Probabilistic interpretation of conservation laws and optimal transport in one dimension

Julien Reygner

École des Ponts
ParisTech

CERMICS – École des Ponts ParisTech

Based on joint works with B. Jourdain.

Conservation laws in one space dimension

Continuum physics is based on **conservation laws**

$$\partial_t \mathbf{u}(t,x) + \mathbf{div}\,\mathbf{J}(t,x) = 0, \qquad t \geq 0, \quad x \in \mathbb{R}^q,$$

where:

- $\mathbf{u} = (u^1, \ldots, u^d) \in \mathbb{R}^d$ is the vector of **densities of conserved quantities**,
- $\mathbf{div}\,\mathbf{J} = (\mathrm{div}\, J^1, \ldots, \mathrm{div}\, J^d)$ with $J^\gamma \in \mathbb{R}^q$ being the **current** of u^γ,

supplemented with **constitutive relations**

$$\mathbf{J} = \mathbf{f}(\mathbf{u}),$$

where $\mathbf{f}$ is the **flux function**.

Throughout this talk, $q = 1$: one space dimension.

- Relevant *e.g.* for **gas dynamics** or **road traffic**.
- Resulting equation is the **nonlinear conservation law**

$$\boxed{\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0,}$$

  with $\mathbf{f} : \mathbb{R}^d \to \mathbb{R}^d$.
- Possibility of introducing a **dissipation mechanism** by adding a viscosity term $\frac{1}{2}\partial_{xx}\mathbf{A}(\mathbf{u})$ on the right-hand side.

Scalar conservation law $d = 1$	**System of conservation laws $d \geq 2$**
Thin notations u, f, ...	Bold notations $\mathbf{u}$, $\mathbf{f}$, ...

Probabilistic interpretation of one-dimensional conservation laws

- Assume that initial conditions $u_0^1, \dots, u_0^d : \mathbb{R} \to \mathbb{R}$ are **monotonic** and **bounded**.
- Up to rescaling: $u_0^1, \dots, u_0^d : \mathbb{R} \to [0,1]$ are **Cumulative Distribution Functions** of $m^1, \dots, m^d \in \mathcal{P}(\mathbb{R})$.
- The coordinates of the solution $u^1(t,\cdot), \dots, u^d(t,\cdot)$ are expected to **remain CDFs**.

Lagrangian description

- Do these CDFs describe the **evolution of a continuum of particles** on $\mathbb{R}$, with an **intrinsic dynamics**?
- What can be deduced from this **Lagrangian description**?

Scalar case with viscosity: **nonlinear Fokker-Planck equation**.

- **McKean-Vlasov** approach: mean-field particle system, propagation of chaos.
- Initiated by **Bossy, Talay** (*Ann. Appl. Probab.* 96, *Math. Comp.* 97), **Jourdain** (*ESAIM: P&S* 97, *Stoch. Proc. Appl.* 00), **Shkolnikov** (*Stoch. Proc. Appl.* 10), ...

Without viscosity: inviscid **Burgers**-like equation.

- Related with pressureless gas dynamics, turbulence models.
- Works by **E, Rykov, Sinai** (*Comm. Math. Phys.* 96), **Brenier, Grenier** (*SIAM J. Numer. Anal.* 98), **Bolley, Brenier, Loeper** (*J. Hyperbolic Differ. Equ.* 05), ...

Outline

The nonlinear Fokker-Planck equation

We consider here the **scalar parabolic Cauchy problem**

$$\begin{cases} \partial_t u + \partial_x f(u) = \frac{1}{2}\partial_{xx} A(u), \\ u(0,x) = u_0(x), \end{cases}$$

- $u_0(x) = H * m(x)$, CDF of $m \in \mathcal{P}(\mathbb{R})$, $H(x) := \mathbb{1}_{\{x \geq 0\}}$,
- $f, A \in C^1([0,1])$, $f'(u) = b(u)$, $A'(u) = \sigma^2(u) > 0$.

The space derivative $P_t := \partial_x u(t, \cdot)$ satisfies the **nonlinear Fokker-Planck equation**

$$\begin{cases} \partial_t P_t = \frac{1}{2}\partial_{xx}(\sigma^2(H * P_t)P_t) - \partial_x(b(H * P_t)P_t), \\ P_0 = m, \end{cases}$$

associated with **nonlinear** (in McKean's sense) **diffusion process** on the line

$$\begin{cases} \mathrm{d}X_t = b(H * P_t(X_t))\mathrm{d}t + \sigma(H * P_t(X_t))\mathrm{d}W_t, \\ X_t \sim P_t, \quad X_0 \sim m. \end{cases}$$

The coefficients of the diffusion **depend on the law** of X_t!

McKean's procedure: the particle system

McKean's idea: replace **nonlinearity** with **interaction**.

- Introduce n copies $X_t^{1,n}, \ldots, X_t^{n,n}$ driven by independent Brownian motions $W_t^1, \ldots, W_t^n$,
- replace P_t with **empirical measure** $\mu_t^n := \frac{1}{n}\sum_{i=1}^n \delta_{X_t^{i,n}}$.

We obtain a system of n particles on the line with **mean-field interaction**

$$\mathrm{d}X_t^{i,n} = b(H * \mu_t^n(X_t^{i,n}))\mathrm{d}t + \sigma(H * \mu_t^n(X_t^{i,n}))\mathrm{d}W_t^i, \qquad X_0^{i,n} \sim m \text{ iid.}$$

Approximation result: **propagation of chaos** in the space of sample-paths.

- **Law of large numbers** for the empirical measure $\mu^n := \frac{1}{n}\sum_{i=1}^n \delta_{(X_t^{i,n})_{t\geq 0}}$, converges to the **unique solution** P **to the nonlinear martingale problem** associated with nonlinear SDE.
- For any fixed $i \geq 1$, the law of $(X_t^{1,n}, \ldots, X_t^{i,n})_{t\geq 0}$ converges to $P^{\otimes i}$.
- **The solution $u(t,\cdot)$ is well approximated by the empirical CDF of the particle system.**

LLN: **Bossy, Talay** (*Ann. Appl. Probab.* 96, *Math. Comp.* 97), **Jourdain** (*ESAIM: P&S* 97, *Stoch. Proc. Appl.* 00), **Shkolnikov** (*Stoch. Proc. Appl.* 10), **Jourdain, R.** (*Stoch. PDE: Anal. Comp.* 13);
CLT: **Jourdain** (*Methodol. Comput. Appl. Probab.* 00);
LDP: **Dembo, Shkolnikov, Varadhan, Zeitouni** (*Comm. Pure Appl. Math.* 15+).

Rank-based interacting diffusions

The particle system rewrites

$$\mathrm{d}X_t^{i,n} = b\left(\frac{1}{n}\sum_{j=1}^{n} \mathbb{1}_{\{X_t^{j,n} \leq X_t^{i,n}\}}\right)\mathrm{d}t + \sigma\left(\frac{1}{n}\sum_{j=1}^{n} \mathbb{1}_{\{X_t^{j,n} \leq X_t^{i,n}\}}\right)\mathrm{d}W_t^i,$$

i.e. **the particle ranked in k-th position has constant coefficients $b(k/n)$ and $\sigma(k/n)$.**

- System of **rank-based interacting diffusions**, example of **competing particles**.
- Occurrence in spin glasses models: **Ruzmaikina, Aizenman** (*Ann. Probab.* 05), **Arguin, Aizenman** (*Ann. Probab.* 09), systems of Brownian queues: **Harrison** 88.
- Important model of equity market in **Stochastic Portfolio Theory**: **Fernholz** 02, **Banner, Fernholz, Karatzas** (*Ann. Appl. Probab.* 05), **Jourdain, R.** (*Ann. Finance* 15) to name a few.
- Interesting ergodic theory: **Pal, Pitman** (*Ann. Appl. Probab.* 08), **Jourdain, Malrieu** (*Ann. Appl. Probab.* 08), **Banner, Fernholz, Ichiba, Karatzas, Papathanakos** (*Ann. Appl. Probab.* 11), **R.** (*Electron. Commun. Probab.* 15), to be discussed now.

Long time behaviour: the particle system

- Our purpose: use the **probabilistic representation** to study the **long time behaviour** of $u(t, \cdot)$.
- We first address the **particle system** by means of classical **ergodic theory** of diffusion processes.

Evolution of centre of mass $\bar{X}_t^n := \frac{1}{n} \sum_{i=1}^n X_t^{i,n}$:

$$\mathrm{d}\bar{X}_t^n = \bar{b}_n \mathrm{d}t + \frac{1}{n} \sum_{k=1}^n \sigma(k/n) \mathrm{d}W_t^i, \qquad \bar{b}_n := \frac{1}{n} \sum_{k=1}^n b(k/n),$$

Brownian motion with constant drift: does not converge!

Necessity of **centering**: define $\tilde{X}_t^{i,n} := X_t^{i,n} - \bar{X}_t^n$, diffusion process in the hyperplane

$$M_n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\}.$$

Ergodicity of $(\tilde{X}_t^{1,n}, \ldots, \tilde{X}_t^{n,n})$?

Ergodicity of the centered particle system

If σ^2 is **constant**, the centered particle system rewrites

$$\mathrm{d}\tilde{X}_t^n = -(\Pi_n \Pi_n^\top)\nabla V_n(\tilde{X}_t^n)\mathrm{d}t + \sigma \Pi_n \mathrm{d}W_t \quad \in M_n,$$

where

$$V_n(x) := -\sum_{k=1}^{n} b(k/n) x_{(k)}, \qquad x_{(1)} \leq \cdots \leq x_{(n)},$$

and Π_n is the orthogonal projection on M_n.

- **Typical gradient system**, candidate equilibrium measure with density $\exp\left(-\frac{2}{\sigma^2}V_n(z)\right)$ with respect to surface measure $\mathrm{d}z$ on M_n.
- **Pal, Pitman** (*Ann. Appl. Probab.* 08):

$$\int_{z \in M_n} \exp\left(-\frac{2}{\sigma^2}V_n(z)\right)\mathrm{d}z < +\infty$$

  if and only if b satisfies the **stability condition**

$$\forall l \in \{1, \ldots, n-1\}, \qquad \frac{1}{l}\sum_{k=1}^{l} b(k/n) > \frac{1}{n-l}\sum_{k=l+1}^{n} b(k/n).$$

If σ^2 is not constant: $|z|^2$ is a **Lyapunov functional** under the stability condition, enough to ensure **ergodicity** (**Banner, Fernholz, Ichiba, Karatzas, Papathanakos** (*Ann. Appl. Probab.* 11), **Jourdain, R.** (*Electron. J. Probab.* 14)).

Intermediary conclusion

For the long time behaviour of the particle system:

- Necessity of **centering** the system around Brownian motion with average velocity

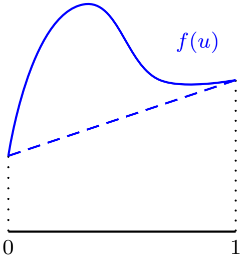
$$\bar{b}_n = \frac{1}{n}\sum_{k=1}^{n} b(k/n) \simeq \int_{u=0}^{1} b(u)\mathrm{d}u = f(1) - f(0) =: \bar{b}.$$

- Convergence to equilibrium if and only if **stability condition**: for all $l \in \{1, \dots, n-1\}$,

$$\frac{1}{l}\sum_{k=1}^{l} b(k/n) > \frac{1}{n-l}\sum_{k=l+1}^{n} b(k/n),$$

  which roughly rewrites: for all $u \in (0,1)$,

$$\frac{f(u)-f(0)}{u-0} > \frac{f(1)-f(u)}{1-u}.$$



- **Jourdain, Malrieu** (*Ann. Appl. Probab.* 08): if f **uniformly concave** and σ^2 is constant, then **Poincaré inequality uniform in** n, which implies **exponential decay to equilibrium at uniform rate**.

What can be extended to nonlinear process?

Average growth of the nonlinear process

Easy computation:

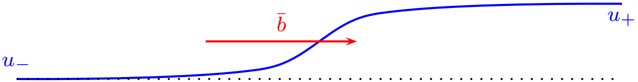
$$\mathbb{E}[X_t] = \mathbb{E}[X_0] + \int_{s=0}^{t} \mathbb{E}[b(H * P_s(X_s))] \mathrm{d}s$$
$$= \mathbb{E}[X_0] + \int_{s=0}^{t} \int_{u=0}^{1} b(u)\mathrm{d}u\mathrm{d}s = \mathbb{E}[X_0] + t\bar{b}.$$

- **Stationary** behaviour can only be observed on **fluctuation process** $\tilde{X}_t = X_t - t\bar{b}$.
- A **stationary distribution** $\tilde{P}_\infty$ for this process has a CDF $\tilde{u}_\infty$ such that

$$u_\infty(t,x) := \tilde{u}_\infty(x - t\bar{b})$$

is a **traveling wave** solution to the original PDE $\partial_t u + \partial_x f(u) = \frac{1}{2}\partial_{xx}A(u)$.

In the 1950s: **Lax, Hopf, Gel'fand, Il'in, Oleinik, ...** interested in $1D$ **viscous shock waves** connecting constant states $u_\pm$ in $\pm\infty$, applications in kinetic chemistry.



Waves are **physically observable** if they are **stable under perturbations**:

- if $u_0(x) = \tilde{u}_\infty(x) + v_0(x)$, do we have $\|u(t,\cdot) - \tilde{u}_\infty(\cdot - t\bar{b})\| \to 0$?
- probabilistic formulation: **convergence of $\tilde{X}_t$ to equilibrium measure $\tilde{P}_\infty$**?

Stability of viscous profiles

Stationary equation for $\tilde{u}_\infty$:

$$\begin{cases} \frac{1}{2}\sigma^2(\tilde{u}_\infty)\partial_x \tilde{u}_\infty = f(\tilde{u}_\infty) - (1-\tilde{u}_\infty)f(0) - \tilde{u}_\infty f(1), \\ \tilde{u}_\infty(-\infty) = 0, \quad \tilde{u}_\infty(+\infty) = 1, \end{cases}$$

$1D$ ODE solvable if and only if f satisfies **Oleinik's entropy condition**

$$\forall u \in (0,1), \qquad f(u) > (1-u)f(0) + uf(1).$$

- **Continuous version of our stability condition!**
- Then all **viscous profiles** $\tilde{u}_\infty$ are translations of each other.

Stability of viscous profiles

Under Oleinik's condition, if $u_0 - \tilde{u}_\infty \in L^1(\mathbb{R})$ and

$$\int_{x\in\mathbb{R}} (u_0(x) - \tilde{u}_\infty(x))\mathrm{d}x = 0 \qquad (i.e.\ \mathbb{E}[\tilde{X}_0] = \mathbb{E}[\tilde{X}_\infty]),$$

then $\lim_{t\to+\infty} \|u(t,\cdot) - \tilde{u}_\infty(\cdot - t\bar{b})\|_{L^1(\mathbb{R})} = 0.$

- **Serre, Freistühler** (*Comm. Pure Appl. Math.* 98), **Gasnikov** (*Izv. Ross. Akad.* 09).
- Rate of convergence: **transfer of information** from **space decay** of initial perturbation to **time decay** to equilibrium, **no general exponential decay!**

The gradient flow approach

The Wasserstein distance on $\mathcal{P}(\mathbb{R}^q)$

For $p \in [1, +\infty)$ and $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^q)$,

$$W_p(\mu, \nu) := \inf \left(\mathbb{E}[|X - Y|^p]\right)^{1/p},$$

taken on **couplings** (X, Y) of μ and ν.

- **Jordan, Kinderlehrer, Otto** (*SIAM J. Math. Anal.* 98): interpret $(\mathcal{P}_2(\mathbb{R}^q), W_2)$ as a infinite-dimensional *Riemannian manifold*, allows to make formal sense of **gradient flow**

$$\partial_t p_t = -\operatorname{Grad} \mathcal{E}[p_t], \qquad \mathcal{E} : \mathcal{P}_2(\mathbb{R}^q) \to \mathbb{R}.$$

  In particular, the **linear Fokker-Planck equation**

$$\partial_t p_t = \Delta p_t + \operatorname{div}(p_t \nabla V)$$

  is the gradient flow of the **free energy**

$$\mathcal{F}[p] := \int_{x\in\mathbb{R}^q} p(x) \log p(x) \mathrm{d}x + \int_{x\in\mathbb{R}^q} V(x) p(x) \mathrm{d}x.$$

- **Carrillo, McCann, Villani** (*Rev. Mat. Iberoamericana* 03, *Arch. Ration. Mech. Anal.* 06) extend to **McKean-Vlasov** (nonlinear Fokker-Planck) models, study long time behaviour, also **Bolley, Gentil, Guillin** (*Arch. Ration. Mech. Anal.* 13).

The free energy of conservation laws

For the sake of simplicity we assume $f(0) = 0$, $\bar{b} = f(1) = 0$, so that $\tilde{X}_t = X_t$.

In our case, it turns out that

$$\partial_t p_t = -\operatorname{Grad} \mathcal{F}[p_t]$$

with

$$\mathcal{F}[p] = \frac{1}{2} \int_{x \in \mathbb{R}} \sigma^2(H * p(x)) p(x) \log p(x) \mathrm{d}x + \int_{x \in \mathbb{R}} f(H * p(x)) \mathrm{d}x.$$

- Formal computation:

  $$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}[p_t] = -|\operatorname{Grad} \mathcal{F}[p_t]|^2 \leq 0,$$

  free energy is a **Lyapunov functional**, p_t converges to **local minimisers**.
- Uniqueness of minimiser (with prescribed expectation) provided by **convexity of $\mathcal{F}$ along geodesics**.
- Existence of an explicit minimiser: **the viscous profile $\tilde{u}_\infty$!**

The 'free energy' approach allows to recover the classical stability results.

Rates of convergence depend on **curvature** of $\mathcal{F}$: currently under investigation.

Outline

Scalar case with viscosity: stability of traveling waves

The inviscid limit: sticky particle dynamics

Wasserstein stability estimates for hyperbolic systems

Kruzkov's Theorem

We let the viscosity be $A(u) = 2\epsilon u$, and call u^ϵ the solution to the Cauchy problem

$$\begin{cases} \partial_t u^\epsilon + \partial_x f(u^\epsilon) = \epsilon \partial_{xx} u^\epsilon, \\ u^\epsilon(0,x) = u_0(x). \end{cases}$$

When $\epsilon \downarrow 0$, u^ϵ should converge to a weak solution to

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \\ u(0,x) = u_0(x), \end{cases}$$

but such solutions are **not unique** in general.

Entropy solution

An **entropy solution** is a function $u : [0,+\infty) \times \mathbb{R} \to [0,1]$ satisfying

$$\partial_t E(u) + \partial_x F(u) \leq 0$$

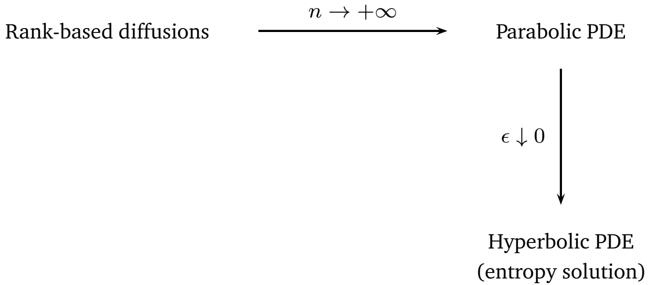
in the distributional sense, for all pair of **entropy-entropy flux** functions (E,F) such that E is convex and $F' = f'E'$.

Kruzkov's Theorem (*Mat. Sb.* 70)

In the **vanishing viscosity limit** $\epsilon \downarrow 0$, u^ϵ converges to the **unique entropy solution** u.

The vanishing viscosity limit in the scalar case I

We have the following diagram:



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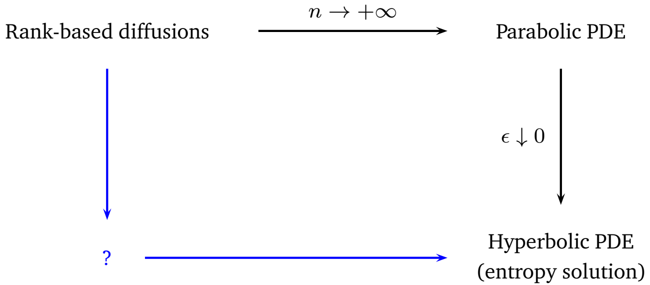
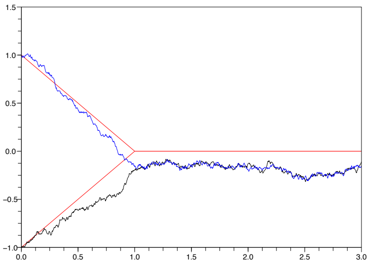


Can we go the other way around?

Small noise limit of rank-based diffusions

Toy example of 2 particles with 'converging' drifts $b_- > b_+$:



- After first collision, particles remain confined around centre of mass.
- Average velocity of centre of mass is $(b_- + b_+)/2$.

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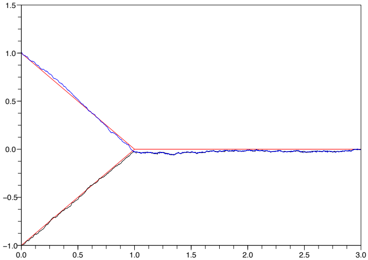

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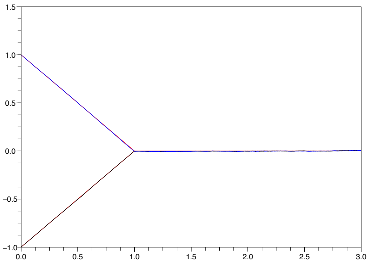

- After first collision, particles remain confined around centre of mass.
- Average velocity of centre of mass is $(b_- + b_+)/2$.
- **Small noise limit: particle stick together and form a cluster.**

Sticky particle dynamics

The sample-paths of the rank-based system converge to the **sticky particle dynamics**:

- the particle in k-th position has initial velocity $b(k/n)$,
- particles **stick together** into clusters at collisions, with **preservation of mass and momentum** but **dissipation of kinetic energy**.

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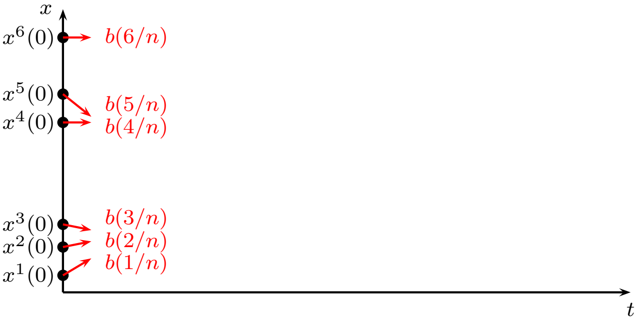

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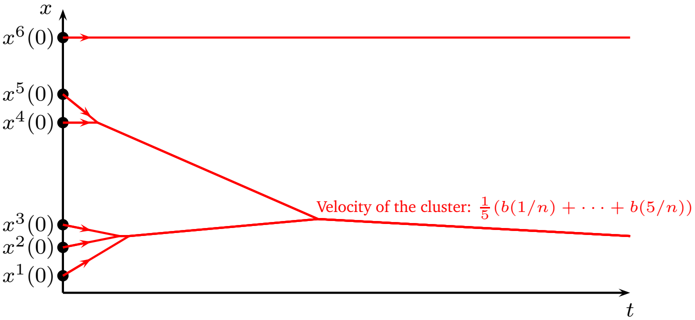


Adhesive dynamics introduced in physics (motion of large structure in the universe, pressureless gases), see in particular **E, Rykov, Sinai** (*Comm. Math. Phys.* 96).

Sticky particle dynamics: large scale behaviour

The Sticky Particle Dynamics defines a flow $(x_1(t), \dots, x_n(t))_{t \geq 0}$ in the polyhedron

$$D_n := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \leq \cdots \leq x_n\}.$$

Large scale behaviour: Brenier, Grenier (*SIAM J. Numer. Anal.* 98)

If the initial positions $(x_1(0), \dots, x_n(0)) \in D_n$ satisfy

$$u_{0,n}(x) := \frac{1}{n} \sum_{k=1}^{n} \mathbb{1}_{\{x_k(0) \leq x\}} \to u_0(x), \quad \mathrm{d}x\text{-a.e.},$$

then for all $t \geq 0$,

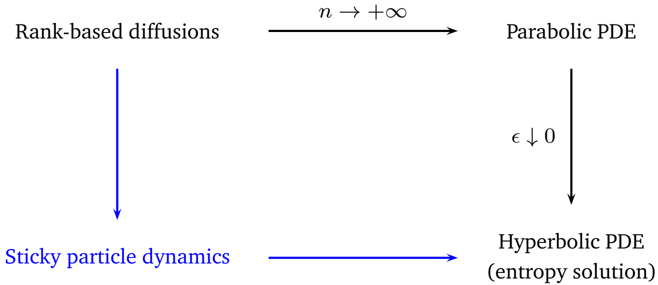
$$u_n(t,x) := \frac{1}{n} \sum_{k=1}^{n} \mathbb{1}_{\{x_k(t) \leq x\}} \to u(t,\cdot), \quad \mathrm{d}x\text{-a.e.},$$

where u is the **entropy solution** of the conservation law with initial condition u_0.

Jourdain, R. (arXiv:1507.01085): L^1 rate of convergence in $\|u_{0,n} - u_0\|_{L^1(\mathbb{R})} + Ct/n$ if $b = f'$ is Lipschitz continuous.

The vanishing viscosity limit in the scalar case II

We can complete the diagram:



Can we now extend this to the case of systems?

Hyperbolic systems: existence and uniqueness theory

We now take $d \geq 2$ and address the **system of conservation laws**

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0,$$

with $\mathbf{u} = (u^1, \ldots, u^d)$ and $\mathbf{f} : \mathbb{R}^d \to \mathbb{R}^d$. Under nonconservative form,

$$\partial_t \mathbf{u} + \mathbf{b}(\mathbf{u})\partial_x \mathbf{u} = 0,$$

with $\mathbf{b}(\mathbf{u})$ being the Jacobian matrix of $\mathbf{f}$. The system is **hyperbolic** when $\mathbf{b}(\mathbf{u})$ has real eigenvalues

$$\lambda^1(\mathbf{u}) \geq \cdots \geq \lambda^d(\mathbf{u}),$$

and **strictly hyperbolic** if $\lambda^1(\mathbf{u}) > \cdots > \lambda^d(\mathbf{u})$.

- Existence of weak solutions goes back to **Glimm** (*Comm. Pure Appl. Math.* 65).
- No theory of uniqueness / identification of the vanishing viscosity limit before **Bianchini, Bressan** (*Ann. of Math.* 05).
- Framework of Bianchini-Bressan theory: strictly hyperbolic system, data $\mathbf{u}_0$ with **small total variation** on the line.

Diagonalisation

In some cases, the system can be written under the **diagonal** form

$$\forall \gamma \in \{1, \ldots, d\}, \qquad \partial_t u^\gamma + \lambda^\gamma(\mathbf{u})\partial_x u^\gamma = 0.$$

We take the **probabilistic** data

$$\forall \gamma \in \{1, \ldots, d\}, \qquad u^\gamma(0, x) = u_0^\gamma(x),$$

with $u_0^\gamma = H * m^\gamma$, $m^\gamma \in \mathcal{P}(\mathbb{R})$.

Remark: data do not have a **small** total variation; we recover the Bianchini-Bressan framework under the **uniformly strict hyperbolicity** (USH) assumption

$$\forall \gamma \in \{1, \ldots, d-1\}, \qquad \inf_{\mathbf{u} \in [0,1]^d} \lambda^\gamma(\mathbf{u}) > \sup_{\mathbf{v} \in [0,1]^d} \lambda^{\gamma+1}(\mathbf{v}).$$

- Can we define a **particle system** that approaches solutions to the system?
- Can we derive **contraction** or **stability** estimates?

The Multitype Sticky Particle Dynamics

Scalar case: $\partial_t u + b(u)\partial_x u = 0$, **system case:** $\partial_t u^\gamma + \lambda^\gamma(\mathbf{u})\partial_x u^\gamma = 0$.

We have d CDFs $u^1, \ldots, u^d$ to approximate.

- We introduce d systems of n particles on the line.
- Each system is associated with a **type** $\gamma \in \{1, \ldots, d\}$.

The k-th particle of type γ has an initial velocity

$$\lambda^\gamma\left(\frac{k^1}{n}, \ldots, \frac{k^d}{n}\right),$$

where k^δ is the **rank** of the particle in the system of type δ.

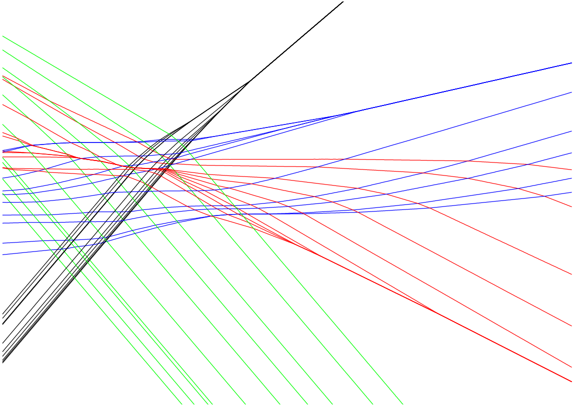
- To reproduce the entropic behaviour of the scalar case, each system evolves according to the typewise **sticky particle dynamics** in D_n.
- At collisions between clusters of **different types**: update of the velocities according to post-collisional order.
- **Assumption USH** ensures that post-collisional is **prescribed without ambiguity**.

The configuration space is $D_n^d = D_n \times \cdots \times D_n$ with typical elements

$$\mathbf{x} = \{x_k^\gamma \in \mathbb{R} : 1 \leq k \leq n, 1 \leq \gamma \leq d\},$$

and the MSPD defines a continuous flow $\mathbf{x}(t) \in D_n^d$.

A trajectory of the Multitype Sticky Particle Dynamics



$d = 4$ types, $n = 10$ particles.

A compactness result

Approximation of solutions by the MSPD

Assume USH and that $\lambda^1, \dots, \lambda^d$ are continuous on $[0,1]^d$. If the sequence of initial configurations $\mathbf{x}(0) \in D_n^d$ is such that

$$\forall \gamma \in \{1, \dots, d\}, \qquad \frac{1}{n} \sum_{k=1}^{n} \delta_{x_k^\gamma(0)} \to m^\gamma,$$

then any subsequence of the sequence of empirical CDFs $\mathbf{u}_n$ has a converging subsequence and its limit is a weak solution to the diagonal hyperbolic system with initial data $u_0^\gamma = H * m^\gamma$.

Proof by a **tightness** argument on the measure

$$\frac{1}{n} \sum_{k=1}^{n} \delta_{(x_k^1(t), \dots, x_k^d(t))_{t \geq 0}}$$

in $\mathcal{P}(C([0,+\infty); \mathbb{R}^d))$, see **Jourdain, R.** (arXiv:1501.01498).

Outline

Scalar case with viscosity: stability of traveling waves

The inviscid limit: sticky particle dynamics

Wasserstein stability estimates for hyperbolic systems

Stability estimates

Purpose of this section: establish **stability estimates** of the form

$$\mathrm{dist}(\mathbf{u}(t,\cdot),\mathbf{v}(t,\cdot)) \leq \mathcal{L} \times \mathrm{dist}(\mathbf{u}_0,\mathbf{v}_0),$$

where $\mathbf{u}$ and $\mathbf{v}$ denote two solutions of:

- either scalar conservation law,
- or hyperbolic system in diagonal form,

with initial conditions $\mathbf{u}_0$ and $\mathbf{v}_0$.

We shall proceed by establishing a **uniform discrete** stability estimate first on the associated **particle system**

$$\mathrm{dist}(\mathbf{u}_n(t,\cdot),\mathbf{v}_n(t,\cdot)) \leq \mathcal{L} \times \mathrm{dist}(\mathbf{u}_{0,n},\mathbf{v}_{0,n}),$$

with $\mathcal{L}$ that does not depend on n.

An appropriate choice of distance is the **Wasserstein distance** between coordinates u^γ and v^γ.

The Wasserstein distance in one dimension

We recall that $W_p(\mu,\nu) := \inf \left(\mathbb{E}[|X-Y|^p]\right)^{1/p}$, with $X \sim \mu$, $Y \sim \nu$ in $\mathbb{R}^q$.

Explicit optimal coupling

If $q = 1$, then an **explicit optimal coupling** is provided by

$$(X, Y) = (H * \mu^{-1}(U), H * \nu^{-1}(U)), \qquad U \sim \mathrm{Unif}[0,1],$$

where $F^{-1}(u) := \inf\{x \in \mathbb{R} : F(x) \geq u\}$.

As a consequence,

$$\boxed{W_p^p(\mu,\nu) = \int_{u=0}^{1} |H * \mu^{-1}(u) - H * \nu^{-1}(u)|^p \mathrm{d}u.}$$

In particular if

$$\mu^n = \frac{1}{n}\sum_{k=1}^{n} \delta_{x_k}, \qquad \nu^n = \frac{1}{n}\sum_{k=1}^{n} \delta_{y_k},$$

with $x_1 \leq \cdots \leq x_n$ and $y_1 \leq \cdots \leq y_n$ in $\mathbb{R}^n$, then

$$W_p^p(\mu^n,\nu^n) = \frac{1}{n}\sum_{k=1}^{n} |x_k - y_k|^p.$$

Contraction for SPD

Given two initial configurations $x_1(0) \leq \cdots \leq x_n(0)$ and $y_1(0) \leq \cdots \leq y_n(0)$, the empirical CDFs u_n and v_n satisfy

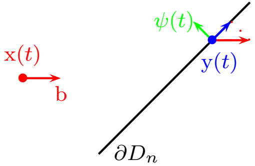
$$\forall t \geq 0, \qquad W_p^p(u_n(t,\cdot), v_n(t,\cdot)) = \frac{1}{n}\sum_{k=1}^n |x_k(t) - y_k(t)|^p.$$

Define $\phi(t), \psi(t) \in \mathbb{R}^n$ by

$$\mathrm{d}t\text{-a.e.}, \qquad \dot{x}_k(t) = b(k/n) + \phi_k(t), \quad \dot{y}_k(t) = b(k/n) + \psi_k(t),$$

so that ϕ and ψ stand for **the constraint of remaining** in D_n.

- Elementary computations show that $\phi(t), \psi(t)$ are **orthogonal to** ∂D_n, $\mathrm{d}t$-a.e.,
- **convexity** of D_n then implies that ℓ^p **distance is nonincreasing** between configurations.



Here with $p = 2$:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathrm{x}(t) - \mathrm{y}(t)|^2 = 2\langle \mathrm{x}(t) - \mathrm{y}(t), -\psi(t)\rangle \leq 0.$$

Contraction in Wasserstein distance for scalar case

We deduce that for all $t \geq 0$, $W_p(u_n(t,\cdot), v_n(t,\cdot)) \leq W_p(u_n(0,\cdot), v_n(0,\cdot))$.

Taking the limit when $n \to +\infty$, we obtain:

Contraction in Wasserstein distance

For all $p \in [1, +\infty)$, the entropy solutions u and v to scalar conservation law with initial condition u_0, v_0 CDFs on $\mathbb{R}$, satisfy

$$\forall t \geq 0, \qquad W_p(u(t,\cdot), v(t,\cdot)) \leq W_p(u_0, v_0).$$

- See also **Bolley, Brenier, Loeper** (*J. Hyperbolic Differ. Equ.* 05) for a different proof.
- When $p = 1$,

$$W_1(u,v) = \|u^{-1} - v^{-1}\|_{L^1(0,1)} = \|u - v\|_{L^1(\mathbb{R})},$$

i.e. we recover **classical L^1 stability**

$$\forall t \geq 0, \qquad \|u(t,\cdot) - v(t,\cdot)\|_{L^1(\mathbb{R})} \leq \|u_0 - v_0\|_{L^1(\mathbb{R})}.$$

ℓ^p stability of MSPD

Purpose: obtain ℓ^p stability estimates as in the scalar case. We assume USH and $\lambda^1, \ldots, \lambda^d$ are Lipschitz continuous on $[0,1]^d$.

Take $\mathbf{x}(0) \in D_n^d$ and $\mathbf{y}(0) = \mathbf{x}(0) + \delta\mathbf{x}(0)$, small perturbation.

- Trajectories $\mathbf{x}(t)$ and $\mathbf{y}(t)$ very similar.
- Most of the time, both configurations are in the same order: ℓ^p distances are **nonincreasing under typewise sticky particle dynamics**.
- On collision intervals: elementary geometric arguments yield

  $$|x_k^\gamma - y_k^\gamma|_{\text{after collision}} \leq \left(1 + \frac{C}{n}\right) |x_k^\gamma - y_k^\gamma|_{\text{before collision}} + \text{lower order terms},$$

  and since each particle sees at most $n(d-1)$ collisions, we conclude that there exists $\mathcal{L}_p \in [1, +\infty)$ **uniform in n** and $\mathbf{x}(0)$, $\mathbf{y}(0)$ such that

  $$\sum_{\gamma=1}^{d} \sum_{k=1}^{n} |x_k^\gamma(t) - y_k^\gamma(t)|^p \leq \mathcal{L}_p^p \sum_{\gamma=1}^{d} \sum_{k=1}^{n} |x_k^\gamma(0) - y_k^\gamma(0)|^p.$$

The inequality is made global in $\mathbf{x}(0), \mathbf{y}(0)$ by **interpolation procedure**.

Global convergence result for MSPD

Taking the limit $n \to +\infty$ in discrete stability estimates, we finally obtain the following main result.

Theorem: convergence of MSPD, Jourdain, R. (arXiv:1501.01498)

Assume USH and Lipschitz continuity.

- The empirical CDFs $\mathbf{u}_n$ of the MSPD converge to the unique **Bianchini-Bressan solution** to the diagonal hyperbolic system.
- This solution induces a **semigroup** on $\mathcal{P}(\mathbb{R})^d$.
- It satisfies the **Wasserstein stability estimates**

$$\forall t \geq 0, \qquad \sum_{\gamma=1}^{d} W_p^p(u^\gamma(t,\cdot), v^\gamma(t,\cdot)) \leq \mathcal{L}_p^p \sum_{\gamma=1}^{d} W_p^p(u_0^\gamma, v_0^\gamma).$$

- Natural **probabilistic construction** of solutions in the framework ‘large data + USH’, alternative to known methods.
- Novel **Wasserstein stability** estimates.
- MSPD naturally leads to **numerical schemes** for simulations, **error estimates** derived in **Jourdain, R.** (arXiv:1507.01085).