A Singular Journey In Optimisation problems Involving Index Processes

Probability, Control, Finance Conference

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The Magic world of optimisation

- At the end of 80'st, Ioannis introduces me at new (for me) optimization problem :

– Singular control problem

– Finite fuel

– Multi armed Bandit problem

- All had in common the same type of methodology :
  - their are convex problems with respsect to some (eventually artificial parameter)
  - the derivatives of the value function with respect to this parameter is easy to compute
  - Come back to the primitive problem by simple integration give new and useful representation

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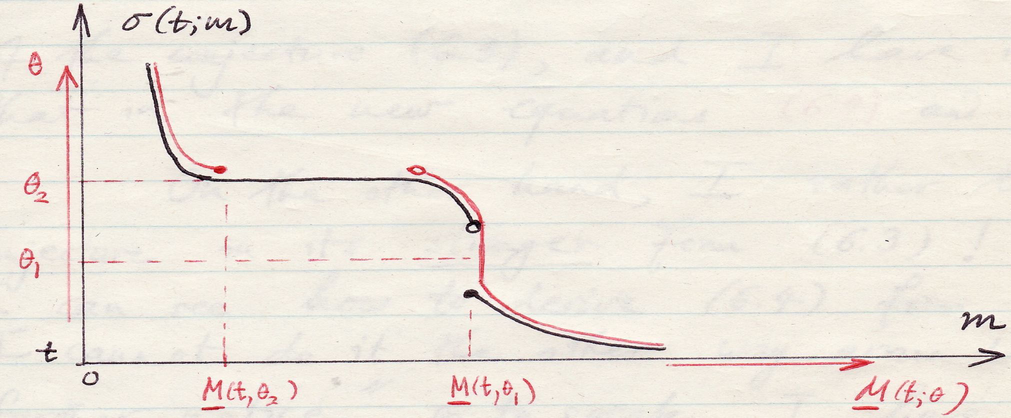
Reply to Remarks of NICOLE (dated 30 AVR '93)
on our CONTINUOUS-TIME DYNAMIC ALLOCATION Paper

(p.5) Have changed everything accordingly.

(p.7)



$\{m / \sigma(t;m) > \theta_2\} = [0, \underline{M}(t,\theta_2))$. Same for θ_1 .

It seems to me that, for this to work, we need to take $\underline{M}(t,\cdot)$ right-continuous, as in the picture (looked at, of course, from the other side of the paper!).

This $\underline{M}(t,\cdot)$ is indeed characterised by

$$\underline{M}(t,\theta) = \sup\{m \geq 0 / \sigma(t;m) > \theta\} = \inf\{m \geq 0 / \sigma(t;m) \leq \theta\}$$

and I am making this correction.

[B] On note $\underline{M}(t) = \inf\{m; \sigma_1(m) + \sigma_2(m) \leq t\} = \inf\{m; \gamma_1[\varphi_1^{-1}(m)] + \gamma_2[\varphi_2^{-1}(m)] \leq t\}$

$T_1(t) = \sigma_1(\underline{M}(t)) \qquad T_2(t) = \sigma_2(\underline{M}(t))$.

$T_1(t) + T_2(t) = t$. , cela est faux. , car il faut faire attention aux paliers de $\underline{M}(t)$.

On a identiquement

$$\underline{M}^1(T_1(t)) = \underline{M}_1 \circ \sigma_1(\underline{M}(t)) = \underline{M}(t) = \underline{M}^2(T_2(t))$$

et $\underline{M}(t) = \varphi^1(W^1_{T_1(t)})$ en un point de croissance de $T_1(t)$: $(\varphi^1)^{-1}(\underline{M}_t) = W^1_{T_1(t)}$

Par suite

$$\underline{M}(t) = \sup\left(\varphi^1(W^1_{T_1(t)}), \varphi^2(W^2_{T_2(t)})\right)$$

On traduit de cette manière que la stratégie $(T_1(t), T_2(t))$ suit l'indice.

Par suite

$$\underline{M}(t) - \varphi^1(W^1_{T_1(t)}) \geqslant 0 \qquad \text{et} \qquad \underline{M}_t - \varphi^2(W^2_{T_2(t)}) \geqslant 0.$$

Posons

$$S^+(t) = \varphi_1^{-1}(\underline{M}_t) - W^1_{T_1(t)} \geqslant 0 \qquad S^-(t) = \varphi_2^{-1}(\underline{M}_t) - W^2_{T_2(t)} \geqslant 0$$

Introduction to Bandit Problem

What is a Multi-Armed bandit problem ?

- There are d-independent projects (investigations, arms) among which effort to be allocated.
- By engaging one project, a stochastic reward is accrued, influencing the time-allocation strategy

⇒ Trade-off between exploration (trying out each arm to find the best one) and exploitation (playing the arm believed to give the best payoff)

- Discrete-time version is well-understood for a long time (Gittins (74-79), Whittle (1980))
- Continuous-time version received also a lot of attention (Karatzas (84), Mandelbaum (87), Menaldi-Robin (90), Tsitsiklis (86), NEK-Karatzas (93,95,97)

Introduction II

Renewed interest in Economy

- RD problems (Weitzman &...(1979,81)
- Strategic experimentation with learning on the quality of some project (Poisson uncertainty) (Keller, Rady, Cripps (2005))
- Learning in matching markets such as labor and consumer good markets : Jovanovic (1979) applies a bandit problem to a competitive labor markets.
- Strategic Trading and Learning about Liquidity (Hong& Rady(2000))

Principle of the solution (Gittins,Whittle)

⇒ To associate to each projet some rate of performance (Gittins index)

⇒ To maximize Gittins indices over all projects and at any time engaged a project with maximal current Gittins index

⇒ The essential idea is that the evolution of each arm does not depends on the running time of the other arms.

General Framework

Several projects $(i = 1, ...d)$ are competing for the attention of a single investigator

- $T_i(t)$ is the total time allocated to project i during the time t, with $\sum_{i=1}^{d} T_i(t) = (\leq)t$
- By engaging project i at time t, the investigator accrues a certain reward $h_i(T_i(t))$ per unit time,

– discounted at the rate $\alpha > 0$ and multiplied by the intensity $i(t) = dT_i(t)/dt$ with which the project is engaged.

– $h_i(t)$ is a progressive process adapted to the filtration $\mathcal{F}_i$, independent of the other.

⇒ The objective is to allocate sequentially the time between these projects optimally

$$\Phi := \sup_{(T_i)} \mathbb{E}\Big[\sum_{i=1}^{d} \int_0^\infty e^{-\alpha t} h_i(T_i(t)) dT_i(t) \Big].$$

Decreasing Rewards

Pathwise solution without probability

Deterministic case and concave analysis (modified pay-off with $\alpha = 0$, and finite horizon T)

– Let $(\underline{h}_i)$ be the family of right-continuous decreasing positive pay-offs, with $\underline{h}_i(0) > 0$ ($\underline{h}_i(t) = 0$ for $t \geq \zeta$. and $H_i(t)$ the primitive of h_i with $H_i(0) = 0$, assumed to be constant after some date ζ.

– H_i is a concave increasing function, with convex decreasing Fenchel conjuguate $G_i(m) = \sup_{t \leq T}\{H_i(t) - tm\}$ with derivative $G_i'(m) = \sigma_i(m)$.
$\mathbf{H_i(t) = \int_0^\infty t \wedge \sigma_i(m) dm}$.

– The criterium is now

$$\Phi_T := \sup_{(T_i)} \sum_{i=1}^{d} \int_0^T \underline{h}_i(T_i(t)) dT_i(t) = \sup J_T(\mathcal{T})$$

over all strategies : $\mathcal{T} = (T_i)$ with $\quad \sum_{i=1}^{d} T_i(t) = t$.

Criterium Transformation

$$\mathbf{J_T}(\mathcal{T}) := \sum_{i=1}^{d} \int_0^T \underline{\mathrm{h}}_i(T_i(t))dT_i(t) = \sum_{\mathbf{i=1}}^{\mathbf{d}} \mathbf{H_i}(\mathbf{T_i}(\mathbf{T}))$$

Proof

- $\underline{\mathrm{h}}_i(T_i(t)) = \int_0^\infty \mathbf{1}_{\{m<\underline{\mathrm{h}}_i(T_i(t))\}}dm = \int_0^\infty \mathbf{1}_{\{T_i(t))<\sigma_i(m)\}}dm$
- $\sum_{i=1}^d \mathbf{1}_{\{T_i(t))<\sigma_i(m)\}}dT_i(t) = \sum_{i=1}^d d(T_i(t) \wedge \sigma_i'(m))$

$\Rightarrow$ $J_T(\mathcal{T}) = \int_0^\infty dm \int_0^T d(T_i(t) \wedge \sigma_i(m)) = \int_0^\infty dm\, T_i(T) \wedge \sigma_i(m)$

Remark : Assume that the reward functions (h_i) are not decreasing. The same properties hold true by using the **concave envelope** of $\int_0^t h_i(s)ds$, defined through its conjugate $G_i(m) = \sup_t\{\int_0^t (h_i(s) - m)ds\}$.

Max-convolution problem

New formulations

- The **bandit problem** becomes

$$\Phi_T := \sup\{\sum_{i=1}^{d} H_i(T_i(T)) |\, T_i \text{ increasing, and } \sum_{i=1}^{d} T_i(t) = t, \forall t \leq T\}$$

- The **Max-Convolution problem** with value function V(t) is :

$$V(t) := \sup_{(\theta_i(t))} \{\sum_{i=1}^{d} H_i(\theta_i(t)) |\, \sum_{i=1}^{d} \theta_i(t) = t, \}$$

- Showing that the problems are equivalent is obtained by constructing a monotone optimal solution for the Max-convolution problem.

Optimal Time Allocation in Max-Convolution Pb

- **Main property** The conjugate $U(m)$ of the Max-Convolate $V(t)$ is the sum of the conjugate functions $U(m) = \sum_{i=1}^{d} G_i(m)$, with derivative $\tau(m) = \sum_{i=1}^{d} \sigma_i(m)$.
- $V(\tau(m)) = \tau(m)m - U(m) = \sum_{i=1}^{d}(m\sigma_i(m) - G_i(m) = \sum_{i=1}^{d} H_i(\sigma_i(m))$

Optimal time allocation

- Let $V'(t) = \underline{\mathrm{M}}_t$ be the decreasing derivative of V, also the inverse of $\tau(m)$, and called the **Gittins Index** of the problem.
- The optimal time allocation is the increasing process $\boldsymbol{\theta^*_i(t) = \sigma_i(V'(t))}$
- The optimal allocation is **of Index type**, i.e. maximizing the index $V'(t) = \sup_i \underline{\mathrm{h}}_i(\theta^*_i(t)) = \sup_i \underline{\mathrm{h}}_i(\sigma_i(V'(t))$.
  In the case of strictly decreasing continuous pay-offs, all projects may be engaged at the same time.

The Stochastic Decreasing case

Pathwise static problem

- Assume the decreasing pay-off as $\underline{\mathrm{h}}_i(t,\omega) = \inf_{0\leq u\leq t} k_i(u,\omega)$ where $k_i(t)$ is $\mathcal{F}_i(t)$-adapted.

– The inverse process of $\underline{\mathrm{h}}_i(t)$ is given by the stopping time $\sigma_i(m) = \sup\{t \mid \underline{\mathrm{h}}_i(t) \leq m\}$

- The strategic allocation $T_i(t)$ is an $\mathcal{F}_i(t)$-adapted non decreasing cadlag process.
- All the previous results hold true, but the optimality is more difficult to establish, because the $\mathcal{F}_i(t)$-mesurability constraint.
- We have to use multi-parameter stochastic calculus, as Mandelbaum (92), Nek.Karatzas(93-97)

Today, we are concerned by the one- dimensional problem, which consists in replacing any adapted and positive process h_i by a decreasing process $\underline{\mathrm{M}}_i(t) = \sup_{s<t} M_i(s)$ where M_i is called **the Index process**.

Max-Plus decomposition

Different Type of Max-Plus decomposition

- In our context, the problem is to find an adapted Index process $M(t)$

$$V_t = \mathbb{E}[\int_t^{\infty} e^{-\alpha s} h(s) ds | \mathcal{F}_t] = \mathbb{E}[\int_t^{\infty} e^{-\alpha s} \sup_{t<u<s} M(u) ds | \mathcal{F}_t] = \mathbb{E}[\int_t e^{-\alpha s} \underline{\mathrm{M}}_{t,s} ds | \mathcal{F}_t]$$

- More generally, in a Markov framework (Foellmer -Nek (05), (Foellmer, Riedel), the problem is to represent any fonction $u(x)$ as

$$u(x) = \mathbb{E}_x[\int_0^{\zeta} \sup_{0<u<t} f(X_t) dB_t], \quad B \text{ additive fonctional}$$

- In Bank-Nek (04), Bank-Riedel (01) the problem motivated by consumption problem is to solve for "any " adapted process X

$$X_t = \mathbb{E}[\int_t^{\infty} G(s, \sup_{t<u<s} L_s) ds | \mathcal{F}_t], \quad G(s,l) \text{ decreasing in l}$$

The class of supermartingale decomposition II

– Nek-Meziou (2002,2005) for general process

– Foellmer Knispel (2006)

See P. Bank, H. Follmer (02), American Options, Multi-armed Bandits, and Optimal Consumption Plans : A Unifying View, Paris-Princeton Lectures on Mathematical Finance 2002, Lecture Notes in Math. no. 1814, Springer, Berlin, 2003, 1-42.

Max-plus algebra Calculus

It is an idempotent **semiring** :

$\Rightarrow$ $\oplus = \max$ is a commutative, associative and **idempotent** operation : $a \oplus a = a$, the **zero** $= \epsilon$, is given by $\epsilon = -\infty$,

$\Rightarrow$ $\otimes$ is an associative **product** distributive over addition, with a unit element $e = 0$. ϵ is absorbing for $\otimes$: $\epsilon \otimes a = a \otimes \epsilon = \epsilon$, $\forall a$.

$\Rightarrow$ $\mathbb{R}_{\max}$ can be equipped with the natural order relation :

$$a \succeq b \Longleftrightarrow a = a \oplus b.$$

$\Rightarrow$ **Linear Equation.** *The set of solutions x of $z \oplus x = m$ is empty if $m \leq z$. If not, the set has a* **greatest element** $x = m$.

Max-Plus Supermartingale Decomposition

Let Z be a càdlàg supermartingale in the class $(\mathcal{D})$ defined on $[,\zeta]$.

- There exists $L = \big(L_t\big)_{\leq t \leq \zeta}$ adapted, with upper-right continuous paths with **running supremum** $L^*_{t,s} = \sup_{t \leq u \leq s} L_u$, s.t.

$$\mathbf{Z_t} = \mathbb{E}\big[(\sup_{t \leq u \leq \zeta} L_u) \vee Z_\zeta | \mathcal{F}_t\big] = \mathbb{E}\big[L^*_{t,\zeta} \oplus Z_\zeta | \mathcal{F}_t\big] = \mathbb{E}\Big[\oint_{\mathbf{t}}^{\zeta} \mathbf{L_u} \oplus \mathbf{Z}_\zeta | \mathcal{F}_\mathbf{t}\Big]$$

- Let $M^\oplus$ be the martingale : $\mathbf{M^\oplus_t} := \mathbb{E}\big[L^*_{\mathbf{0},\zeta} \oplus Z_\zeta \big| \mathcal{F}_t)\big]$.Then,

$$M^\oplus_t \geq \max(Z_t, L^*_{\mathbf{0},t}) = Z_t \oplus L^*_{\mathbf{0},t} \quad \leq t \leq \zeta$$

and the equality holds at times when L^* **increases** or at **maturity** ζ :

$$M^\oplus_S = \max(Z_S, L^*_{\mathbf{0},S}) = Z_S \oplus L^*_{0,S} \quad \text{for all stopping times } S \in \mathcal{A}_{L^\star} \cup \{\zeta\}.$$

Uniqueness in the Max-Plus decomposition

Let $Z \in \mathcal{D}$ be a cadlag **supermartingale** and assume that

- there exist two increasing adapted processes Λ^1_t and Λ^2_t $(\Lambda^i_{-0} = -\infty)$ and two u.i. martingales M^1 and M^2 such that $\mathbf{M^i_\zeta = \Lambda^i_\zeta \vee Z_\zeta}$ and $\mathbf{M^i_0 = Z_0}$
- Λ^i only increases at times when the martingale M^i hits the supermartingale Z, (**flat-off condition**)

$$\int_{[0,\zeta]} (M^i_t - Z^i_t) d\Lambda^i_t = 0$$

- (M^i, Λ^i) are two (max-+) decompositions of Z ($\oplus = \vee = \max$)

$$\mathbf{M^1_t \geq Z_t \oplus \Lambda^1_t}, \quad \mathbf{M^2_t \geq Z_t \oplus \Lambda^2_t}.$$

$\Rightarrow$ M^1 and M^2 are **indistinguishable** processes.

$\Rightarrow$ Given such a martingale $M^\oplus$, the set $\mathcal{K}$ of Λ satisfying the above conditions has a **maximal** element $\Lambda^{\max}$ which is also in $\mathcal{K}$.

If Z is bounded by below, $\Lambda^{\max}$ is also bounded by below with the same constant.

Sketch of the proof when Z and Λ are bounded by below

Recall the assumption $\int_0^\zeta (M^i_s - Z_s)d\Lambda^i_s = 0$ with $\Lambda^i_\zeta \geq Z_\zeta$

Then, for any regular **convex** function ($\mathcal{C}^2$ with linear growth)g, $\mathbf{g(0) = 0}$.

$$g(\mathbf{M^1_\zeta} - \mathbf{M^2_\zeta}) \leq g'(\mathbf{M^1_\zeta} - \mathbf{M^2_\zeta})(M^1_\zeta - M^2_\zeta) = g'(\mathbf{\Lambda^1_\zeta} - \mathbf{\Lambda^2_\zeta})(M^1_\zeta - M^2_\zeta)$$

$$\mathbb{E}\big[g(\mathbf{M^1_\zeta} - \mathbf{M^2_\zeta})\big] \leq$$

$$\mathbb{E}\big[g'(\Lambda^1_{\mathbf{0}} - \Lambda^2_{\mathbf{0}})(\mathbf{M^1_\zeta} - \mathbf{M^2_\zeta})\big] + \mathbb{E}\big[(M^1_\zeta - M^2_\zeta)\int_0^\zeta g''_d(\Lambda^1_t - \Lambda^2_t)(d\Lambda^1_t - \Lambda^2_t)\big]$$

$$= \mathbb{E}\big[\int_0^\zeta (M^1_{\mathbf{t}} - M^2_{\mathbf{t}})g''_d(\Lambda^1_t - \Lambda^2_t)(d\Lambda^1_t - \Lambda^2_t)\big]$$

$$= \mathbb{E}\big[\int_0^\zeta (\mathbf{Z}_t - M^2_t)g''_d(\Lambda^1_t - \Lambda^2_t)d\Lambda^1_t - \int_0^\zeta (M^1_t - \mathbf{Z}_t)g''_d(\Lambda^1_t - \Lambda^2_t)d\Lambda^2_t\big] \leq \mathbf{0}$$

by the **flat condition and the convexity of** g.

In particular, $\mathbb{E}\big[g(\mathbf{M^1_\zeta} - \mathbf{M^2_\zeta})\big] = 0$ for $\mathbf{g(x) = x^+}$

Darling, Ligget, Taylor Point of View,(1972)

Introduction DLT have studied American Call options with infinite horizon on discrete time supermartingale, sum of iid r.v. with negative expectation. They gave a large place to the running supremum of these variables.

- Z is a supermartingale on $[0,\zeta]$ and $\mathbb{E}\big[|Z^*_{0,\zeta}|\big] < +\infty$ $\mathbb{E}\big[|Z^*_{t,\zeta}|\big] < +\infty$
- Assume Z to be **a conditional expectation of some running supremum** process $L^*_{s,t} = \sup_{\{s\leq u\leq t\}} L_u$, such that $\mathbb{E}\big[|L^*_{0,\zeta}|\big] < +\infty$ and $\mathbf{Z_t} = \mathbb{E}\big[L^*_{t,\zeta}|\mathcal{F}_t\big]$

American Call options Let $C_t(Z,m)$ be the American Call option with strike m, $\mathbf{C_t(Z,m)} = \operatorname{ess\,sup}_{t\leq S\leq\zeta} \mathbb{E}\big[(\mathbf{Z_S} - m)^+|\mathcal{F}_t\big]$. Then

$$\mathbf{C_t(Z,m)} = \mathbb{E}\Big[\Big(\mathbf{L^*_{t,\zeta}} \vee \mathbf{Z_\zeta} - \mathbf{m}\Big)^+|\mathcal{F}_\mathbf{t}\Big]$$

and the stopping time $\mathbf{D_t(m)} = \inf\{s \in [t,\zeta];\ L_s \geq m\}$ is optimal.

Proof

$\Rightarrow$ $\mathbb{E}\big[\big(L^*_{t,\zeta} - m\big)^+ | \mathcal{F}_t\big]$ is a supermartingale dominating $\mathbb{E}\big[L^*_{t,\zeta} | \mathcal{F}_t\big] - m = Z_t - m$, and so $C_t(Z, m)$

$\Rightarrow$ Conversely, since on $\{\theta = D_t(m) < \infty\}$, $L^*_{\theta,\zeta} \geq m$, at time $\theta = D_t(m)$, we can omit the sign $+$, and replace $(L^*_{\theta,\zeta} - m)$ by its conditional expectation $Z_{D_t(m)} - m$, still nonnegative.

Main question :

To find numerical method to calculate a Max-Plus Index

– Directly by using AY-martingale (elementary)

– By characterization through optimization problems (Gittins, Karatzas, Foellmer)

Closed Formulae
based on Azéma-Yor martingales

Azéma-Yor Martingales (1979)

Definition Let X be a càdlàg local semimartingale with $X_0 = a$ and $X_t^* = \sup_{0 \le s \le t} X_s$ its running supremum assumed to be nonnegative. Then for any finite variation function u, with locally integrable right-hand derivative u', the process $M^{\mathbf{u}}(X)$

$$M_t^{\mathbf{u}}(X) = u(X_t^*) + u'(X_t^*)(X_t - X_t^*)$$

is a local martingale, called the **Azéma-Yor** martingale associated with (u, X).

Main properties

⇒ $M_t^{\mathbf{u}}(X) = M_0^{\mathbf{u}}(X) + \int_0^t u'(X_t^*)\, dX_s,$ (1)

⇒ If u' is only defined on $[a, b)$, $M^{\mathbf{u}}(X)$ may be defined up to the exit time ζ of $[a, b)$ by X.

⇒ Assume u' to be non negative. Then the running supremum of $M^{\mathbf{u}}(X)$ is given by $u(N_t^*)$

Bachelier equation

First introduced by Bachelier in 1906.

Def : Let $\phi : [a^*, \infty)$ be a locally bounded away from 0 function and X a local martingale with continuous running supremum. The Bachelier equation is

$$dY_t = \phi(Y_t^*)dX_t$$

Example Let u be a increasing function, v the inverse function of u, and $\phi = u' \circ v = 1/v'$. Then $M^u(X)$ the AY-martingale associated with u is a solution of the Bachelier equation.

Bachelier equation, (suite)

Th : Let $\phi : [a^*, \infty) \to (0, \infty)$ be a Borel function locally bounded away from zero, and $(X_t : t \geq 0)$, $X_0 = a$, a càdlàg semimartingale as before.

- Define $v(y) = a + \int_{a^*}^{y} \frac{ds}{\phi(s)}$ and $u(x) = v^{-1}(x)$. So $u'(x) = (v^{-1})'(x) = \phi \circ v(x)$.

$\Rightarrow$ Then the Bachelier equation

$$dY_t = \phi(Y_t^*)\, dX_t, \quad Y_0 = a^* \tag{1}$$

has a strong, pathwise unique, solution defined up to its explosion time $\zeta_Y = T_{V(\infty)}$.

- The solution is given by $\mathbf{Y_t = M_t^u(X)}$, $t < T_{V(\infty)}$.

For any process X as before, and any increasing function u function (with locally bounded derivative) with inverse function v, we have

$$\mathbf{X_t = M_t^u(M^v(X))}$$

Maximum distribution

Well-known result.

Th : Let (N_t), $N_0 = 1$ be a non-negative local martingale with a continuous running supremum and with $N_t \to 0$ a.s. Then $1/N^*_\infty$ has **a uniform distribution on** $[0, 1]$.

Proof : Let $u(x) = (K - x)+$ the "Put "function. Then, $M^U(N)$ is bounded and u.i. martingale, such that

$$\mathbb{E}\big((K - N^*_\infty)^+ + \mathbf{1}_{\{K>N^*_\infty\}} N^*_\infty big) = K\mathbb{P}(K \geq N^*{}_\infty) = K - 1$$

• Moreover if $b \geq 1$ is a constant such that for $\zeta = T_b$, $N_\zeta \in \{0, b\}$, then $\mathbb{P}(N^*_\zeta = b) = 1/b$ and conditionally to $\{N^*_\zeta < b\}$, $\mathbf{1/N^*_\zeta}$ is uniformly distributed on $[1/b, 1]$.

Surmartingale decomposition and running supremum

- Let N be a local martingale with continuous running supremum, and going to 0 at ß
- Let u be a increasing convave function, such that $\mathbb{E}(|u(N)|^*_\infty) < \infty$

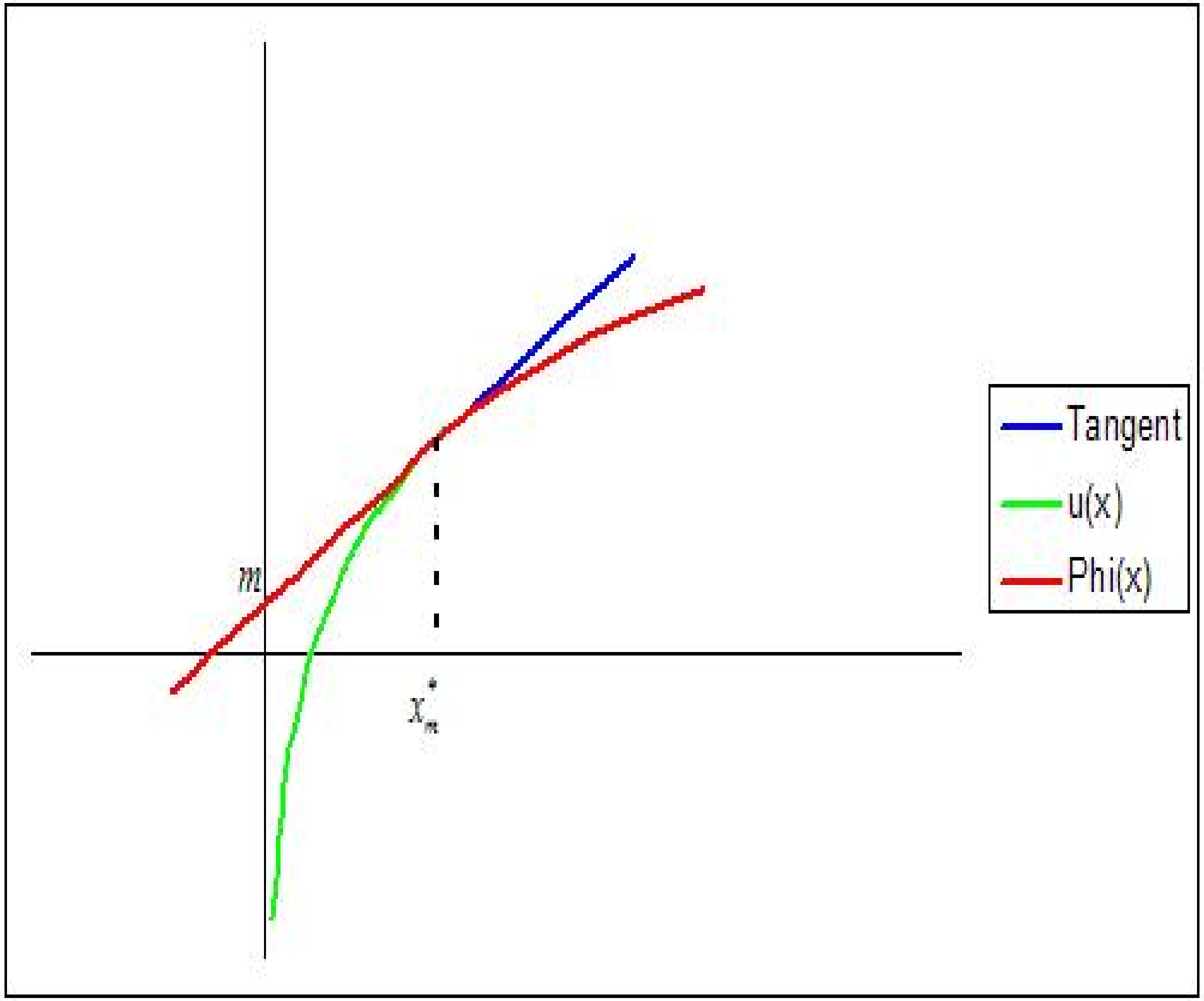
$\Rightarrow$ The supermartingale $\mathbf{u(N_t)}$ is the conditional expectation of the running supremum between t and ∞ of $L_t = v(N_t)$ where $v(x) = u(x) - x\,u'(x)$ is an non decreasing function, that is

$$\mathbf{Z_t} = \mathbf{u}(\mathbf{N_t}) = \mathbb{E}\Big(\sup_{\mathbf{t},\infty} \mathbf{v}(\mathbf{N_u})|\mathcal{F}_\mathbf{t}$$

- More generally, **g** is a continuous increasing function on $\mathbb{R}^+$ whose increasing concave envelope **u** is finite.

– **Galtchouk, Mirochnitchenko Result (1994)** : The process $\mathbf{Z_t} = u(N_t)$ is the Snell envelope of $Y = g(N)$.

Concave envelop of $u \vee m$



Max-Plus decomposition of Supermartingales with Independent Increments

Continuous case Let N be a geometric Brownian motion with return=0 and volatility to be specified. Let Z be a supermartingale defined on $[0, \infty]$ such that

- a **geometric** Brownian motion with **negative drift** ,
  $\frac{dZ_t}{Z_t} = -rdt + \sigma dW_t, \quad Z_0 = z > 0.$
– Setting $\gamma = 1 + \frac{2r}{\sigma^2}$, $N_t = Z_t^\gamma$ is a local martingale, with volatility $\gamma\sigma$
– $Z_t = u(N_t)$ where u is the increasing concave function $u(x) = x^{1/\gamma}$.
- $v(x) = u(x) - xu'(x) = \frac{\gamma-1}{\gamma} x^{1/\gamma} = \frac{\gamma-1}{\gamma} z,$
- Let Z be a **Brownian motion** with negative drift $-(r + \frac{1}{2}\sigma^2) \geq 0$
  $dZ_t = -(r + \frac{1}{2}\sigma^2)dt + \sigma dW_t, \quad Z_0 = z.$
  Then $Z_t = \frac{1}{\gamma}\ln(N_t)$, $v(z) = z - \frac{1}{\gamma}$ and the Call American boundary is $y^*(m) = m + \frac{1}{\gamma}$.
- **the exponentional of a Lévy process with jumps**
  Assume Z to be a supermartingale with a continuous and integrable supremum.

Then the same result holds with a modified coefficient γ_{Levy}, such that $Z_t^{\gamma_{Levy}}$ defines a local martingale that goes to 0 at ∞.

- **Finite horizon** T without Azéma-Yor martingale

  Same kind of solution : we have to find a function b(.) such that at any time t

$$Z_t = \mathbb{E}\Big[\sup_{t \le u \le T} \mathbf{b(T-u)} Z_u \Big| \mathcal{F}_t \Big]$$

Can we find a direct and efficient method to calculate the boundary $b(T-t)$ **?**.

References : Previous papers are relative to processes with independent increments (E. Mordecki (2001) - S. Asmussen, F. Avram and M. Pistorius (2004) - L. Alili and A. E. Kiprianou (2005)).

Universal Index and Pricing Rule

Framework : Let $Z = u(N_.)$ be a increasing concave function of the cadlag local martingale N going to 0 at infinity, with continuous running supremum. Assume $\mathbb{E}[|Z^*_{0,\infty}|] < +\infty$.

- Let φ be the increasing convex, inverse function of u, such that $\varphi(Z) = N$ is a local martingale and $\psi(\mathbf{z}) = \mathbf{v}\,\mathbf{o}\,\varphi(\mathbf{z}) = \mathbf{z} - \dfrac{\phi(\mathbf{z})}{\phi'(\mathbf{z})}$. Then

$$Z_t = \mathbb{E}[\psi(Z^*_{t,\infty})|\mathcal{F}_t], \qquad C^Z_t(m) = \mathbb{E}[(\psi(Z^*_{t,\infty}) - m)^+|\mathcal{F}_t]$$

$$\begin{aligned} y^*(m) &= \psi^{-1}(m) = m + \frac{\varphi(y^*(m))}{\varphi'(y^*(m))} \\ C^Z_t(m) &= \begin{cases} (Z_t - m) & \text{if} \quad Z_t \geq y^*(m) \\ \frac{y^*(m)-m}{\varphi(y^*(m))}\varphi(Z_t) & \text{if} \quad Z_t \leq y^*(m) \end{cases}. \end{aligned}$$

Optimality of Azema-Yor martingale

Martingale optimization problem

The optimization problem

Let $Y_t = g(N_t)$ be a floor process and $Z_t^Y = u(N_t)$ the Snell envelope of Y where u is the concave envelope of g.

The following problem is motivated by portfolio insurance :

$\mathcal{M}(x) = \Big\{(M_t)_{t\geq 0}\text{u.i.martingale}\,|M_0 = x \text{ and } \mathbf{M_t \geq g(N_t)}\ \forall t \in [0,\zeta]\Big\}$

- We aim at finding a martingale (M_t^*) in $\mathcal{M}(x)$ such that for all martingales (M_t) in $\mathcal{M}(x)$, and for any utility function,(concave, increasing) V such that the following quantities have sense

$$\mathbb{E}(\mathbf{V}(\mathbf{M}_\zeta^*)) \geq \mathbb{E}(\mathbf{V}(\mathbf{M}_\zeta))$$

- The initial value of any martingale dominating Y must be **at least** equal to the one of the Snell envelope Z^Y, that is $u(N_0)$.

The u-Azéma-Yor martingale is optimal

The martingale $M_t^{AY} = u(N_t^*) + u'((N_t^*)(N_t - N_t^*)$ *martingale is optimal for the concave order of the terminal value.*
In particular, $dM_\zeta^{Y,\oplus} = u'(N_t^*)dN_t$ *is less variable than the martingale of the Doob Meyer Decomposition* $dM^{DM} = u'(N_t)dN_t$.

Sketch of proof : Let M be in $\mathcal{M}^Y(Z_0^Y)$. Since M dominates Z^Y, the American Call option $C_t(M, m)$ also dominates $C_t(Z^Y, m)$. By convexity,

$$C_t(M, m) = \mathbb{E}\big[(M_\zeta - m)^+ | \mathcal{F}_S\big] \geq \mathbb{E}\big[(L_{S,\zeta}^{Y,*} \vee Y_\zeta - m)^+ | \mathcal{F}_S\big] \quad \forall S \in \mathcal{T}.$$

More generally, this inequality holds true for any convex function g, and

$$\mathbb{E}\big[g\big(M_\zeta\big)\big] \geq \mathbb{E}\big[g\big(L_{0,\zeta}^{Y,*} \vee Y_\zeta\big)\big] = \mathbb{E}\big[g(M_\zeta^{Y,\oplus})\big]$$

Initial condition $x \geq Z_0^Y$ Same result by using $L^{Y,*}S, \zeta \vee m$ in place of $L_{S,\zeta}^{Y,*}$.

Skew Brownian Motion

Strategic Process

Mandelbaum (1993), Nek.Karatzas (1997)

Framework : Two arms Brownian Bandit

- With two independent Brownian motions (W^1, W^2) we associate two pay-off strictly increasing functions $\eta^i(W^i_t)$ and their Gittins Index $\nu^i(W^i)$

– ν^i is also positive strictly increasing, with inverse function μ^i (assumed to have the same domain.)

- Let $\sigma^i(m) = \inf\{t; \nu^i(W^i_t) \leq m\}$, $\gamma^i(\alpha) = \inf\{t; W^i_t \leq \alpha\}((\alpha \leq 0)$, $\sigma^i(m) = \gamma^i(\mu^i(m))$.
- The minimum rewards : $\underline{\mathrm{W}}^i(t) = \inf_{u \leq t} W^i(u)$ is the inverse function of $\gamma^i(\alpha)$, and is flat on the excursions of the reflected Brownian motion $\mathcal{R}(W^i)(t) = W^i(t) - \underline{\mathrm{W}}^i(t)$
- $\underline{\mathrm{M}}^i_t = \inf_{u \leq t} \nu^i(W^i_u) = \nu^i(\underline{\mathrm{W}}^i(t))$ is the inverse function of σ^i

A reflected Brownian motion

Optimal strategies

- Let $\underline{\mathrm{M}}$ the continuous inverse of $\sigma^1(m) + \sigma^2(m)$, $T^i(t) = \sigma^i(\underline{\mathrm{M}}(t))$ when t is a decreasing time of $\underline{\mathrm{M}}$, and $\underline{\mathbf{M}}(t) = \underline{\mathbf{M}}^i(T^i(t)) \quad i = 1, 2$ **at any time.**
- Let $S^i = \mathcal{R}(W^i)(T_i) = W^i(T_i) - \underline{\mathrm{W}}^i(T_i) = \mu^i(M_i(T_i)) - \mu^i(\underline{\mathrm{M}}) \geq 0$. Then $S^i(t) > 0$ if $T_i(t)$ belongs to an excursion of $\mathcal{R}(W^i)$, and does not belongs to the support of $\underline{\mathrm{M}}$.

$\Rightarrow$ **Lemma** Let ν the inverse fonction of $\mu^1 + \mu^2$ and μ the inverse function of ν. Put $\mathcal{S}(t) = S^1(t) + S^2(t) = W_t + L_t^W$. Then $W(t) = W^1(T_1(t)) + W^2(T_2(t))$ is a $\mathcal{G} = \mathcal{F}^1 \vee \mathcal{F}^2$-Brownian motion, and $L^W = -\sum_{i=1}^2 \mu^i(\underline{\mathrm{M}}) = -\mu(\underline{\mathrm{M}})$.

– By the previous remarks, L^W only increases when $\mathcal{S}(t) = 0$ and $\mathcal{S}(t) = S^1(t) + S^2(t)$ is a reflected Brownian motion ;

– by uniqueness of the Skohorod problem $\mu(\underline{\mathrm{M}})(t) = \inf_{u \leq t} W^1(T_1(t)) + W^2(T_2(t))$.

By classical result, the distribution of L_t^W is well-known.

Skew Brownian motion

Pathwise construction

- Let $\mathbf{X} = \mathbf{S^1} - \mathbf{S^2} = \mathbf{B} + \mathbf{V}$ where $B = W^1(T_1) - W^2(T_2)$ is a Brownian motion, and $V = \mu^1(\underline{\mathrm{M}}) - \mu^2(\underline{\mathrm{M}}) = \phi(L^W)$ where

$$\phi(\mathbf{l}) = (\mu^1 - \mu^2)(\nu)(-l) = (\mu^1 - \mu^2)((\mu^1 + \mu^2)^{-1})(-l).$$

– Because $S^1_t S^2_t = 0$, $S^1 = X^+$ and $S^2 = X^-$, and $|X| = \mathcal{S}$ is a reflected Brownian motion, with local time $L^X = L^W$. Then, X is solution of the following problem, involving the local time L^X, where the function $\phi \in \mathcal{C}^1$ and $|\phi| \leq 1$

$$X_t = \phi(L^X)_t + B_t$$

- **Examples** :
  - $\nu^1 = \nu^2$, $\phi \equiv 0$, and X is a Brownian motion
  - $\nu^1(x) = \nu^2(\alpha x)$, $\alpha \in (0, 1]$, then $\phi(l) = \beta l$ with $\beta = \frac{1-\alpha}{1+\alpha}$, and X is the Skew Brownian motion (Harrison Kreps(1981), Walsh(78))

Multidimensional case

Assume a bandit problems with d projects

- By the same way, we still have that $S^i(t) > 0$ only outside of the open support of $\underline{\mathrm{M}}$, and $\mathcal{S}(t) = \sum_{i=0}^{d} S^j(t)$ is a reflected Brownian motion, with intrinsic local time $-\mu(\underline{\mathrm{M}})$

- **How describe the muti-dimensional process $\mathcal{S}$ which are reflected independent Brownian motions with different scales of times**

To finish...

- In 1993, my daughter Imen (6 years) asks me :
  but Mom, why do you argue with Ioannis always bandit problems with multiple guns, you are not police ?
  She was really surprised.
- Explanation : in french the word bandit is the same, but **the word arm means weapon**

Thank you Ioannis for these moments

so stimulating and friendly

Happy Birthday

Next Year in Paris