in Λ.

We claim that B is a solution for $R \to A \to \Lambda \supset \mathfrak q$. First note that B_z is isomorphic to $C[y_1,\ldots,y_r,z,z^{-1}]$ and hence is smooth. On the other hand, $B_{y_\ell} \cong A[x_i,y_i,y_\ell^{-1},t_{ij},i \neq \ell]$ which is smooth over A. Thus we see that z and $a_\ell y_\ell$ (compositions of smooth maps are smooth) are all elements of $H_{B/R}$. This proves the lemma. □

10. Separable residue fields

In this section we explain how to solve a local problem in the case of a separable residue field extension.

Lemma 10.1 (Ogoma)**.** *Let A be a Noetherian ring and let M be a finite A-module. Let $S \subset A$ be a multiplicative set. If $\pi \in A$ and $Ker(\pi : S^{-1}M \to S^{-1}M) = Ker(\pi^2 : S^{-1}M \to S^{-1}M)$ then there exists an $s \in S$ such that for any $n > 0$ we have $Ker(s^n\pi : M \to M) = Ker((s^n\pi)^2 : M \to M)$.*

Proof. Let $K = \mathrm{Ker}(\pi : M \to M)$ and $K' = \{m \in M \mid \pi^2 m = 0 \text{ in } S^{-1}M\}$ and $Q = K'/K$. Note that $S^{-1}Q = 0$ by assumption. Since A is Noetherian we see that Q is a finite A-module. Hence we can find an $s \in S$ such that s annihilates Q. Then s works. □

Lemma 10.2. *Let Λ be a Noetherian ring. Let $I \subset \Lambda$ be an ideal. Let $I \subset \mathfrak q$ be a prime. Let n, e be positive integers Assume that $\mathfrak q^n \Lambda_{\mathfrak q} \subset I\Lambda_{\mathfrak q}$ and that $\Lambda_{\mathfrak q}$ is a regular local ring of dimension d. Then there exists an $n > 0$ and $\pi_1, \ldots, \pi_d \in \Lambda$ such that*

1. *$(\pi_1, \ldots, \pi_d)\Lambda_{\mathfrak q} = \mathfrak q \Lambda_{\mathfrak q}$,*
2. *$\pi_1^n, \ldots, \pi_d^n \in I$, and*
3. *for $i = 1, \ldots, d$ we have*

$$Ann_{\Lambda/(\pi_1^e,\ldots,\pi_{i-1}^e)\Lambda}(\pi_i) = Ann_{\Lambda/(\pi_1^e,\ldots,\pi_{i-1}^e)\Lambda}(\pi_i^2).$$

Proof. Set $S = \Lambda \setminus \mathfrak q$ so that $\Lambda_{\mathfrak q} = S^{-1}\Lambda$. First pick $\pi_1, \ldots, \pi_d$ with (1) which is possible as $\Lambda_{\mathfrak q}$ is regular. By assumption $\pi_i^n \in I\Lambda_{\mathfrak q}$. Thus we can find $s_1, \ldots, s_d \in S$ such that $s_i \pi_i^n \in I$. Replacing π_i by $s_i\pi_i$ we get (2). Note that (1) and (2) are preserved by further multiplying by elements of S. Suppose that (3) holds for $i = 1, \ldots, t$ for some $t \in \{0, \ldots, d\}$. Note that $\pi_1, \ldots, \pi_d$ is a regular sequence in $S^{-1}\Lambda$, see Algebra, Lemma 99.3. In particular $\pi_1^e, \ldots, \pi_t^e, \pi_{t+1}$ is a regular sequence in $S^{-1}\Lambda = \Lambda_{\mathfrak q}$ by Algebra, Lemma 66.10. Hence we see that

$$\mathrm{Ann}_{S^{-1}\Lambda/(\pi_1^e,\ldots,\pi_{i-1}^e)}(\pi_i) = \mathrm{Ann}_{S^{-1}\Lambda/(\pi_1^e,\ldots,\pi_{i-1}^e)}(\pi_i^2).$$

Thus we get (3) for $i = t+1$ after replacing π_{t+1} by $s\pi_{t+1}$ for some $s \in S$ by Lemma 10.1. By induction on t this produces a sequence satisfying (1), (2), and (3). □

Lemma 10.3. *Let $k \to A \to \Lambda \supset \mathfrak q$ be as in Situation 9.1 where*

1. *k is a field,*
2. *Λ is Noetherian,*
3. *$\mathfrak q$ is minimal over $\mathfrak h_A$,*
4. *$\Lambda_{\mathfrak q}$ is a regular local ring, and*
5. *the field extension $k \subset \kappa(\mathfrak q)$ is separable.*

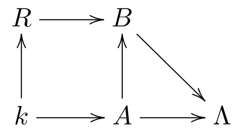
Then $k \to A \to \Lambda \supset \mathfrak q$ can be resolved.

Proof. Set $d = \dim \Lambda_{\mathfrak q}$. Set $R = k[x_1, \ldots, x_d]$. Choose $n > 0$ such that $\mathfrak q^n\Lambda_{\mathfrak q} \subset \mathfrak h_A \Lambda_{\mathfrak q}$ which is possible as $\mathfrak q$ is minimal over $\mathfrak h_A$. Choose generators $a_1, \ldots, a_r$ of $H_{A/R}$. Set

$$B = A[x_1, \ldots, x_d, z_{ij}]/(x_i^n - \sum z_{ij}a_j)$$

Each B_{a_j} is smooth over R it is a polynomial algebra over $A_{a_j}[x_1, \ldots, x_d]$ and A_{a_j} is smooth over k. Hence B_{x_i} is smooth over R. Let $B \to C$ be the R-algebra map constructed in Lemma 4.1 which comes with a R-algebra retraction $C \to B$. In particular a map $C \to \Lambda$ fitting into the diagram above. By construction C_{x_i} is a smooth R-algebra with $\Omega_{C_{x_i}/R}$ free. Hence we can find $c > 0$ such that x_i^c is strictly standard in C/R, see Lemma 4.7. Now choose $\pi_1, \ldots, \pi_d \in \Lambda$ as in Lemma 10.2 where $n = n$, $e = 8c$, $\mathfrak q = \mathfrak q$ and $I = \mathfrak h_A$. Write $\pi_i^n = \sum \lambda_{ij}a_j$ for some $\pi_{ij} \in \Lambda$. There is a map $B \to \Lambda$ given by $x_i \mapsto \pi_i$ and $z_{ij} \mapsto \lambda_{ij}$. Set $R = k[x_1, \ldots, x_d]$.

Diagram



Now we apply Lemma 9.2 to $R \to C \to \Lambda \supset \mathfrak{q}$ and the sequence of elements $x_1^c, \ldots, x_d^c$ of R. Assumption (2) is clear. Assumption (1) holds for R by inspection and for Λ by our choice of $\pi_1, \ldots, \pi_d$. (Note that if $\mathrm{Ann}_\Lambda(\pi) = \mathrm{Ann}_\Lambda(\pi^2)$, then we have $\mathrm{Ann}_\Lambda(\pi) = \mathrm{Ann}_\Lambda(\pi^c)$ for all $c > 0$.) Thus it suffices to resolve

$$R/(x_1^e, \ldots, x_d^e) \to C/(x_1^e, \ldots, x_d^e) \to \Lambda/(\pi_1^e, \ldots, \pi_d^e) \supset \mathfrak{q}/(\pi_1^e, \ldots, \pi_d^e)$$

for $e = 8c$. By Lemma 9.4 it suffices to resolve this after localizing at $\mathfrak{q}$. But since $x_1, \ldots, x_d$ map to a regular sequence in $\Lambda_\mathfrak{q}$ we see that $R \to \Lambda$ is flat, see Algebra, Lemma 120.2. Hence

$$R/(x_1^e, \ldots, x_d^e) \to \Lambda_\mathfrak{q}/(\pi_1^e, \ldots, \pi_d^e)$$

is a flat ring map of Artinian local rings. Moreover, this map induces a separable field extension on residue fields by assumption. Thus this map is a filtered colimit of smooth algebras by Algebra, Lemma 142.10 and Proposition 5.3. Existence of the desired solution follows from Lemma 2.1. □

11. Inseparable residue fields

In this section we explain how to solve a local problem in the case of an inseparable residue field extension.

Lemma 11.1. *Let k be a field of characteristic $p > 0$. Let $(\Lambda, \mathfrak{m}, K)$ be an Artinian local k-algebra. Assume that $\dim H_1(L_{K/k}) < \infty$. Then Λ is a filtered colimit of Artinian local k-algebras A with each map $A \to \Lambda$ flat, with $\mathfrak{m}_A\Lambda = \mathfrak{m}$, and with A essentially of finite type over k.*

Proof. Note that the flatness of $A \to \Lambda$ implies that $A \to \Lambda$ is injective, so the lemma really tells us that Λ is a directed union of these types of subrings $A \subset \Lambda$. Let n be the minimal integer such that $\mathfrak{m}^n = 0$. We will prove this lemma by induction on n. The case $n = 1$ is clear as a field extension is a union of finitely generated field extensions.

Pick $\lambda_1, \ldots, \lambda_d \in \mathfrak{m}$ which generate $\mathfrak{m}$. As K is formally smooth over $\mathbf{F}_p$ (see Algebra, Lemma 142.6) we can find a ring map $\sigma : K \to \Lambda$ which is a section of the quotient map $\Lambda \to K$. In general σ is **not** a k-algebra map. Given σ we define

$$\Psi_\sigma : K[x_1, \ldots, x_d] \longrightarrow \Lambda$$

using σ on elements of K and mapping x_i to λ_i. Claim: there exists a $\sigma : K \to \Lambda$ and a subfield $k \subset F \subset K$ finitely generated over k such that the image of k in Λ is contained in $\Psi_\sigma(F[x_1, \ldots, x_d])$.

We will prove the claim by induction on the least integer n such that $\mathfrak{m}^n = 0$. It is clear for $n = 1$. If $n > 1$ set $I = \mathfrak{m}^{n-1}$ and $\Lambda' = \Lambda/I$. By induction we may assume given $\sigma' : K \to \Lambda'$ and $k \subset F' \subset K$ finitely generated such that the image of $k \to \Lambda \to \Lambda'$ is contained in $A' = \Psi_{\sigma'}(F'[x_1, \ldots, x_d])$. Denote $\tau' : k \to A'$ the induced map. Choose a lift $\sigma : K \to \Lambda$ of σ' (this is possible by the formal smoothness of $K/\mathbf{F}_p$ we mentioned above). For later reference we note that we can change σ

to $\sigma + D$ for some derivation $D : K \to I$. Set $A = F[x_1, \ldots, x_d]/(x_1, \ldots, x_d)^n$. Then Ψ_σ induces a ring map $\Psi_\sigma : A \to \Lambda$. The composition with the quotient map $\Lambda \to \Lambda'$ induces a surjective map $A \to A'$ with nilpotent kernel. Choose a lift $\tau : k \to A$ of τ' (possible as $k/\mathbf{F}_p$ is formally smooth). Thus we obtain two maps $k \to \Lambda$, namely $\Psi_\sigma \circ \tau : k \to \Lambda$ and the given map $i : k \to \Lambda$. These maps agree modulo I, whence the difference is a derivation $\theta = i - \Psi_\sigma \circ \tau : k \to I$. Note that if we change σ into $\sigma + D$ then we change θ into $\theta - D|_k$.

Choose a set of elements $\{y_j\}_{j \in J}$ of k whose differentials $\text{d}y_j$ form a basis of $\Omega_{k/\mathbf{F}_p}$. The Jacobi-Zariski sequence for $\mathbf{F}_p \subset k \subset K$ is

$$0 \to H_1(L_{K/k}) \to \Omega_{k/\mathbf{F}_p} \otimes K \to \Omega_{K/\mathbf{F}_p} \to \Omega_{K/k} \to 0$$

As $\dim H_1(L_{K/k}) < \infty$ we can find a finite subset $J_0 \subset J$ such that the image of the first map is contained in $\bigoplus_{j \in J_0} K\text{d}y_j$. Hence the elements $\text{d}y_j$, $j \in J \setminus J_0$ map to K-linearly independent elements of $\Omega_{K/\mathbf{F}_p}$. Therefore we can choose a $D : K \to I$ such that $\theta - D|_k = \xi \circ \text{d}$ where ξ is a composition

$$\Omega_{k/\mathbf{F}_p} = \bigoplus\nolimits_{j \in J} k\text{d}y_j \longrightarrow \bigoplus\nolimits_{j \in J_0} k\text{d}y_j \longrightarrow I$$

Let $f_j = \xi(\text{d}y_j) \in I$ for $j \in J_0$. Change σ into $\sigma + D$ as above. Then we see that $\theta(a) = \sum_{j \in J_0} a_j f_j$ for $a \in k$ where $\text{d}a = \sum a_j \text{d}y_j$ in $\Omega_{k/\mathbf{F}_p}$. Note that I is generated by the monomials $\lambda^E = \lambda_1^{e_1} \ldots \lambda_d^{e_d}$ of total degree $|E| = \sum e_i = n - 1$ in $\lambda_1, \ldots, \lambda_d$. Write $f_j = \sum_E c_{j,E} \lambda^E$ with $c_{j,E} \in K$. Replace F' by $F = F'(c_{j,E})$. Then the claim holds.

Choose σ and F as in the claim. The kernel of Ψ_σ is generated by finitely many polynomials $g_1, \ldots, g_t \in K[x_1, \ldots, x_d]$ and we may assume their coefficients are in F after enlarging F by adjoining finitely many elements. In this case it is clear that the map $A = F[x_1, \ldots, x_d]/(g_1, \ldots, g_t) \to K[x_1, \ldots, x_d]/(g_1, \ldots, g_t) = \Lambda$ is flat. By the claim A is a k-subalgebra of Λ. It is clear that Λ is the filtered colimit of these algebras, as K is the filtered union of the subfields F. Finally, these algebras are essentially of finite type over k by Algebra, Lemma 51.3. □

Lemma 11.2. *Let k be a field of characteristic $p > 0$. Let Λ be a Noetherian geometrically regular k-algebra. Let $\mathfrak{q} \subset \Lambda$ be a prime ideal. Let $n \geq 1$ be an integer and let $E \subset \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$ be a finite subset. Then we can find $m \geq 0$ and $\varphi : k[y_1, \ldots, y_m] \to \Lambda$ with the following properties*

1. *setting $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$ we have $\mathfrak{q}\Lambda_\mathfrak{q} = \mathfrak{p}\Lambda_\mathfrak{q}$ and $k[y_1, \ldots, y_m]_\mathfrak{p} \to \Lambda_\mathfrak{q}$ is flat,*
2. *there is a factorization by homomorphisms of local Artinian rings*

$$k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p} \to D \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$$

   where the first arrow is essentially smooth and the second is flat,
3. *E is contained in D modulo $\mathfrak{q}^n\Lambda_\mathfrak{q}$.*

Proof. Set $\bar{\Lambda} = \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$. Note that $\dim H_1(L_{\kappa(\mathfrak{q})/k}) < \infty$ by More on Algebra, Proposition 26.1. Pick $A \subset \bar{\Lambda}$ containing E such that A is local Artinian, essentially of finite type over k, the map $A \to \bar{\Lambda}$ is flat, and $\mathfrak{m}_A$ generates the maximal ideal of $\bar{\Lambda}$, see Lemma 11.1. Denote $F = A/\mathfrak{m}_A$ the residue field so that $k \subset F \subset K$. Pick $\lambda_1, \ldots, \lambda_t \in \Lambda$ which map to elements of A in $\bar{\Lambda}$ such that moreover the images of $\text{d}\lambda_1, \ldots, \text{d}\lambda_t$ form a basis of $\Omega_{F/k}$. Consider the map $\varphi' : k[y_1, \ldots, y_t] \to \Lambda$ sending y_j to λ_j. Set $\mathfrak{p}' = (\varphi')^{-1}(\mathfrak{q})$. By More on Algebra, Lemma 26.2 the ring map $k[y_1, \ldots, y_t]_{\mathfrak{p}'} \to \Lambda_\mathfrak{q}$ is flat and $\Lambda_\mathfrak{q}/\mathfrak{p}'\Lambda_\mathfrak{q}$ is regular. Thus we can choose further elements $\lambda_{t+1}, \ldots, \lambda_m \in \Lambda$ which map into $A \subset \bar{\Lambda}$ and which map to a regular system of parameters of $\Lambda_{\mathfrak q}/\mathfrak p'\Lambda_{\mathfrak q}$. We obtain $\varphi : k[y_1, \ldots, y_m] \to \Lambda$ having property (1) such that $k[y_1, \ldots, y_m]_{\mathfrak p}/\mathfrak p^n k[y_1, \ldots, y_m]_{\mathfrak p} \to \bar{\Lambda}$ factors through A. Thus $k[y_1, \ldots, y_m]_{\mathfrak p}/\mathfrak p^n k[y_1, \ldots, y_m]_{\mathfrak p} \to A$ is flat by Algebra, Lemma 36.8. By construction the residue field extension $\kappa(\mathfrak p) \subset F$ is finitely generated and $\Omega_{F/\kappa(\mathfrak p)} = 0$. Hence it is finite separable by More on Algebra, Lemma 25.1. Thus $k[y_1, \ldots, y_m]_{\mathfrak p}/\mathfrak p^n k[y_1, \ldots, y_m]_{\mathfrak p} \to A$ is finite by Algebra, Lemma 51.3. Finally, we conclude that it is étale by Algebra, Lemma 133.7. Since an étale ring map is certainly essentially smooth we win. □

Lemma 11.3. *Let $\varphi : k[y_1, \ldots, y_m] \to \Lambda$, n, $\mathfrak q$, $\mathfrak p$ and*

$$k[y_1, \ldots, y_m]_{\mathfrak p}/\mathfrak p^n \to D \to \Lambda_{\mathfrak q}/\mathfrak q^n\Lambda_{\mathfrak q}$$

be as in Lemma 11.2. Then for any $\lambda \in \Lambda \setminus \mathfrak q$ there exists an integer $q > 0$ and a factorization

$$k[y_1, \ldots, y_m]_{\mathfrak p}/\mathfrak p^n \to D \to D' \to \Lambda_{\mathfrak q}/\mathfrak q^n\Lambda_{\mathfrak q}$$

such that $D \to D'$ is an essentially smooth map of local Artinian rings, the last arrow is flat, and λ^q is in D'.

Proof. Set $\bar{\Lambda} = \Lambda_{\mathfrak q}/\mathfrak q^n\Lambda_{\mathfrak q}$. Let $\bar{\lambda}$ be the image of λ in $\bar{\Lambda}$. Let $\alpha \in \kappa(\mathfrak q)$ be the image of λ in the residue field. Let $k \subset F \subset \kappa(\mathfrak q)$ be the residue field of D. If α is in F then we can find an $x \in D$ such that $x\bar{\lambda} = 1 \bmod \mathfrak q$. Hence $(x\bar{\lambda})^q = 1 \bmod (\mathfrak q)^q$ if q is divisible by p. Hence $\bar{\lambda}^q$ is in D. If α is transcendental over F, then we can take $D' = (D[\bar{\lambda}])_{\mathfrak m}$ equal to the subring generated by D and $\bar{\lambda}$ localized at $\mathfrak m = D[\bar{\lambda}] \cap \mathfrak q\bar{\Lambda}$. This works because $D[\bar{\lambda}]$ is in fact a polynomial algebra over D in this case. Finally, if $\lambda \bmod \mathfrak q$ is algebraic over F, then we can find a p-power q such that α^q is separable algebraic over F, see Algebra, Section 39. Note that D and $\bar{\Lambda}$ are henselian local rings, see Algebra, Lemma 140.11. Let $D \to D'$ be a finite étale extension whose residue field extension is $F \subset F(\alpha^q)$, see Algebra, Lemma 140.8. Since $\bar{\Lambda}$ is henselian and $F(\alpha^q)$ is contained in its residue field we can find a factorization $D' \to \bar{\Lambda}$. By the first part of the argument we see that $\bar{\lambda}^{qq'} \in D'$ for some $q' > 0$. □

Lemma 11.4. *Let $k \to A \to \Lambda \supset \mathfrak q$ be as in Situation 9.1 where*

1. *k is a field of characteristic $p > 0$,*
2. *Λ is Noetherian and geometrically regular over k,*
3. *$\mathfrak q$ is minimal over $\mathfrak h_A$.*

Then $k \to A \to \Lambda \supset \mathfrak q$ can be resolved.

Proof. The lemma is proven by the following steps in the given order. We will justify each of these steps below.

1. Pick an integer $N > 0$ such that $\mathfrak q^N\Lambda_{\mathfrak q} \subset H_{A/k}\Lambda_{\mathfrak q}$.
2. Pick generators $a_1, \ldots, a_t \in A$ of the ideal $H_{A/R}$.
3. Set $d = \dim(\Lambda_{\mathfrak q})$.
4. Set $B = A[x_1, \ldots, x_d, z_{ij}]/(x_i^{2N} - \sum z_{ij}a_j)$.
5. Consider B as a $k[x_1, \ldots, x_d]$-algebra and let $B \to C$ be as in Lemma 4.1. We also obtain a section $C \to B$.
6. Choose $c > 0$ such that each x_i^c is strictly standard in C over $k[x_1, \ldots, x_d]$.
7. Set $n = N + dc$ and $e = 8c$.
8. Let $E \subset \Lambda_{\mathfrak q}/\mathfrak q^n\Lambda_{\mathfrak q}$ be the images of generators of A as a k-algebra.

(9) Choose an integer m and a k-algebra map $\varphi : k[y_1, \ldots, y_m] \to \Lambda$ and a factorization by local Artinian rings

$$k[y_1, \ldots, y_m]_{\mathfrak p}/\mathfrak p^n k[y_1, \ldots, y_m]_{\mathfrak p} \to D \to \Lambda_{\mathfrak q}/\mathfrak q^n \Lambda_{\mathfrak q}$$

such that the first arrow is essentially smooth, the second is flat, E is contained in D, with $\mathfrak p = \varphi^{-1}(\mathfrak q)$ the map $k[y_1, \ldots, y_m]_{\mathfrak p} \to \Lambda_{\mathfrak q}$ is flat, and $\mathfrak p \Lambda_{\mathfrak q} = \mathfrak q \Lambda_{\mathfrak q}$.

(10) Choose $\pi_1, \ldots, \pi_d \in \mathfrak p$ which map to a regular system of parameters of $k[y_1, \ldots, y_m]_{\mathfrak p}$.

(11) Let $R = k[y_1, \ldots, y_m, t_1, \ldots, t_m]$ and $\gamma_i = \pi_i t_i$.

(12) If necessary modify the choice of π_i such that for $i = 1, \ldots, d$ we have

$$\mathrm{Ann}_{R/(\gamma_1^e, \ldots, \gamma_{i-1}^e)R}(\gamma_i) = \mathrm{Ann}_{R/(\gamma_1^e, \ldots, \gamma_{i-1}^e)R}(\gamma_i^2)$$

(13) There exist $\delta_1, \ldots, \delta_d \in \Lambda$, $\delta_i \not\in \mathfrak q$ and a factorization $D \to D' \to \Lambda_{\mathfrak q}/\mathfrak q^n \Lambda_{\mathfrak q}$ with D' local Artinian, $D \to D'$ essentially smooth, the map $D' \to \Lambda_{\mathfrak q}/\mathfrak q^n \Lambda_{\mathfrak q}$ flat such that, with $\pi_i' = \delta_i \pi_i$, we have for $i = 1, \ldots, d$
   (a) $(\pi_i')^{2N} = \sum a_j \lambda_{ij}$ in Λ where λ_{ij} mod $\mathfrak q^n \Lambda_{\mathfrak q}$ is an element of D',
   (b) $\mathrm{Ann}_{\Lambda/({\pi'}_1^e, \ldots, {\pi'}_{i-1}^e)}({\pi'}_i) = \mathrm{Ann}_{\Lambda/({\pi'}_1^e, \ldots, {\pi'}_{i-1}^e)}({\pi'}_i^2)$,
   (c) δ_i mod $\mathfrak q^n \Lambda_{\mathfrak q}$ is an element of D'.

(14) Define $B \to \Lambda$ by sending x_i to π_i' and z_{ij} to λ_{ij} found above. Define $C \to \Lambda$ by composing the map $B \to \Lambda$ with the retraction $C \to B$.

(15) Map $R \to \Lambda$ by φ on $k[y_1, \ldots, y_m]$ and by sending t_i to δ_i. Further introduce a map

$$k[x_1, \ldots, x_d] \longrightarrow R = k[y_1, \ldots, y_m, t_1, \ldots, t_d]$$

by sending x_i to $\gamma_i = \pi_i t_i$.

(16) It suffices to resolve

$$R \to C \otimes_{k[x_1, \ldots, x_d]} R \to \Lambda \supset \mathfrak q$$

(17) Set $I = (\gamma_1^e, \ldots, \gamma_d^e) \subset R$.

(18) It suffices to resolve

$$R/I \to C \otimes_{k[x_1, \ldots, x_d]} R/I \to \Lambda/I\Lambda \supset \mathfrak q/I\Lambda$$

(19) We denote $\mathfrak r \subset R = k[y_1, \ldots, y_m, t_1, \ldots, t_d]$ the inverse image of $\mathfrak q$.

(20) It suffices to resolve

$$(R/I)_{\mathfrak r} \to C \otimes_{k[x_1, \ldots, x_d]} (R/I)_{\mathfrak r} \to \Lambda_{\mathfrak q}/I\Lambda_{\mathfrak q} \supset \mathfrak q \Lambda_{\mathfrak q}/I\Lambda_{\mathfrak q}$$

(21) Set $J = (\pi_1^e, \ldots, \pi_d^e)$ in $k[y_1, \ldots, y_m]$.

(22) It suffices to resolve

$$(R/JR)_{\mathfrak p} \to C \otimes_{k[x_1, \ldots, x_d]} (R/JR)_{\mathfrak p} \to \Lambda_{\mathfrak q}/J\Lambda_{\mathfrak q} \supset \mathfrak q \Lambda_{\mathfrak q}/J\Lambda_{\mathfrak q}$$

(23) It suffices to resolve

$$(R/\mathfrak p^n R)_{\mathfrak p} \to C \otimes_{k[x_1, \ldots, x_d]} (R/\mathfrak p^n R)_{\mathfrak p} \to \Lambda_{\mathfrak q}/\mathfrak q^n \Lambda_{\mathfrak q} \supset \mathfrak q \Lambda_{\mathfrak q}/\mathfrak q^n \Lambda_{\mathfrak q}$$

(24) It suffices to resolve

$$(R/\mathfrak p^n R)_{\mathfrak p} \to B \otimes_{k[x_1, \ldots, x_d]} (R/\mathfrak p^n R)_{\mathfrak p} \to \Lambda_{\mathfrak q}/\mathfrak q^n \Lambda_{\mathfrak q} \supset \mathfrak q \Lambda_{\mathfrak q}/\mathfrak q^n \Lambda_{\mathfrak q}$$

(25) The ring $D'[t_1, \ldots, t_d]$ is given the structure of an $R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p}$-algebra by the given map $k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p} \to D'$ and by sending t_i to t_i. It suffices to find a factorization

$$B \otimes_{k[x_1,\ldots,x_d]} (R/\mathfrak{p}^n R)_\mathfrak{p} \to D'[t_1, \ldots, t_d] \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$$

where the second arrow sends t_i to δ_i and induces the given homomorphism $D' \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$.

(26) Such a factorization exists by our choice of D' above.

We now give the justification for each of the steps, except that we skip justifying the steps which just introduce notation.

Ad (1). This is possible as $\mathfrak{q}$ is minimal over $\mathfrak{h}_A = \sqrt{H_{A/k}\Lambda}$.

Ad (6). Note that A_{a_i} is smooth over k. Hence B_{a_j}, which is isomorphic to a polynomial algebra over $A_{a_j}[x_1, \ldots, x_d]$, is smooth over $k[x_1, \ldots, x_d]$. Thus B_{x_i} is smooth over $k[x_1, \ldots, x_d]$. By Lemma 4.1 we see that C_{x_i} is smooth over $k[x_1, \ldots, x_d]$ with finite free module of differentials. Hence some power of x_i is strictly standard in C over $k[x_1, \ldots, x_n]$ by Lemma 4.7.

Ad (9). This follows by applying Lemma 11.2.

Ad (10). Since $k[y_1, \ldots, y_m]_\mathfrak{p} \to \Lambda_\mathfrak{q}$ is flat and $\mathfrak{p}\Lambda_\mathfrak{q} = \mathfrak{q}\Lambda_\mathfrak{q}$ by construction we see that $\dim(k[y_1, \ldots, y_m]_\mathfrak{p}) = d$ by Algebra, Lemma 104.7. Thus we can find $\pi_1, \ldots, \pi_d \in \Lambda$ which map to a regular system of parameters in $\Lambda_\mathfrak{q}$.

Ad (12). By Algebra, Lemma 99.3 any permutation of the sequence $\pi_1, \ldots, \pi_d$ is a regular sequence in $k[y_1, \ldots, y_m]_\mathfrak{p}$. Hence $\gamma_1 = \pi_1 t_1, \ldots, \gamma_d = \pi_d t_d$ is a regular sequence in $R_\mathfrak{p} = k[y_1, \ldots, y_m]_\mathfrak{p}[t_1, \ldots, t_d]$, see Algebra, Lemma 66.11. Let $S = k[y_1, \ldots, y_m] \setminus \mathfrak{p}$ so that $R_\mathfrak{p} = S^{-1}R$. Note that $\pi_1, \ldots, \pi_d$ and $\gamma_1, \ldots, \gamma_d$ remain regular sequences if we multiply our π_i by elements of S. Suppose that

$$\mathrm{Ann}_{R/(\gamma_1^e,\ldots,\gamma_{i-1}^e)R}(\gamma_i) = \mathrm{Ann}_{R/(\gamma_1^e,\ldots,\gamma_{i-1}^e)R}(\gamma_i^2)$$

holds for $i = 1, \ldots, t$ for some $t \in \{0, \ldots, d\}$. Note that $\gamma_1^e, \ldots, \gamma_t^e, \gamma_{t+1}$ is a regular sequence in $S^{-1}R$ by Algebra, Lemma 66.10. Hence we see that

$$\mathrm{Ann}_{S^{-1}R/(\gamma_1^e,\ldots,\gamma_{i-1}^e)}(\gamma_i) = \mathrm{Ann}_{S^{-1}R/(\gamma_1^e,\ldots,\gamma_{i-1}^e)}(\gamma_i^2).$$

Thus we get

$$\mathrm{Ann}_{R/(\gamma_1^e,\ldots,\gamma_t^e)R}(\gamma_{t+1}) = \mathrm{Ann}_{R/(\gamma_1^e,\ldots,\gamma_t^e)R}(\gamma_{t+1}^2)$$

after replacing π_{t+1} by $s\pi_{t+1}$ for some $s \in S$ by Lemma 10.1. By induction on t this produces the desired sequence.

Ad (13). Let $S = \Lambda \setminus \mathfrak{q}$ so that $\Lambda_\mathfrak{q} = S^{-1}\Lambda$. Set $\bar{\Lambda} = \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$. Suppose that we have a $t \in \{0, \ldots, d\}$ and $\delta_1, \ldots, \delta_t \in S$ and a factorization $D \to D' \to \bar{\Lambda}$ as in (13) such that (a), (b), (c) hold for $i = 1, \ldots, t$. We have $\pi_{t+1}^N \in H_{A/k}\Lambda_\mathfrak{q}$ as $\mathfrak{q}^N\Lambda_\mathfrak{q} \subset H_{A/k}\Lambda_\mathfrak{q}$ by (1). Hence $\pi_{t+1}^N \in H_{A/k}\bar{\Lambda}$. Hence $\pi_{t+1}^N \in H_{A/k}D'$ as $D' \to \bar{\Lambda}$ is faithfully flat, see Algebra, Lemma 77.11. Recall that $H_{A/k} = (a_1, \ldots, a_t)$. Say $\pi_{t+1}^N = \sum a_j d_j$ in D' and choose $c_j \in \Lambda_\mathfrak{q}$ lifting $d_j \in D'$. Then $\pi_{t+1}^N = \sum c_j a_j + \epsilon$ with $\epsilon \in \mathfrak{q}^n\Lambda_\mathfrak{q} \subset \mathfrak{q}^{n-N}H_{A/k}\Lambda_\mathfrak{q}$. Write $\epsilon = \sum a_j c'_j$ for some $c'_j \in \mathfrak{q}^{n-N}\Lambda_\mathfrak{q}$. Hence $\pi_{t+1}^{2N} = \sum(\pi_{t+1}^N c_j + \pi_{t+1}^N c'_j)a_j$. Note that $\pi_{t+1}^N c'_j$ maps to zero in $\bar{\Lambda}$; this trivial but key observation will ensure later that (a) holds. Now we choose $s \in S$ such that there exist $\mu_{t+1j} \in \Lambda$ such that on the one hand $\pi_{t+1}^N c_j + \pi_{t+1}^N c'_j = \mu_{t+1j}/s^{2N}$ in $S^{-1}\Lambda$ and on the other $(s\pi_{t+1})^{2N} = \sum \mu_{t+1j} a_j$ in Λ (minor detail omitted). We may further replace s by a power and enlarge D' such that s maps to an element of D'. With these choices μ_{t+1j} maps to $s^{2N}d_j$ which is an element of D'. Note that $\pi_1, \ldots, \pi_d$ are a regular sequence of parameters in $S^{-1}\Lambda$ by our choice of φ. Hence $\pi_1, \ldots, \pi_d$ forms a regular sequence in $\Lambda_\mathfrak{q}$ by Algebra, Lemma 99.3. It follows that ${\pi'}_1^e, \ldots, {\pi'}_t^e, s\pi_{t+1}$ is a regular sequence in $S^{-1}\Lambda$ by Algebra, Lemma 66.10. Thus we get

$$\mathrm{Ann}_{S^{-1}\Lambda/({\pi'}_1^e, \ldots, {\pi'}_t^e)}(s\pi_{t+1}) = \mathrm{Ann}_{S^{-1}\Lambda/({\pi'}_1^e, \ldots, {\pi'}_t^e)}((s\pi_{t+1})^2).$$

Hence we may apply Lemma 10.1 to find an $s' \in S$ such that

$$\mathrm{Ann}_{\Lambda/({\pi'}_1^e, \ldots, {\pi'}_t^e)}((s')^q s\pi_{t+1}) = \mathrm{Ann}_{\Lambda/({\pi'}_1^e, \ldots, {\pi'}_t^e)}(((s')^q s\pi_{t+1})^2).$$

for any $q > 0$. By Lemma 11.3 we can choose q and enlarge D' such that $(s')^q$ maps to an element of D'. Setting $\delta_{t+1} = (s')^q s$ and we conclude that (a), (b), (c) hold for $i = 1, \ldots, t+1$. For (a) note that $\lambda_{t+1j} = (s')^{2Nq}\mu_{t+1j}$ works. By induction on t we win.

Ad (16). By construction the radical of $H_{(C \otimes_{k[x_1, \ldots, x_d]} R)/R}\Lambda$ contains $\mathfrak{h}_A$. Namely, the elements $a_j \in H_{A/k}$ map to elements of $H_{B/k[x_1, \ldots, x_n]}$, hence map to elements of $H_{C/k[x_1, \ldots, x_n]}$, hence $a_j \otimes 1$ map to elements of $H_{C \otimes_{k[x_1, \ldots, x_d]} R/R}$. Moreover, if we have a solution $C \otimes_{k[x_1, \ldots, x_n]} R \to T \to \Lambda$ of

$$R \to C \otimes_{k[x_1, \ldots, x_d]} R \to \Lambda \supset \mathfrak{q}$$

then $H_{T/R} \subset H_{T/k}$ as R is smooth over k. Hence T will also be a solution for the original situation $k \to A \to \Lambda \supset \mathfrak{q}$.

Ad (18). Follows on applying Lemma 9.2 to $R \to C \otimes_{k[x_1, \ldots, x_d]} R \to \Lambda \supset \mathfrak{q}$ and the sequence of elements $\gamma_1^c, \ldots, \gamma_d^c$. We note that since x_i^c are strictly standard in C over $k[x_1, \ldots, x_d]$ the elements γ_i^c are strictly standard in $C \otimes_{k[x_1, \ldots, x_d]} R$ over R by Lemma 3.7. The other assumption of Lemma 9.2 holds by steps (12) and (13).

Ad (20). Apply Lemma 9.4 to the situation in (18). In the rest of the arguments the target ring is local Artinian, hence we are looking for a factorization by a smooth algebra T over the source ring.

Ad (22). Suppose that $C \otimes_{k[x_1, \ldots, x_d]} (R/JR)_\mathfrak{p} \to T \to \Lambda_\mathfrak{q}/J\Lambda_\mathfrak{q}$ is a solution to

$$(R/JR)_\mathfrak{p} \to C \otimes_{k[x_1, \ldots, x_d]} (R/JR)_\mathfrak{p} \to \Lambda_\mathfrak{q}/J\Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}/J\Lambda_\mathfrak{q}$$

Then $C \otimes_{k[x_1, \ldots, x_d]} (R/I)_\mathfrak{r} \to T_\mathfrak{r} \to \Lambda_\mathfrak{q}/I\Lambda_\mathfrak{q}$ is a solution to the situation in (20).

Ad (23). Our $n = N + dc$ is large enough so that $\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p} \subset J_\mathfrak{p}$ and $\mathfrak{q}^n\Lambda_\mathfrak{q} \subset J\Lambda_\mathfrak{q}$. Hence if we have a solution $C \otimes_{k[x_1, \ldots, x_d]} (R/\mathfrak{p}^n R)_\mathfrak{p} \to T \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$ of (22 then we can take T/JT as the solution for (23).

Ad (24). This is true because we have a section $C \to B$ in the category of R-algebras.

Ad (25). This is true because D' is essentially smooth over the local Artinian ring $k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p}$ and

$$R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p} = k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p}[t_1, \ldots, t_d].$$

Hence $D'[t_1, \ldots, t_d]$ is a filtered colimit of smooth $R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p}$-algebras and $B \otimes_{k[x_1, \ldots, x_d]} (R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p})$ factors through one of these.

Ad (26). The final twist of the proof is that we cannot just use the map $B \to D'$ which maps x_i to the image of π'_i in D' and z_{ij} to the image of λ_{ij} in D' because we need the diagram

$$\begin{array}{ccc} B & \longrightarrow & D'[t_1, \ldots, t_d] \\ \uparrow & & \uparrow \\ k[x_1, \ldots, x_d] & \longrightarrow & R_{\mathfrak p}/\mathfrak p^n R_{\mathfrak p} \end{array}$$

to commute and we need the compostion $B \to D'[t_1, \ldots, t_d] \to \Lambda_{\mathfrak q}/\mathfrak q^n \Lambda_{\mathfrak q}$ to be the map of (14). This requires us to map x_i to the image of $\pi_i t_i$ in $D'[t_1, \ldots, t_d]$. Hence we map z_{ij} to the image of $\lambda_{ij} t_i^{2N}/\delta_i^{2N}$ in $D'[t_1, \ldots, t_d]$ and everything is clear. □

12. The main theorem

In this section we wrap up the discussion.

Theorem 12.1 (Popescu)**.** *Any regular homomorphism of Noetherian rings is a filtered colimit of smooth ring maps.*

Proof. By Lemma 8.4 it suffices to prove this for $k \to \Lambda$ where Λ is Noetherian and geometrically regular over k. Let $k \to A \to \Lambda$ be a factorization with A a finite type k-algebra. It suffices to construct a factorization $A \to B \to \Lambda$ with B of finite type such that $\mathfrak h_B = \Lambda$, see Lemma 3.8. Hence we may perform Noetherian induction on the ideal $\mathfrak h_A$. Pick a prime $\mathfrak q \supset \mathfrak h_A$ such that $\mathfrak q$ is minimal over $\mathfrak h_A$. It now suffices to resolve $k \to A \to \Lambda \supset \mathfrak q$ (as defined in the text following Situation 9.1). If the characteristic of k is zero, this follows from Lemma 10.3. If the characteristic of k is $p > 0$, this follows from Lemma 11.4. □

13. The approximation property for G-rings

Let R be a Noetherian local ring. In this case R is a G-ring if and only if the ring map $R \to R^\wedge$ is regular, see More on Algebra, Lemma 39.7. In this case it is true that the henselization R^h and the strict henselization R^{sh} of R are G-rings, see More on Algebra, Lemma 39.8. Moreover, any algebra essentially of finite type over a field, over a complete local ring, over $\mathbf{Z}$, or over a characteristic zero Dedekind ring is a G-ring, see More on Algebra, Proposition 39.12. This gives an ample supply of rings to which the result below applies.

Let R be a ring. Let $f_1, \ldots, f_m \in R[x_1, \ldots, x_n]$. Let S be an R-algebra. In this situation we say a vector $(a_1, \ldots, a_n) \in S^n$ is a *solution in S* if and only if

$$f_j(a_1, \ldots, a_n) = 0 \text{ in } S, \text{ for } j = 1, \ldots, m$$

Of course an important question in algebraic geometry is to see when systems of polynomial equations have solutions. The following theorem tells us that having solutions in the completion of a local Noetherian ring is often enough to show there exist solutions in the henselization of the ring.

Theorem 13.1. *Let R be a Noetherian local ring. Let $f_1, \ldots, f_m \in R[x_1, \ldots, x_n]$. Suppose that $(a_1, \ldots, a_n) \in (R^\wedge)^n$ is a solution in $R^\wedge$. If R is a henselian G-ring, then for every integer N there exists a solution $(b_1, \ldots, b_n) \in R^n$ in R such that $a_i - b_i \in \mathfrak m^N R^\wedge$.*

Proof. Let $c_i \in R$ be an element such that $a_i - c_i \in \mathfrak{m}^N$. Choose generators $\mathfrak{m}^N = (d_1, \ldots, d_M)$. Write $a_i = c_i + \sum a_{i,l} d_l$. Consider the polynomial ring $R[x_{i,l}]$ and the elements

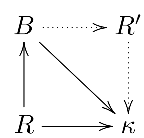
$$g_j = f_j(c_1 + \sum x_{1,l} d_l, \ldots, c_n + \sum x_{n,l} d_{n,l}) \in R[x_{i,l}]$$

The system of equations $g_j = 0$ has the solution $(a_{i,l})$. Suppose that we can show that g_j as a solution $(b_{i,l})$ in R. Then it follows that $b_i = c_i + \sum b_{i,l} d_l$ is a solution of $f_j = 0$ which is congruent to a_i modulo $\mathfrak{m}^N$. Thus it suffices to show that solvability over $R^\wedge$ implies solvability over R.

Let $A \subset R^\wedge$ be the R-subalgebra generated by $a_1, \ldots, a_n$. Since we've assumed R is a G-ring, i.e., that $R \to R^\wedge$ is regular, we see that there exists a factorization

$$A \to B \to R^\wedge$$

with B smooth over R, see Theorem 12.1. Denote $\kappa = R/\mathfrak{m}$ the residue field. It is also the residue field of $R^\wedge$, so we get a commutative diagram



Since the vertical arrow is smooth, More on Algebra, Lemma 9.12 implies that there exists an étale ring map $R \to R'$ which induces an isomorphism $R/\mathfrak{m} \to R'/\mathfrak{m}R'$ and an R-algebra map $B \to R'$ making the diagram above commute. Since R is henselian we see that $R \to R'$ has a section, see Algebra, Lemma 140.3. Let $b_i \in R$ be the image of a_i under the ring maps $A \to B \to R' \to R$. Since all of these maps are R-algebra maps, we see that $(b_1, \ldots, b_n)$ is a solution in R. □

Theorem 13.2. *Let R be a Noetherian local ring. Let $f_1, \ldots, f_m \in R[x_1, \ldots, x_n]$. Suppose that $(a_1, \ldots, a_n) \in (R^\wedge)^n$ is a solution. If R is a G-ring, then for every integer N there exist*

1. *an étale ring map $R \to R'$,*
2. *a maximal ideal $\mathfrak{m}' \subset R'$ lying over $\mathfrak{m}$*
3. *a solution $(b_1, \ldots, b_n) \in (R')^n$ in R'*

such that $\kappa(\mathfrak{m}) = \kappa(\mathfrak{m}')$ and $a_i - b_i \in (\mathfrak{m}')^N R'_{\mathfrak{m}'}$.

Proof. We could deduce this theorem from Theorem 13.1 using that the henselization R^h is a G-ring by More on Algebra, Lemma 39.8 and writing R^h as a directed colimit of étale extension R'. Instead we prove this by redoing the proof of the previous theorem in this case.

Let $c_i \in R$ be an element such that $a_i - c_i \in \mathfrak{m}^N$. Choose generators $\mathfrak{m}^N = (d_1, \ldots, d_M)$. Write $a_i = c_i + \sum a_{i,l} d_l$. Consider the polynomial ring $R[x_{i,l}]$ and the elements

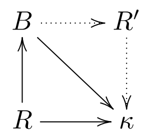
$$g_j = f_j(c_1 + \sum x_{1,l} d_l, \ldots, c_n + \sum x_{n,l} d_{n,l}) \in R[x_{i,l}]$$

The system of equations $g_j = 0$ has the solution $(a_{i,l})$. Suppose that we can show that g_j as a solution $(b_{i,l})$ in R' for some étale ring map $R \to R'$ endowed with a maximal ideal $\mathfrak{m}'$ such that $\kappa(\mathfrak{m}) = \kappa(\mathfrak{m}')$. Then it follows that $b_i = c_i + \sum b_{i,l} d_l$ is a solution of $f_j = 0$ which is congruent to a_i modulo $(\mathfrak{m}')^N$. Thus it suffices to show that solvability over $R^\wedge$ implies solvability over some étale ring extension which induces a trivial residue field extension at some prime over $\mathfrak{m}$.

Let $A \subset R^\wedge$ be the R-subalgebra generated by $a_1, \ldots, a_n$. Since we've assumed R is a G-ring, i.e., that $R \to R^\wedge$ is regular, we see that there exists a factorization

$$A \to B \to R^\wedge$$

with B smooth over R, see Theorem 12.1. Denote $\kappa = R/\mathfrak{m}$ the residue field. It is also the residue field of $R^\wedge$, so we get a commutative diagram



Since the vertical arrow is smooth, More on Algebra, Lemma 9.12 implies that there exists an étale ring map $R \to R'$ which induces an isomorphism $R/\mathfrak{m} \to R'/\mathfrak{m}R'$ and an R-algebra map $B \to R'$ making the diagram above commute. Let $b_i \in R'$ be the image of a_i under the ring maps $A \to B \to R'$. Since all of these maps are R-algebra maps, we see that $(b_1, \ldots, b_n)$ is a solution in R'. □

14. Other chapters

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Commutative Algebra
(8) Brauer Groups
(9) Sites and Sheaves
(10) Homological Algebra
(11) Derived Categories
(12) More on Algebra
(13) Smoothing Ring Maps
(14) Simplicial Methods
(15) Sheaves of Modules
(16) Modules on Sites
(17) Injectives
(18) Cohomology of Sheaves
(19) Cohomology on Sites
(20) Hypercoverings
(21) Schemes
(22) Constructions of Schemes
(23) Properties of Schemes
(24) Morphisms of Schemes
(25) Coherent Cohomology
(26) Divisors
(27) Limits of Schemes
(28) Varieties
(29) Chow Homology
(30) Topologies on Schemes
(31) Descent
(32) Adequate Modules
(33) More on Morphisms
(34) More on Flatness
(35) Groupoid Schemes
(36) More on Groupoid Schemes
(37) Étale Morphisms of Schemes
(38) Étale Cohomology
(39) Crystalline Cohomology
(40) Algebraic Spaces
(41) Properties of Algebraic Spaces
(42) Morphisms of Algebraic Spaces
(43) Decent Algebraic Spaces
(44) Topologies on Algebraic Spaces
(45) Descent and Algebraic Spaces
(46) More on Morphisms of Spaces
(47) Quot and Hilbert Spaces
(48) Spaces over Fields
(49) Cohomology of Algebraic Spaces
(50) Stacks
(51) Formal Deformation Theory
(52) Groupoids in Algebraic Spaces
(53) More on Groupoids in Spaces
(54) Bootstrap
(55) Examples of Stacks
(56) Quotients of Groupoids

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