Random Partitions and the Quantum Benjamin-Ono Hierarchy

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Outline

I. Random Partitions via Circular β-Ensembles

II. Limit Shapes and Gaussian Fluctuations

III. Nazarov-Sklyanin Lax Operator

IV. From Hierarchies to Scaling Limits

Elevator Version

Random Partitions	β-Ensembles
$\Uparrow$	$\Uparrow$
$\Uparrow$	$\Uparrow$
$\Uparrow$	$\Uparrow$
$\Uparrow$	$\Uparrow$
$\Uparrow$	$\Uparrow$
Lax Operator	Random Matrices

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I. Random Partitions via Circular β-Ensembles

Throughout, live on the unit circle

$$\mathbb{T} := \{w \in \mathbb{C} \ : \ |w| = 1\}.$$

Define an inner product on $\mathbb{C}[w_1^{\pm}, \ldots, w_N^{\pm}]$ by

$$\langle F^{\mathrm{out}}, F^{\mathrm{in}} \rangle_{\mathbb{T};\beta;N} := \oint_{\mathbb{T}^N} \overline{F^{\mathrm{out}}(\vec{w})} \cdot F^{\mathrm{in}}(\vec{w}) \prod_{i<j} \left| w_i - w_j \right|^{\beta} \prod_{i=1}^{N} \frac{dw_i}{2\pi \mathbf{i} w_i}$$

This involves $|\Psi_\circ|^2$ for multi-valued

$$\Psi_\circ(w_1, \ldots, w_N | \beta) := \prod_{i<j} (w_i - w_j)^{\beta/2}.$$

Restrict $\langle \cdot, \cdot \rangle_{\mathbb{T};\beta,N}$ to:

1. **Chirality**: ordinary polynomials $\mathbb{C}[w_1, \ldots, w_N]$
2. **Symmetry**: symmetric polynomials $\mathbb{C}[w_1, \ldots, w_N]^{S(N)}$

I. Random Partitions via Circular β-Ensembles

Conjugating

$$\mathcal{H} = \Psi_\circ^{-1} \widehat{\mathcal{H}} \Psi_\circ$$

the generator $\widehat{\mathcal{H}}$ of *circular-β Dyson Brownian motion* by

$$\Psi_\circ := \prod_{i<j} (w_i - w_j)^{\beta/2}$$

arrive at

$$\mathcal{H} \;=\; \sum_{i=1}^{N} \left(w_i \frac{\partial}{\partial w_i} \right)^2 + \frac{\beta}{2} \sum_{i<j} \frac{w_i + w_j}{w_i - w_j} \left(w_i \frac{\partial}{\partial w_i} - w_j \frac{\partial}{\partial w_j} \right).$$

1. $\mathcal{H}$ is self-adjoint operator on $\mathbb{C}[w_1, \ldots, w_N]^{S(N)}$ for $\langle \cdot, \cdot \rangle_{\mathbb{T};\beta,N}$
2. $\mathcal{H}$ commutes with degree operator $\mathcal{D} = \sum_{i=1}^{N} w_i \frac{\partial}{\partial w_i}$
3. Eigenspaces of $\mathcal{D}$ on $\mathbb{C}[w_1, \ldots, w_N]^{S(N)}$ are dim $< \infty$

I. Random Partitions via Circular β-Ensembles

Spectral Theorem: $\mathcal{H}$ diagonalized in $\mathbb{C}[w_1, \ldots, w_N]^{S(N)}$ by

$$P_\lambda(w_1, \ldots, w_N|\beta)$$

a basis of multivariate homogeneous symmetric orthogonal polynomials for $\langle \cdot, \cdot \rangle_{\mathbb{T};\beta,N}$.

Definition: These P_λ are the *Jack symmetric polynomials*, indexed by $0 \leq \lambda_N \leq \cdots \leq \lambda_1$ with $\lambda_i \in \mathbb{N}$.

Collective Variables: work with Jacks $P_\lambda(p|\beta)$ *only* via

$$p_k(\vec{w}) = w_1^k + \cdots + w_N^k$$

and $\frac{\partial}{\partial p_k}$. Abandon original w_i and $\frac{\partial}{\partial w_i}$ for stability in $N \to \infty$

I. Random Partitions via Circular β-Ensembles

Recall our pairing on symmetric polynomials

$$\langle F^{\text{out}}, F^{\text{in}} \rangle_{\mathbb{T};\beta;N} := \oint_{\mathbb{T}^N} \overline{F^{\text{out}}(\vec{w})} \cdot F^{\text{in}}(\vec{w}) \prod_{i<j} \left| w_i - w_j \right|^{\beta} \prod_{i=1}^{N} \frac{dw_i}{2\pi \mathbf{i} w_i}$$

and call $p_k(\vec{w}) = w_1^k + \cdots + w_N^k$ the *power sums*.

Amazing Fact: Unlike Jacks $P_\lambda(w_1, \ldots, w_N | \beta)$, for $\# : \mathbb{N} \to \mathbb{N}$ compact support

$$p_\mu = p_1^{\#_1} p_2^{\#_2} \cdots p_k^{\#_k} \cdots$$

are **not** orthogonal with respect to $\langle \cdot, \cdot \rangle_{\mathbb{T};\beta,N}$,
but **ARE** orthogonal in the limit

$$\langle F^{\text{out}}, F^{\text{in}} \rangle_{\mathbb{T};\beta,\infty} := \lim_{N \to \infty} \frac{\langle F^{\text{out}}, F^{\text{in}} \rangle_{\mathbb{T};\beta;N}}{\langle 1, 1 \rangle_{\mathbb{T};\beta;N}}.$$

- Can define $\langle \cdot, \cdot \rangle_{\mathbb{T};\beta,\infty}$ on $\mathbb{C}[p_1, p_2, \ldots]$ by declaring $p_{-k} = \frac{\beta}{2} k \frac{\partial}{\partial p_k}$ to be the adjoint of multiplication by p_k.

I. Random Partitions via Circular β-Ensembles

To better understand asymptotic orthogonality of p_μ, rescale

$$p_k = \frac{V_k}{-\varepsilon_2}$$

to new variables $V_1, V_2, \ldots$ via real parameters $\varepsilon_2 < 0 < \varepsilon_1$ chosen so that $\frac{\beta}{2} = -\frac{\varepsilon_2}{\varepsilon_1}$. The limit

$$\langle \cdot, \cdot \rangle_{-\varepsilon_1 \varepsilon_2} := \langle \cdot, \cdot \rangle_{\mathbb{T};\beta,\infty} := \lim_{N \to \infty} \frac{\langle \cdot, \cdot \rangle_{\mathbb{T};\beta;N}}{\langle 1, 1 \rangle_{\mathbb{T};\beta;N}}.$$

is now defined on $\mathcal{F} = \mathbb{C}[V_1, V_2, \ldots]$ by declaring

$$V_{-k} = -\varepsilon_1 \varepsilon_2 \, k \frac{\partial}{\partial V_k}$$

to be adjoint of multiplication by V_k on $\mathcal{F}$. If $\deg V_k = k$,

$$\mathcal{F} = \bigoplus_{d=0}^{\infty} \mathcal{F}_d$$

decomposes via degree operator $\mathcal{D} = \frac{1}{-\varepsilon_1 \varepsilon_2} \sum_{k=1}^{\infty} V_k V_{-k}$.

I. Random Partitions via Circular β-Ensembles

Write Jacks $P_\lambda(w_1, \dots, w_N|\beta)$ via rescaled power sums V_k,

$$P_\lambda(V|\varepsilon_2, \varepsilon_1) = \sum_{\mu \in \mathbb{Y}_d} \chi_\lambda^\mu(\varepsilon_2, \varepsilon_1) V_\mu$$

strange superpositions of $V_1^{\#_1} V_2^{\#_2} \cdots$ diagonalizing $\mathcal{H}$.

- $V_\mu \in \mathcal{F}$ indexed by *partitions* $\mu = 1^{\#_1} 2^{\#_2} \cdots k^{\#_k}$
- $P_\lambda \in \mathcal{F}$ indexed by *partitions* $\lambda \in \mathbb{Y}_d$ of degree d

$$0 \le \cdots \le \lambda_2 \le \lambda_1 \qquad \deg(\lambda) := \sum_{i=1}^{\infty} \lambda_i$$

- inner product $-\varepsilon_1 \varepsilon_2$ vs. anisotropy $\varepsilon_1 + \varepsilon_2$
- compare $\frac{\beta}{2} = -\frac{\varepsilon_2}{\varepsilon_1}$ inverse Jack parameter
- at the isotropic point $\varepsilon_1 + \varepsilon_2 = 0$ or $\beta = 2$

$$(\varepsilon_2, \varepsilon_1) \to (-\varepsilon, \varepsilon)$$

Jacks become *Schur functions*.

Since V_μ and P_λ are **both** orthogonal for $\langle\cdot,\cdot\rangle_{-\varepsilon_1\varepsilon_2}$, write the resolution of the identity in two ways

$$\prod_{k=1}^{\infty} \exp\left(\frac{\overline{V_k^{\mathrm{out}}}V_k^{\mathrm{in}}}{-\varepsilon_1\varepsilon_2 k}\right) = \sum_{\lambda\in\mathbb{Y}} \frac{P_\lambda(\overline{V^{\mathrm{out}}}|\varepsilon_2,\varepsilon_1)P_\lambda(V^{\mathrm{in}}|\varepsilon_2,\varepsilon_1)}{\langle P_\lambda, P_\lambda\rangle_{-\varepsilon_1\varepsilon_2}}$$

to define **Stanley-Cauchy kernel** $\Pi(\overline{V^{\mathrm{out}}}, V^{\mathrm{in}}|\frac{1}{-\varepsilon_1\varepsilon_2})$.

Definition: If $V_k^{\mathrm{out}} = V_k^{\mathrm{in}}$ for all $k \in \mathbb{Z}_+$, define $M_V(\varepsilon_2,\varepsilon_1)$ *Jack Measure* on partitions λ by

$$\mathrm{Prob}_{V;\varepsilon_2,\varepsilon_1}(\lambda) = \frac{1}{\Pi(\overline{V}, V|\frac{1}{-\varepsilon_1\varepsilon_2})} \cdot \frac{\left|P_\lambda(V|\varepsilon_2,\varepsilon_1)\right|^2}{\langle P_\lambda, P_\lambda\rangle_{-\varepsilon_1\varepsilon_2}}$$

I. Random Partitions via Circular β-Ensembles

Motivation: On $\mathbb{T}$ the unit circle,

$$V(w) = \sum_{k=1}^{\infty} \frac{\overline{V_k^{\mathrm{out}}} w^{-k}}{k} + \sum_{k=1}^{\infty} \frac{V_k^{\mathrm{in}} w^k}{k}$$

defines $V : \mathbb{T} \to \mathbb{R}$ iff $V^{\mathrm{out}} \equiv V^{\mathrm{in}}$.

Random Matrices: class functions on $U(N)$ lead to

$$\mathrm{Z}_{\mathbb{T};V}(N|2,t) \quad = \quad \oint_{\mathbb{T}^N} e^{-\frac{N}{t}\sum_{i=1}^{N} V(w_i)} \prod_{i<j} \left| w_i - w_j \right|^2 \prod_{i=1}^{N} \frac{dw_i}{2\pi \mathbf{i} w_i}$$

Gessel-Heine-Szegö-Weyl: Z is a *Toeplitz determinant*

$$\det T_N\left(e^{-\frac{N}{t}V(w)}\right) = \mathrm{Z}_{\mathbb{T};V}(N|2,t) = \Pi \cdot \mathbb{P}\left(\lambda_1' \leq N\right).$$

and *law of first column* of λ from Schur measure $M_V(-\varepsilon, \varepsilon)$.

- Random Matrices $\to$ Random Partitions: N *formal variable!*

I. Random Partitions via Circular β-Ensembles

Motivation: Jack measures $M_V(\varepsilon_2, \varepsilon_1)$ on partitions

$$\prod_{k=1}^{\infty} \exp\left(\frac{\overline{V_k^{\text{out}}} V_k^{\text{in}}}{-\varepsilon_1 \varepsilon_2 k}\right) = \Pi(\overline{V}, V | \tfrac{1}{-\varepsilon_1\varepsilon_2}) = \sum_{\lambda \in \mathbb{Y}} \frac{|P_\lambda(V|\varepsilon_2, \varepsilon_1)|^2}{\langle P_\lambda, P_\lambda \rangle_{-\varepsilon_1\varepsilon_2}}$$

unify β, V deformations of *Poissonized Plancherel measures:*

$$\begin{tikzcd}
& M_V(\varepsilon_2, \varepsilon_1) \arrow[dl] \arrow[dr] & \\
M_V(-\varepsilon, \varepsilon) \arrow[dr] & & M_\bullet(\varepsilon_2, \varepsilon_1) \arrow[dl] \\
& M_\bullet(-\varepsilon, \varepsilon) &
\end{tikzcd}$$

$(\varepsilon_2, \varepsilon_1) \longrightarrow (-\varepsilon, \varepsilon)$ Schur measures Okounkov (2001)

$V_\bullet(w) = w + \frac{1}{w}$ abelian pure Nekrasov-Okounkov (2006)

I. Random Partitions via Circular β-Ensembles

Motivation: Circular β-Ensembles, random $\vec{w} \in \mathbb{T}^N$:

$$Z_{\mathbb{T};V}(N|\beta,t) \quad := \quad \oint_{\mathbb{T}^N} e^{-\frac{N}{t}\sum_{i=1}^{N} V(w_i)} \prod_{i<j} \left| w_i - w_j \right|^{\beta} \prod_{i=1}^{N} \frac{dw_i}{2\pi \mathbf{i} w_i}$$

From *Probability* to *Integrability*:

- strong-weak duality + phase transitions $\qquad \frac{\beta}{2} \longleftrightarrow \frac{2}{\beta}$
- $\left|\Psi_\circ\right|^2$ Calogero-Sutherland QMBP $\qquad \Psi_\circ = \prod_{i<j}(w_i - w_j)^{+\beta/2}$

β-**Gessel-Heine-Szegö-Weyl:** Can write Z via expectation

$$\text{Toeplitz operators} = Z_{\mathbb{T};V}(N|\beta,t) = \Pi \cdot \mathbb{E}\Big[o_\lambda(t|N,\beta)\Big]$$

against Jack measure on partitions so that N *formal variable!*

$$-\varepsilon_1\varepsilon_2 = \frac{2}{\beta} \cdot \frac{t^2}{N^2} \qquad \varepsilon_1 + \varepsilon_2 = \left(\frac{2}{\beta} - 1\right)\frac{t}{N}.$$

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II. Limit Shapes and Gaussian Fluctuations

III. Nazarov-Sklyanin Lax Operator

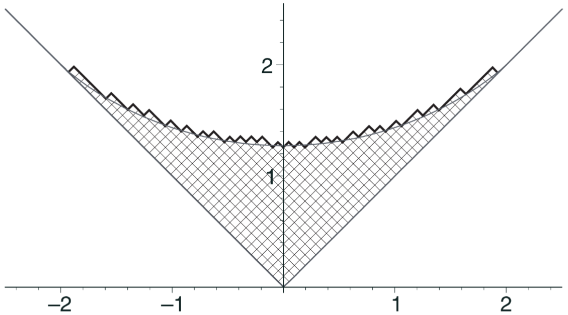
IV. From Hierarchies to Scaling Limits

II. Limit Shapes and Gaussian Fluctuations

For Jack measures $M_V(\varepsilon_2, \varepsilon_1)$, random $\deg(\lambda) = \lambda_1 + \lambda_2 \cdots$

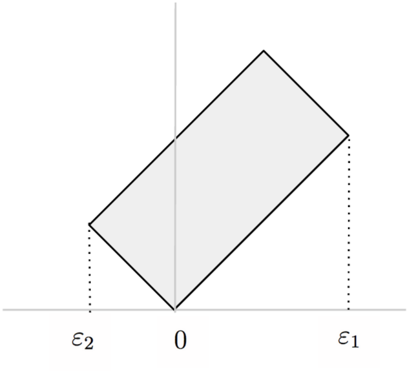
$$\mathbb{E}[\deg(\lambda)] = \frac{1}{-\varepsilon_1\varepsilon_2} \sum_{k=1}^{\infty} |V_k|^2 < \infty$$

For $V_\bullet(w) = w + \frac{1}{w}$, typical λ from $M_\bullet(-\varepsilon, \varepsilon)$ as $\varepsilon \to 0$:



II. Limit Shapes and Gaussian Fluctuations

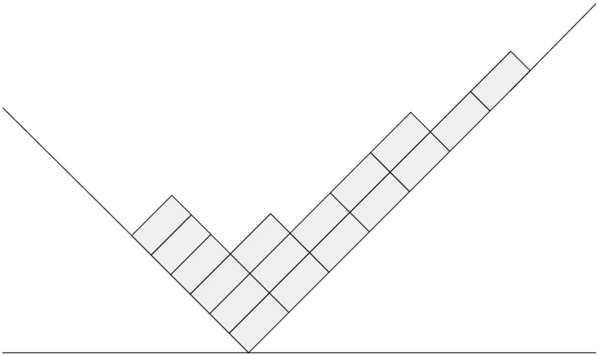
For $\varepsilon_2 < 0 < \varepsilon_1$ real parameters, consider *anisotropic box*



- Area$\left(\square_{\varepsilon_2,\varepsilon_1}\right) = 2(-\varepsilon_1\varepsilon_2)$.
- Isotropic boxes $(\varepsilon_2, \varepsilon_1) \to (-\varepsilon, \varepsilon)$ are squares.

II. Limit Shapes and Gaussian Fluctuations

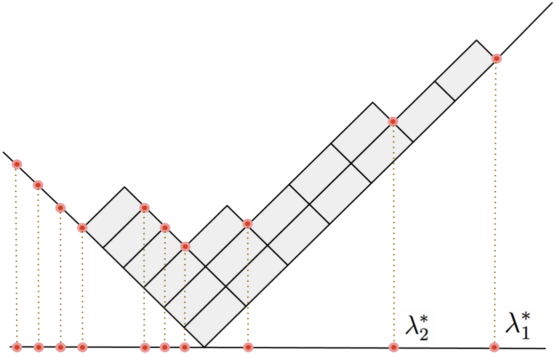
A partition λ ...



Rows: $0 \leq \cdots \leq 0 \leq 1 \leq 1 \leq 1 \leq 2 \leq 5 \leq 7$

II. Limit Shapes and Gaussian Fluctuations

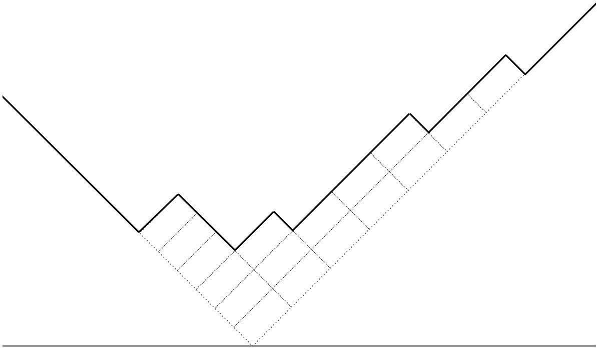
A partition λ ...



Shifted variables: $\lambda_i^* = \varepsilon_2(i-1) + \varepsilon_1\,\lambda_i$

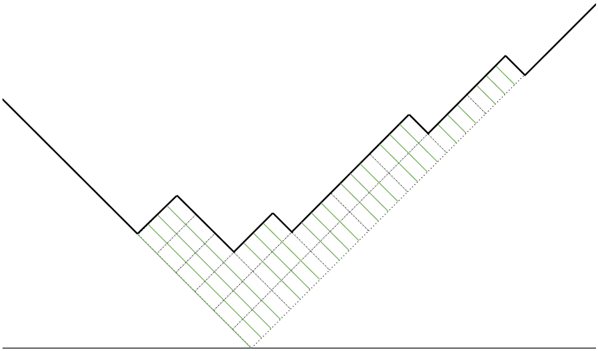
II. Limit Shapes and Gaussian Fluctuations

A partition λ ... and its anisotropic profile $f_\lambda(c|\varepsilon_2, \varepsilon_1)$



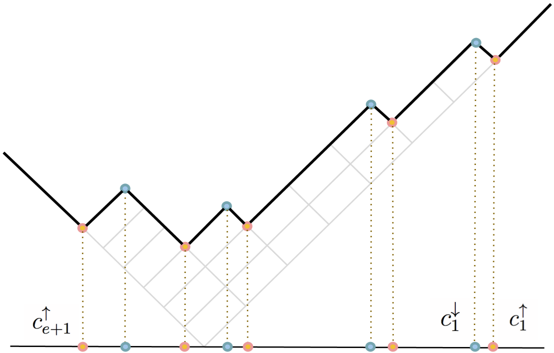
II. Limit Shapes and Gaussian Fluctuations

A partition λ ... and its anisotropic profile $f_\lambda(c|\varepsilon_2, \varepsilon_1)$



II. Limit Shapes and Gaussian Fluctuations

A partition λ ... and its anisotropic profile $f_\lambda(c|\varepsilon_2, \varepsilon_1)$



Linear statistics: $\mathrm{ch}_\ell[f] = \int_{-\infty}^{\infty} c^\ell \frac{1}{2} f''(c) dc$

II. Limit Shapes and Gaussian Fluctuations

Partitions: sequences

$$0 \leq \cdots \leq \lambda_2 \leq \lambda_1$$

of non-negative integers $\lambda_i \in \mathbb{N}$ such that

$$\deg(\lambda) := \sum_{i=1}^{\infty} \lambda_i < \infty \qquad \mathbb{Y} = \bigcup_{d=0}^{\infty} \mathbb{Y}_d$$

Profiles: functions $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ of $c \in \mathbb{R}$ such that

$$|f(c_1) - f(c_2)| \leq 1 \cdot |c_1 - c_2|$$

$$\text{Area}(f) := \int_{-\infty}^{\infty} (f(c) - |c|) dc < \infty \qquad \mathcal{Y} = \bigcup_{A=0}^{\infty} \mathcal{Y}(A).$$

II. Limit Shapes and Gaussian Fluctuations

Recall

$$\mathbb{E}[\deg(\lambda)] = \frac{1}{-\varepsilon_1\varepsilon_2} \sum_{k=1}^{\infty} |V_k|^2$$

Macroscopic Scaling: If we *choose* to represent λ as *anisotropic partition* $\lambda \in \mathbb{Y}(\varepsilon_2, \varepsilon_1)$ with the *same* $\varepsilon_2, \varepsilon_1$ defining $M_V(\varepsilon_2, \varepsilon_1)$, then

$$\mathbb{E}\big[\mathrm{Area}(f_\lambda(\cdot|\varepsilon_2, \varepsilon_1))\big] = 2 \sum_{k=1}^{\infty} |V_k|^2$$

is $\underline{\text{independent of both } \varepsilon_1 \text{ and } \varepsilon_2!}$.

Thus, have scaled to “see something" as either $\varepsilon_2 \to 0 \leftarrow \varepsilon_1$.

II. Limit Shapes and Gaussian Fluctuations

Motivation: Recall the change of variables

$$-\varepsilon_1\varepsilon_2 = \frac{2}{\beta} \cdot \frac{t^2}{N^2} \qquad \varepsilon_1 + \varepsilon_2 = \Big(\frac{2}{\beta} - 1\Big)\frac{t}{N}.$$

- The *scaling limit* of Jack measures

  $$\varepsilon_2 \to 0 \leftarrow \varepsilon_1$$

  at rate $\frac{\beta}{2} = -\frac{\varepsilon_2}{\varepsilon_1}$ is the *thermodynamic limit*

  $$N \to \infty$$

  of circular β-ensembles in background V at $V^{\text{out}} = V^{\text{in}}$:

  $$V(w) = \sum_{k=1}^{\infty} \frac{\overline{V_k^{\text{out}}} w^{-k}}{k} + \sum_{k=1}^{\infty} \frac{V_k^{\text{in}} w^k}{k}$$

<u>However</u>, will state results in terms of **symbol** $v : \mathbb{T} \to \mathbb{R}$

$$v(w) = \sum_{k=1}^{\infty} \overline{V_k^{\text{out}}} w^{-k} + \sum_{k=1}^{\infty} V_k^{\text{in}} w^k.$$

II. Limit Shapes and Gaussian Fluctuations

For random λ sampled from $M_V(\varepsilon_2, \varepsilon_1)$ with analytic symbol v, in the limit $\varepsilon_2 \to 0 \leftarrow \varepsilon_1$ taken so that $\beta/2 = -\varepsilon_2/\varepsilon_1 > 0$ is fixed,

Theorem 1 [M. 2015] (LLN) *The random profile*

$$f_\lambda(c|\varepsilon_2, \varepsilon_1) \to f_{\star|v}(c)$$

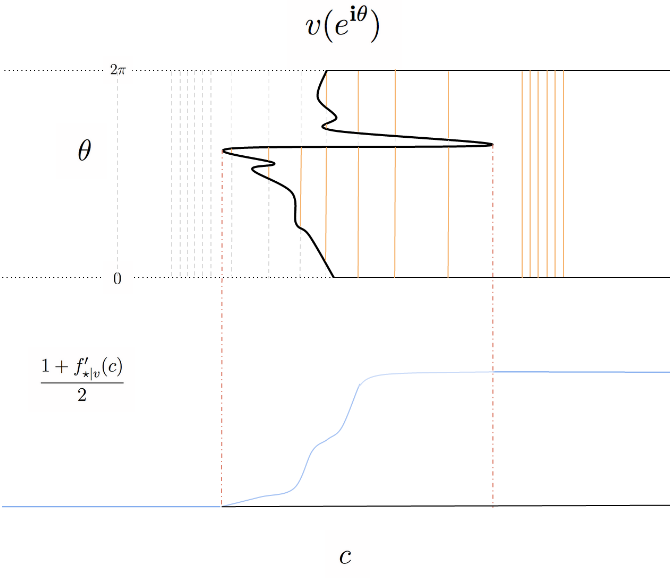
concentrates on a limit shape $f_{\star|v}(c) \in \mathcal{Y}$, independent of β:

$$2\pi \cdot \frac{1+f'_{\star|v}(c)}{2} = (v_* d\theta)\Big((-\infty, c)\Big)$$

is the distribution function of the **push-forward along** $v : \mathbb{T} \to \mathbb{R}$ of the **uniform measure on the circle**.

- Recover: $\beta = 2$ Okounkov (2003).

LLN: despite valleys of v, we are in one-cut regime.

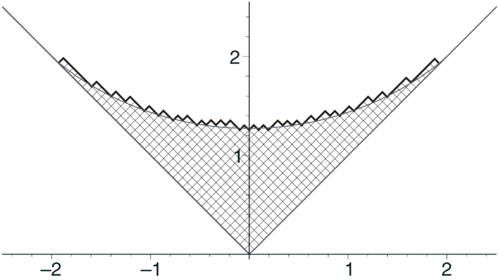


II. Limit Shapes and Gaussian Fluctuations

<u>Recover:</u> Poissonized Plancherel $v_\bullet(e^{\mathbf{i}\theta}) = 2\cos\theta$, new proof

$$f'_{\star|\bullet}(c) = \tfrac{2}{\pi}\arcsin\tfrac{c}{2}$$

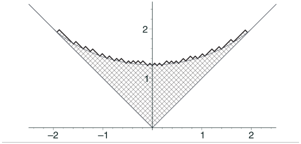
of Vershik-Kerov + Logan-Shepp (1977) without LDP.



Corrections: (CLT) Kerov (1993), Ivanov-Olshanski (2003)

II. Limit Shapes and Gaussian Fluctuations

Application: Since VKLS is supported on $[-2, 2]$



$$\det T_N\left(e^{-\frac{N}{t}(w+\frac{1}{w})}\right) = Z_{\mathbb{T};w+\frac{1}{w}}(N|2, t) = \Pi \cdot \mathbb{P}\left(\varepsilon\,\lambda_1 \le t\right)$$

implies the Gross-Witten phase transition at $t = 2$ (1980).

- Forrester, Majumdar, Schehr *Non-intersecting Brownian walkers and Yang-Mills theory on the sphere* (2010)

Corrections: At $\beta = 2$ **and** $V_\bullet(w) = w + \frac{1}{w}$, as $\varepsilon \to 0$

$$\mathbb{P}\left(\frac{\varepsilon\,\lambda_1 - 2}{\varepsilon^{2/3}} \le s\right) \to \mathrm{TW}_2(s)$$

via double scaling limit in Baik-Deift-Johansson (1999).

II. Limit Shapes and Gaussian Fluctuations

For random λ sampled from $M_V(\varepsilon_2, \varepsilon_1)$ with analytic symbol v, in the limit $\varepsilon_2 \to 0 \leftarrow \varepsilon_1$ taken so that $\beta/2 = -\varepsilon_2/\varepsilon_1 > 0$ is fixed,

Theorem 2 [M. 2015] (CLT) *Profile fluctuations*

$$\phi_\lambda(c|\varepsilon_2, \varepsilon_1) := \frac{1}{\sqrt{-\varepsilon_1\varepsilon_2}}\Big(f_\lambda(c|\varepsilon_2, \varepsilon_1) - f_{\star|v}(c)\Big)$$

converge to a Gaussian field: besides explicit shift $X_v(c)$,

$$\phi_v(c) = (v_* \Phi^{\mathbb{H}_+}\Big|_{\mathbb{T}_+})(c) + \left(\sqrt{\frac{2}{\beta}} - \sqrt{\frac{\beta}{2}}\right) X_v(c).$$

this is **push-forward along** $v : \mathbb{T} \to \mathbb{R}$ **of the restriction to** $\mathbb{T}_+ = \mathbb{T} \cap \mathbb{H}_+$ **of the *Gaussian free field* on** $\mathbb{H}_+$

$$\operatorname{Cov}\Big[\Phi^{\mathbb{H}_+}(w_1), \Phi^{\mathbb{H}_+}(w_2)\Big] = \frac{1}{4\pi} \log \left| \frac{w_1 - \overline{w_2}}{w_1 - w_2} \right|^2$$

with zero boundary conditions.

II. Limit Shapes and Gaussian Fluctuations

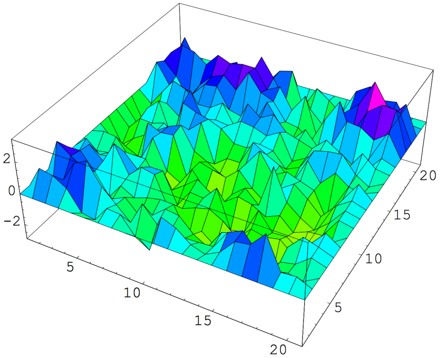


Figure 4.1: Discrete Gaussian free field on 20 by 20 grid with zero boundary conditions.

- Compare: Borodin *Gaussian free field in β-ensembles and random surfaces* (Lecture C.M.I. 2013)
- Recover: Breuer-Duits CLT for biorthogonal ensembles at $\beta = 2$ with "symbol" v (2013)

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III. Nazarov-Sklyanin Lax Operator

IV. From Hierarchies to Scaling Limits

III. Nazarov-Sklyanin Lax Operator

Recall $\mathcal{F} = \mathbb{C}[V_1, V_2, \ldots]$ with $\langle \cdot, \cdot \rangle_{-\varepsilon_1\varepsilon_2}$ via

$$V_{-k} := -\varepsilon_1\varepsilon_2 k \frac{\partial}{\partial V_k}.$$

Define the $\widehat{\mathfrak{gl}_1}$ current

$$\mathbf{v}(w|-\varepsilon_1\varepsilon_2) = \sum_{k=-\infty}^{\infty} V_k \otimes w^{-k}$$

at level $-\varepsilon_1\varepsilon_2$ and $V_0 = 0$.

Auxiliary Hardy Space: $\mathbb{C}[w]$ has basis $w^h = |h\rangle$ for $h \in \mathbb{N}$

$$\pi_\bullet : \mathbb{C}[w, w^{-1}] \to \mathbb{C}[w].$$

is self-adjoint for pairing

$$\langle h_+ | h_- \rangle = \delta(h_+ - h_-)$$

III. Nazarov-Sklyanin Lax Operator

Proposition: The *Toeplitz operator*

$$\mathsf{T}(\mathbf{v}|-\varepsilon_1\varepsilon_2) := (\mathbb{1}\otimes\pi_\bullet)\mathbf{v}(w|-\varepsilon_1\varepsilon_2)(\mathbb{1}\otimes\pi_\bullet)$$

with symbol $\mathbf{v}$ is well-defined $\mathsf{T}(\mathbf{v}) : \mathcal{F}\otimes\mathbb{C}[w]\to\mathcal{F}\otimes\mathbb{C}[w]$ and self-adjoint with respect to $\langle\cdot,\cdot\rangle_{-\varepsilon_1\varepsilon_2}\otimes\delta(h_+-h_-)$.

Compare: if $V_{-k}, V_k\in\mathbb{C}$ are modes of scalar symbol

$$v(w)=\sum_{k=-\infty}^{\infty}V_k w^{-k}$$

can define a *Toeplitz operator*

$$\mathsf{T}(v)=\pi_\bullet v(w)\pi_\bullet$$

which gives $\mathsf{T}(v):\mathbb{C}[w]\to\mathbb{C}[w]$ if $v(w)\in\mathbb{C}[w,w^{-1}]$.

III. Nazarov-Sklyanin Lax Operator

Definition: The *Nazarov-Sklyanin Lax operator*

$$\mathcal{L} : \mathcal{F} \otimes \mathbb{C}[w] \to \mathcal{F} \otimes \mathbb{C}[w]$$

is given by

$$\mathcal{L} = \mathcal{L}(\varepsilon_2, \varepsilon_1) = \mathsf{T}(\mathbf{v}| -\varepsilon_1\varepsilon_2) + (\varepsilon_1 + \varepsilon_2)\mathcal{D}_{\mathsf{aux}}$$

- $\mathsf{T}(\mathbf{v}|-\varepsilon_1\varepsilon_2) = (\mathbb{1} \otimes \pi_\bullet)\mathbf{v}(w|-\varepsilon_1\varepsilon_2)(\mathbb{1} \otimes \pi_\bullet)$ Toeplitz
- $\mathcal{D}_{\mathsf{aux}} = \mathbb{1} \otimes w\frac{\partial}{\partial w}$ auxiliary degree operator.

Even at the isotropic point

$$(\varepsilon_2, \varepsilon_1) \to (-\varepsilon, \varepsilon)$$

$\mathsf{T}(\mathbf{v})$ still unbounded operator on $\mathcal{F} \otimes \mathbb{C}[w]$.

III. Nazarov-Sklyanin Lax Operator

$\mathcal{L}$ is operator on $\mathbb{C}[w]$ with coefficients $\langle h_+|\mathcal{L}|h_-\rangle : \mathcal{F} \to \mathcal{F}$:

$$\mathcal{L} = \begin{bmatrix} 0 & v_1 & v_2 & v_3 & \cdots & v_h & \cdots \\ v_{-1} & \mathbf{1}(\varepsilon_1+\varepsilon_2) & v_1 & v_2 & \ddots & v_{h-1} & \ddots \\ v_{-2} & v_{-1} & \mathbf{2}(\varepsilon_1+\varepsilon_2) & v_1 & \ddots & \ddots & \ddots \\ v_{-3} & v_{-2} & v_{-1} & \mathbf{3}(\varepsilon_1+\varepsilon_2) & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ v_{-h} & v_{-(h-1)} & \ddots & \ddots & \ddots & \mathbf{h}(\varepsilon_1+\varepsilon_2) & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

- (pairings) When $k_+ = k_- = \mathbf{k}$, $[V_{-k}, V_k] = -\varepsilon_1\varepsilon_2\mathbf{k}$
- (slides) When $h_+ = h_- = \mathbf{h}$, $\mathcal{L}_{h,h} = (\varepsilon_1+\varepsilon_2)\mathbf{h}$.
- $\mathcal{L}_{h,h} \equiv 0$ if $\varepsilon_1 + \varepsilon_2 = 0$ (*no slides iff* $\beta = 2$).

III. Nazarov-Sklyanin Lax Operator

Recall $V_{-k} = -\varepsilon_1\varepsilon_2 k \frac{\partial}{\partial V_k}$. Notice that

$$\langle 0|\mathcal{L}^2|0\rangle = \sum_{h=0}^{\infty} \mathcal{L}_{0,h}\mathcal{L}_{h,0} = \sum_{k=1}^{\infty} V_k V_{-k}$$

is a multiple of degree operator, and

$$\begin{aligned}\langle 0|\mathcal{L}^3|0\rangle &= \sum_{h_1,h_2=0}^{\infty} \mathcal{L}_{0,h_1}\mathcal{L}_{h_1,h_2}\mathcal{L}_{h_2,0} \\ &= \sum_{h_1,h_2=0}^{\infty} V_{+h_1}V_{h_2-h_1}V_{-h_2} + (\varepsilon_1+\varepsilon_2)\sum_{h=0}^{\infty} hV_hV_{-h}\end{aligned}$$

is the Hamiltonian of the quantum Benjamin-Ono equation.

- Recover: for $V_k = p_k(-\varepsilon_2)$ at $p_k = w_1^k + \cdots + w_N^k$, these $\langle 0|\mathcal{L}^2|0\rangle$ and $\langle 0|\mathcal{L}^3|0\rangle$ are $\mathcal{D}$ and $\mathcal{H}$ for circular β-ensembles!

III. Nazarov-Sklyanin Lax Operator

Theorem [NS], First Part: *For all* $\ell = 0, 1, 2 \ldots$, *the VEVs*

$$\langle 0|\mathcal{L}^{\ell}|0\rangle : \mathcal{F} \to \mathcal{F}$$

commute and give the quantum Benjamin-Ono hierarchy.

Gather these operators as modes of the *resolvent*

$$\mathcal{R}(u) = (u - \mathcal{L})^{-1} := \sum_{\ell=0}^{\infty} u^{-\ell-1}\mathcal{L}^{\ell}$$

for u formal parameter. The family of operators

$$\mathcal{T}(u|\, \varepsilon_2, \varepsilon_1) := \langle 0|\mathcal{R}(u|\, \varepsilon_2, \varepsilon_1)|0\rangle$$

commute $[\mathcal{T}(u_1), \mathcal{T}(u_2)] = 0$.

- Compare: **R** for $\mathrm{Y}_{\varepsilon_1+\varepsilon_2}(\widehat{\mathfrak{gl}_1})$ Maulik-Okounkov (2012)

III. Nazarov-Sklyanin Lax Operator

As $\dim \mathcal{F}_d < \infty$ and all $\langle 0|\mathcal{L}^\ell|0\rangle$ self-adjoint w.r.t. $\langle\cdot,\cdot\rangle_{-\varepsilon_1\varepsilon_2}$ commute with degree operator $\langle 0|\mathcal{L}^2|0\rangle$,

Theorem [NS], Second Part: *The basis of $\mathcal{F}_d$*

$$P_\lambda(V|\varepsilon_2,\varepsilon_1) = \sum_{|\mu|=d} \chi^\lambda_\mu(\varepsilon_2,\varepsilon_1) V_\mu$$

simultaneously diagonalizing all $\langle 0|\mathcal{L}^\ell|0\rangle$ are the Jack symmetric functions with inverse Jack parameter $\frac{\beta}{2} = -\frac{\varepsilon_2}{\varepsilon_1}$.

This gives a **new definition** of Jacks that is *stable* (no N).

- Compare: Sergeev-Veselov *Dunkl operators at infinity and Calogero-Moser systems* (2013)
- Compare: Dubrovin *Symplectic field theory of a disk, quantum integrable systems, and Schur polynomials* (2014)

III. Nazarov-Sklyanin Lax Operator

Theorem [NS] Third Part: *The eigenvalues of*

$$\mathcal{T}(u|\,\varepsilon_2, \varepsilon_1) = \langle 0|(u - \mathcal{L})^{-1}|0\rangle$$

on Jacks $P_\lambda(V|\varepsilon_2, \varepsilon_1)$ *are Stieltjes transforms*

$$T_\lambda(u|\,\varepsilon_2, \varepsilon_1) = \int_{-\infty}^{+\infty} \frac{\tau_\lambda(c|\,\varepsilon_2, \varepsilon_1)dc}{u - c}$$

of the anisotropic ***transition measure*** $\tau_\lambda(c|\,\varepsilon_2, \varepsilon_1)$ *of* λ*.*

This $\tau_\lambda(c|\,\varepsilon_2, \varepsilon_1)$ appears naturally in Pieri rule for Jacks.

- Compare: Jucy-Murphy elements for $\mathbb{C}[S(d)]$ and transition measures $\tau_\lambda(c|-\varepsilon, \varepsilon)$ at $\varepsilon_1 + \varepsilon_2 = 0$ Biane (1998)

III. Nazarov-Sklyanin Lax Operator

Alternatively, the relation

$$f_\lambda(c|\,\varepsilon_2, \varepsilon_1) \longleftrightarrow \tau_\lambda(c|\,\varepsilon_2, \varepsilon_1)$$

is an incarnation of the **Kerov-Markov-Krein transform** which associates to *any* profile $f \in \mathcal{Y}$ a probability measure τ_f on $\mathbb{R}$ defined by

$$\exp\left(\int_{-\infty}^{\infty} \log \left[\frac{1}{u - c} \right] \tfrac{1}{2} f''(c) dc \right) = \int_{-\infty}^{\infty} \frac{\tau_f(c) dc}{u - c}.$$

Corollary: *The spectral measure* $\tau_\lambda(c|\varepsilon_2, \varepsilon_1)$ *of*

$$\mathcal{L} : \mathcal{F} \otimes \mathbb{C}[w] \to \mathcal{F} \otimes \mathbb{C}[w]$$

at the vector $P_\lambda(v|\varepsilon_2, \varepsilon_1) \otimes |0\rangle$ *is the Kerov-Markov-Krein transform of the profile* $f_\lambda(c|\varepsilon_2, \varepsilon_1)$.

III. Nazarov-Sklyanin Lax Operator

Theorem [NS] Third Part + KMK: *The eigenvalues of*

$$\mathcal{T}(u|\varepsilon_2, \varepsilon_1) = \langle 0|(u - \mathcal{L})^{-1}|0\rangle$$

on Jacks $P_\lambda(v|\varepsilon_2, \varepsilon_1)$ *depend* ***only*** *on the profile* $f_\lambda(c|\varepsilon_2, \varepsilon_1)$.

Explicitly, the logarithmic derivative

$$\frac{\partial}{\partial u} \log \mathcal{T}(u)$$

acts on Jacks with eigenvalues

$$\int_{-\infty}^{\infty} \frac{\frac{1}{2} f_\lambda''(c|\varepsilon_2, \varepsilon_1) dc}{u - c}.$$

- Compare: log derivative of transmission coefficient for KdV
- Compare: log derivative of transfer matrix for XXZ

Outline

I. Random Partitions via Circular β-Ensembles

II. Limit Shapes and Gaussian Fluctuations

III. Nazarov-Sklyanin Lax Operator

IV. From Hierarchies to Scaling Limits

IV. From Hierarchies to Scaling Limits: **Recipe**

Probability: To regard object $\lambda \in \mathbb{Y}$ as random with weight $W_\lambda(V|\varepsilon) \geq 0$ in variables V and ε, the *partition function*

$$\Pi(V|\varepsilon) := \sum_{\lambda} W_\lambda(V|\varepsilon)$$

must converge so that $\mathrm{Prob}_{V,\varepsilon}(\lambda) = \frac{1}{\Pi(V|\varepsilon)} W_\lambda(V|\varepsilon)$.

Integrability: Given a family of operators $\mathcal{T}(u|\varepsilon)$ indexed by $u \in \mathbb{U}$, acting in variables V, and simultaneously diagonal

$$\mathcal{T}(u|\varepsilon) W_\lambda(V|\varepsilon) = T_\lambda(u|\varepsilon) W_\lambda(V|\varepsilon)$$

on weights $W_\lambda(V|\varepsilon)$ with eigenvalues $T_\lambda(u|\varepsilon)$, we have

$$\Pi(V|\varepsilon)^{-1} \mathcal{T}(u_1|\varepsilon) \cdots \mathcal{T}(u_n|\varepsilon) \Pi(V|\varepsilon) = \mathbb{E}\Big[T_\lambda(u_1|\varepsilon) \cdots T_\lambda(u_n|\varepsilon) \Big].$$

IV. From Hierarchies to Scaling Limits: **Ingredients**

1. Random partition $\lambda \in \mathbb{Y}$ sampled with relative weight

$$W_\lambda(V|\varepsilon) := \left.\frac{P_\lambda(\overline{V^{\mathrm{out}}}|\,\varepsilon_2,\varepsilon_1)P_\lambda(V^{\mathrm{in}}|\,\varepsilon_2,\varepsilon_1)}{\langle P_\lambda, P_\lambda\rangle_{-\varepsilon_1\varepsilon_2}}\right|_{V^{\mathrm{out}}=V^{\mathrm{in}}}$$

2. Partition function: decoupled ("Amazing Fact")

$$\prod_{k=1}^{\infty}\exp\left(\frac{\overline{V_k^{\mathrm{out}}}V_k^{\mathrm{in}}}{-\varepsilon_1\varepsilon_2\,k}\right) = \Pi = \sum_{\lambda\in\mathbb{Y}} W_\lambda(V|\varepsilon)$$

3. Observables: $\mathcal{L} : \mathcal{F}\otimes\mathbb{C}[w] \to \mathcal{F}\otimes\mathbb{C}[w]$ resolvent VEV

$$\langle 0|\frac{1}{(u-\mathcal{L}(\varepsilon_2,\varepsilon_1))}|0\rangle P_\lambda = \left(\int_{-\infty}^{\infty}\frac{\tau_\lambda(c|\,\varepsilon_2,\varepsilon_1)}{u-c}\right)P_\lambda$$

- Joint moments of linear statistics computable via

$$\Pi^{-1}\langle 0|(u_1-\mathcal{L}^{\mathrm{out}})^{-1}|0\rangle\cdots\langle 0|(u_n-\mathcal{L}^{\mathrm{out}})^{-1}|0\rangle\Pi$$

IV. From Hierarchies to Scaling Limits: Cooking

$$\Pi^{-1}\langle 0|(u_1-\mathcal{L}^{\mathrm{out}})^{-1}|0\rangle\cdots\langle 0|(u_n-\mathcal{L}^{\mathrm{out}})^{-1}|0\rangle\Pi$$

1. $\mathcal{L}(\varepsilon_2,\varepsilon_1)=\mathsf{T}(\mathbf{v}|-\varepsilon_1\varepsilon_2)+(\varepsilon_1+\varepsilon_2)\mathcal{D}$ made of V_k and

$$V_{-k}=-\varepsilon_1\varepsilon_2 k\frac{\partial}{\partial V_k}.$$

2. Keep track of pairings (Leibniz rule) and slides
3. Exchange relation

$$[\overline{V_{-k}^{\mathrm{out}}},\Pi]=V_k^{\mathrm{in}}.$$

from Kac-Moody symbol to scalar symbol $v:\mathbb{T}\to\mathbb{R}$

$$\begin{aligned}\mathbf{v}^{\mathrm{out}}(w|-\varepsilon_1\varepsilon_2) &= \sum_{k=1}^{\infty}\overline{V_k^{\mathrm{out}}}w^{-k}+\sum_{k=1}^{\infty}\overline{V_{-k}^{\mathrm{out}}}w^{k}\\ v(w) &= \sum_{k=1}^{\infty}\overline{V_k^{\mathrm{out}}}w^{-k}+\sum_{k=1}^{\infty}V_k^{\mathrm{in}}w^{k}.\end{aligned}$$

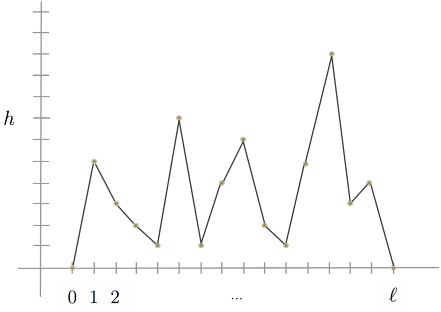
IV. From Hierarchies to Scaling Limits: **Symbols**

At $\varepsilon_2 = 0 = \varepsilon_1$, $n = 1$ case reduces to

$$\Pi^{-1}\langle 0|\mathcal{L}(\varepsilon_2, \varepsilon_1)^{\ell}|0\rangle\Pi \longrightarrow \langle 0|T(v)^{\ell}|0\rangle$$

VEV of power of Toeplitz operator with symbol

$$v(w) = \sum_{k=1}^{\infty} \overline{V_k^{\text{out}}} w^{-k} + \sum_{k=1}^{\infty} V_k^{\text{in}} w^k.$$

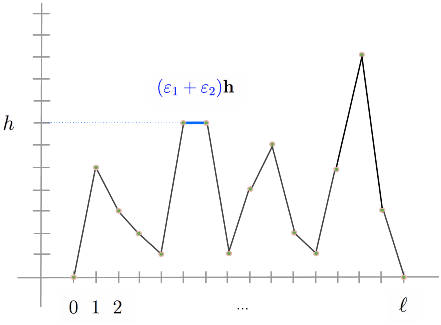


IV. From Hierarchies to Scaling Limits: **Slides**

If $-\varepsilon_1\varepsilon_2 = 0$ but $\varepsilon = \varepsilon_1 + \varepsilon_2 \neq 0$, $n = 1$ case reduces to

$$\Pi^{-1}\langle 0|\mathcal{L}(\varepsilon_2, \varepsilon_1)^{\ell}|0\rangle\Pi \longrightarrow \langle 0|\big(T(v) + \varepsilon\mathcal{D}_{\mathrm{aux}}\big)^{\ell}|0\rangle$$

VEV of power of unbounded perturbation of $T(v)$.



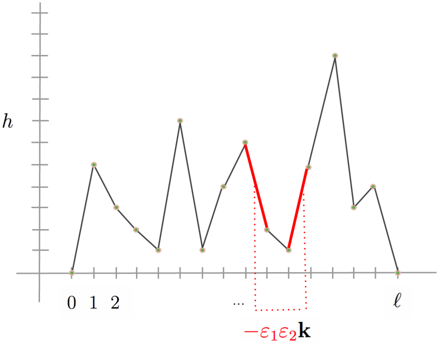
IV. From Hierarchies to Scaling Limits: **Pairings**

If $-\varepsilon_1\varepsilon_2 \neq 0$ but $\varepsilon_1 + \varepsilon_2 = 0$, $n = 1$ case reduces to

$$\Pi^{-1}\langle 0|\mathcal{L}(\varepsilon_2, \varepsilon_1)^{\ell}|0\rangle\Pi \longrightarrow \langle 0|T(\mathbf{v}|-\varepsilon_1\varepsilon_2)^{\ell}|0\rangle$$

VEV of Toeplitz operator with $\widehat{\mathfrak{gl}}_1$ current

$$\mathbf{v}(w|-\varepsilon_1\varepsilon_2) = \sum_{k=-\infty}^{\infty} V_k \otimes w^{-k}$$



IV. From Hierarchies to Scaling Limits: Regularity

Estimates: although the partition function

$$\Pi = \prod_{k=1}^{\infty} \exp\left(\frac{\overline{V_k^{\mathrm{out}}} V_k^{\mathrm{in}}}{-\varepsilon_1 \varepsilon_2 \, k}\right)$$

is convergent if the potential

$$V(w) = \sum_{k=1}^{\infty} \frac{\overline{V_k^{\mathrm{out}}} w^{-k}}{k} + \sum_{k=1}^{\infty} \frac{V_k^{\mathrm{in}} w^k}{k}$$

lies in $H^{1/2}(\mathbb{T})$, our operators $\langle 0|\mathcal{L}^{\ell}|0\rangle$ only defined on $\mathcal{F}$.

<u>Lemma:</u> Π is in the domain of definition of $\langle 0|\mathcal{L}^{\ell}|0\rangle$ iff

$$V \in H^s(\mathbb{T}) \qquad \text{for} \quad \ell = 2s.$$

In particular, moment method requires all Sobolev norms.

- $V : \mathbb{T} \to \mathbb{C}$ analytic around $\mathbb{T}$ lies in all $H^s(\mathbb{T})$

IV. From Hierarchies to Scaling Limits: **Expansions**

For random λ sampled from $M_V(\varepsilon_2, \varepsilon_1)$ with *analytic* symbol v,

Theorem 3 [M. 2015]: $\widehat{W}_n^v(\ell_1, \ldots, \ell_n | \varepsilon_1, \varepsilon_2)$ joint cumulants of $\langle 0|\mathcal{L}^{\ell_i}|0\rangle$ have *convergent* expansion:

$$\sum_{g=0}^{\infty} \sum_{m=0}^{\infty} (-\varepsilon_1\varepsilon_2)^{(\mathbf{n}-1)+g} (\varepsilon_1 + \varepsilon_2)^m \widehat{W}_{n,g,m}^v(\ell_1, \ldots, \ell_n)$$

$\widehat{W}_{n,g,m}^v(\ell_1, \ldots, \ell_n)$ weighted enumeration of **connected "ribbon paths"** on n sites of lengths $\ell_1, \ldots, \ell_n$ with $(n-1) + g$ *pairings* and m *slides*, and are expressed solely through ***matrix elements of the classical Toeplitz operator*** $T(v)$ ***with scalar symbol*** $v(w)$**.**

- Recover: double Hurwitz numbers for Schur measures at $\beta = 2$ or $\varepsilon_1 + \varepsilon_2 = 0$ Okounkov (2000)
- Compare: loop equations and refined topological recursion Chekhov-Eynard (2006), Borot-Guionnet (2012)

IV. From Hierarchies to Scaling Limits: Expansions

Compare: expansion of β-Ensembles on $\mathbb{R}$ for one-cut V:

Theorem [C-E + B-G]: After the change of variables

$$-\varepsilon_1\varepsilon_2 = \frac{2}{\beta}\cdot\frac{t^2}{N^2} \qquad \varepsilon_1+\varepsilon_2 = \left(\frac{2}{\beta}-1\right)\frac{t}{N}.$$

the joint cumulants $\widehat{W}_n^V(\ell_1,\ldots,\ell_n|\varepsilon_2,\varepsilon_1)$ of linear statistics $\int_{-\infty}^{\infty} x^\ell \rho(x)dx$ have *asymptotic* expansion:

$$\sum_{g=0}^{\infty}\sum_{m=0}^{\infty}(-\varepsilon_1\varepsilon_2)^{(\mathbf{n}-1)+g}(\varepsilon_1+\varepsilon_2)^m\widehat{W}^V_{n,g,m}(\ell_1,\ldots,\ell_n)$$

where $\widehat{W}^V_{n,g,m}(\ell_1,\ldots,\ell_n)$ is V-weighted enumeration of **connected ribbon graphs** on n vertices of degree $\ell_1,\ldots,\ell_n$ of genus g and m *Möbius strips*, and are expressed solely through ***geometry of the spectral curve*** Σ_V**.**

IV. From Hierarchies to Scaling Limits: **Expansions**

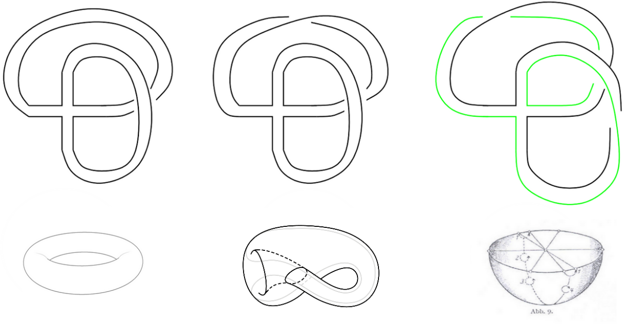


Figure 7: Examples of ribbon graphs of genus $g = 1$.

- Source: Chekhov-Eynard-Marchal *Topological expansion of the Bethe ansatz and quantum algebraic geometry* (2009)
- Compare: $-\varepsilon_1\varepsilon_2$ as “handle-gluing element"

IV. From Hierarchies to Scaling Limits: **Limit Shapes**

Proof Sketch LLN:

- *LLN occurs:* no pairings or slides to leading order.
- *LLN form:* to determine limit shape $f_{\star|v}(c)$, must invert

$$W^{v}_{1,0,0}(u) := \sum_{\ell=0}^{\infty} u^{-\ell-1} \widehat{W}^{v}_{1,0,0}(\ell)$$

which requires the analytic continuation

$$\langle 0|(u - T(v))^{-1}|0\rangle = \exp\left(\frac{1}{2\pi\mathbf{i}} \oint_{\mathbb{T}} \log\left[\frac{1}{u - v(w)}\right] \frac{dw}{w}\right)$$

Krein (1958) and Calderón-Spitzer-Widom (1958), via Wiener-Hopf factorization of the *family of loops*

$$\gamma(w;u) : \mathbb{T} \to GL(1) \qquad\qquad \gamma(w;u) = u - v(w)$$

- *LLN meaning*: Lifshitz-Krein spectral shift function.

IV. From Hierarchies to Scaling Limits: GFF

Proof Sketch CLT:

- *CLT occurs:* it requires $n-1$ pairings to connect n sites, and slides cannot connect correlators
- *CLT form:* to compute covariance of limiting Gaussian field, need $W^{v}_{2,0,0}(u_1, u_2)$: introduce **welding operator**

$$\mathcal{W} := \sum_{k=1}^{\infty} k \frac{\partial}{\partial V^{(1)}_{-k}} \frac{\partial}{\partial V^{(2)}_{k}} \bigg|_{v^{(1)}=v^{(2)}} .$$

Duplicate variables, simulates all ways of creating one pairing of any type k, hence

$$\mathcal{W} : W^{v^{(1)}}_{1,0,0}(u_1) \times W^{v^{(2)}}_{1,0,0}(u_2) \to W^{v}_{2,0,0}(u_1, u_2)$$

have $LLN \times LLN \Rightarrow CLT$ covariance.

- Compare: **loop insertion operator** $\mathcal{K}(u)$ and **unstable correlators** $W^{V}_{1,0,0}(u)$, $W^{V}_{1,0,1}(u)$, $W^{V}_{2,0,0}(u_1, u_2)$

Conclusion

Review:

1. Jack measures $M_V(\varepsilon_2, \varepsilon_1)$ for every symbol $v : \mathbb{T} \to \mathbb{R}$ via N-body circular β-ensembles in background V
2. Limit shapes and global fluctuations as push-forwards
3. Lax operator $\mathcal{L} = \mathsf{T}(\mathbf{v}|-\varepsilon_1\varepsilon_2) + (\varepsilon_1 + \varepsilon_2)\mathcal{D}$ plays role of matrix model for Jack measure arbitrary β and V
4. Toeplitz operators with symbol v and ribbon path expansion: a non-recursive analog of the Chekhov-Eynard refined topological recursion

(Q) Why do global fluctuations involve Gaussian Free Field?

Conclusion

Why do Jack Measure fluctuations involve Gaussian Free Field?

Circular β-ensembles $V \equiv 0 \Rightarrow$ random empirical measure

$$\rho(w) = \frac{t}{N} \sum_{i=1}^{N} \delta(w - w_i).$$

Compare: $N \to \infty$ *limit shape* and *Gaussian fluctuations*:

$$\rho \sim \rho_\star + \tfrac{1}{N}(-\Delta)\left(\Phi\Big|_{\mathbb{T}}\right)$$

where $\rho_\star$ is uniform measure on $\mathbb{T}$ and Φ is a GFF on $\mathbb{C}$.

- $\beta = 2$: Szegö (1915), Diaconis-Shahshahani (1994)
- $\beta > 0$: $\approx$ Johansson (1998), or "obvious" since law circular β-ensembles depends only on empirical measure via

$$\mathrm{Prob}[w_1, \ldots, w_N] = \frac{1}{Z} \exp\Big(- \beta N^2 I_{\mathbb{C}}[\rho] \Big)$$

which is just a conditioned $\rho = -\Delta\Phi$ for Φ GFF on $\mathbb{C}$.

Conclusion

Why do Jack Measure fluctuations involve Gaussian Free Field?

Wait, did we already see Circular β-Ensembles $\to$ GFF?!?!

- **Amazing Fact:** $p_\mu = p_1^{\#_1} p_2^{\#_2} \cdots p_k^{\#_k} \cdots$ are **not** orthogonal with respect to $\langle \cdot, \cdot \rangle_{\mathbb{T};\beta,N}$, but they ***are*** *orthogonal* in limit

$$\langle F^{\text{out}}, F^{\text{in}} \rangle_{-\varepsilon_1\varepsilon_2} := \lim_{N\to\infty} \frac{\langle F^{\text{out}}, F^{\text{in}} \rangle_{\mathbb{T};\beta;N}}{\langle 1, 1 \rangle_{\mathbb{T};\beta;N}}.$$

Conclusion: Stanley-Cauchy identity defining Jack measures

$$\prod_{k=1}^{\infty} \exp\left(\frac{\overline{V_k^{\text{out}}} V_k^{\text{in}}}{-\varepsilon_1\varepsilon_2 k} \right) = \sum_{\lambda\in\mathbb{Y}} \frac{P_\lambda(\overline{V^{\text{out}}}|\varepsilon_2, \varepsilon_1) P_\lambda(V^{\text{in}}|\varepsilon_2, \varepsilon_1)}{\langle P_\lambda, P_\lambda \rangle_{-\varepsilon_1\varepsilon_2}}$$

factorizes in variables V_k *precisely due to GFF fluctuations of circular β-ensembles with $V \equiv 0$.*

Conclusion

Review:

1. Jack measures $M_V(\varepsilon_2, \varepsilon_1)$ for every symbol $v : \mathbb{T} \to \mathbb{R}$ via N-body circular β-ensembles in background V
2. Limit shapes and global fluctuations as <u>push-forwards</u>
3. Lax operator $\mathcal{L} = \mathrm{T}(\mathbf{v}|-\varepsilon_1\varepsilon_2) + (\varepsilon_1 + \varepsilon_2)\mathcal{D}$ plays role of <u>matrix model</u> for Jack measure arbitrary β and V
4. Toeplitz operators with symbol v and ribbon path expansion: a <u>non-recursive analog</u> of the Chekhov-Eynard refined topological recursion

Outlook:

- Our v analytic, yet have **new spectral theory** grounded in classical analysis: *towards symbols of low regularity*.
- $\mathcal{L}$ via quantization of classical of Benjamin-Ono fluid with periodic v ... **dynamical meaning of results?**

Thank you!