SOME SURPRISING SIMPLE COMBINED CONTROL AND STOPPING PROBLEMS

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Abstract

We study a variational inequality describing the optimal control and stopping of a Brownian motion with the control as drift.

The form of the solution is the same for wide classes of final charge and control cost functions that are symmetric, smooth, and convex.

SETTING

Let us suppose given a probability space, on which there is a Brownian motion. Here we can formulate the task of solving a Stochastic Differential Equation

$$\mathrm{d}x(t) = u(t)\,\mathrm{d}t + \mathrm{d}w(t)\,,$$

where $x(0)$ is given, $u(\cdot)$ is an "admissible" control process, adapted to $x(\cdot)$, and $w(\cdot)$ is a Brownian motion.

In addition to an operating cost $c > 0$ per unit time, we also assume as given a "final cost" function $k(\cdot)$, and a "running cost of control" function $\psi(\cdot)$. Both these functions are symmetric about the origin, smooth, convex, increasing on the positive half-line, and zero at the origin.

PROBLEM

To find an admissible control process $u(\cdot)$, and a stopping time τ of the filtration of $x(\cdot)$, so as to minimize the total expected cost

$$\mathbb{E}\left(k(x(\tau)) + \int_0^{\tau} \psi\big(u(t)\big)\,\mathrm{d}t + c\,\tau\right).$$

VARIATIONAL INEQUALITY

The problem can be cast as solving a Variational Inequality, with a BELLMAN term describing the optimization of the controlled drift:

$$0 = \min\left(k(x) - V(x)\,,\ \frac{1}{2} V''(x) + \min_{\mathrm{u}\in\mathbb{R}} \left[\mathrm{u}\, V'(x) + \psi(\mathrm{u}) \right] + c \right).$$

- Let us denote by $\xi(\cdot)$ the inverse function of $\psi'(\cdot)$, guess that we have $V'(0) = 0$ as well as

$$V'(x) > 0 \quad \text{for} \quad x > 0\,, \qquad V'(x) < 0 \quad \text{for} \quad x < 0\,,$$

and that the minimum over $\mathrm{u} \in \mathbb{R}$ is attained by

$$\mathbf{u}^*(x) = \xi\big(- V'(x)\big)\,, \quad x \in \mathbb{R}\,.$$

We define the function

$$\lambda(z) := \min_{\mathrm{u}\in\mathbb{R}} \big[\mathrm{u}\, z + \psi(\mathrm{u})\big] = z\,\xi(-z) + \psi\big(\xi(-z)\big)\,, \quad z \in \mathbb{R}$$

and note that it satisfies

$$\lambda(0) = 0\,; \quad \text{and} \quad \lambda'(z) = \xi(-z) < 0\,, \quad \lambda(z) < 0 \quad \text{for} \quad z > 0\,.$$

- The variational inequality thus becomes

$$0 = \min\left(k(\cdot) - V(\cdot)\,,\ \frac{1}{2}\,V''(\cdot) + \lambda\big(V'(\cdot)\big) + c\right)$$

$$= \min\Big(k(\cdot) - V(\cdot)\,, \mathcal{L}\big(V(\cdot)\big) + c\Big)\,,$$

where we set

$$\mathcal{L}\,V := \frac{1}{2}\,V'' + \lambda\big(V'\big)\,.$$

By symmetry, we need only consider this equation on the half-line $(0, \infty)$.

Observation: If for some $\gamma \in \mathbb{R}$ we have

$$\lambda(\gamma) + c = 0\,,$$

then a linear function with slope γ solves $\mathcal{L}V(\cdot) + c = 0$.

• More specifically, let $s > 0$ (for "stop") be a solution of the scalar equation

$$\lambda(k'(s)) + c = 0\,.$$

Let

$$\begin{aligned} V(x) &= k(x)\,, \quad \text{for } 0 \le x \le s\,, \\ V(x) &= k(s) + k'(s)\,(x - s)\,, \quad \text{for } x > s\,. \\ V(x) &= V(-x)\,, \quad \text{for } x < 0\,. \end{aligned}$$

Since $\lambda(\cdot)$ is negative and decreasing on $(0, \infty)$, and $k'(\cdot)$ is increasing on $(0, \infty)$, we have

$$\mathcal{L}k(x) + c \ \geq \ 0 \,, \quad \text{for} \ \ x \leq s \,.$$

- Thus, the function $V(\cdot)$ defined on $[0, \infty)$ as before

$$V(x) \ = \ k(x) \,, \quad \text{for} \ \ x \leq s \,,$$

$$V(x) \ = \ k(s) + k'(s)\,(x - s) \,, \quad \text{for} \ \ x > s \,,$$

and by even symmetry on $(-\infty, 0)$, solves the variational inequality and has a C^1 (smooth) fit to $k(\cdot)$ at the point s.

The situation is depicted in the following picture.

THE VALUE FUNCTION

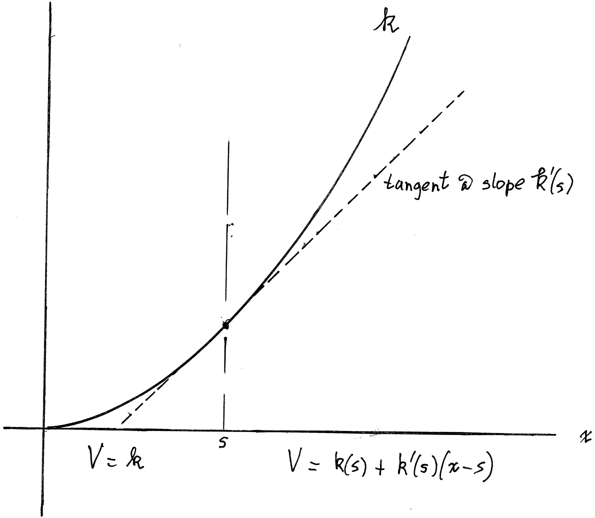

Figure 1: The function $V(\cdot)$ from the previous slide.

THE OPTIMAL STRATEGIES

- The optimal stopping time is

$$\tau_{opt} := \inf\left\{t \geq 0 : x(t) \leq s\right\}, \quad \text{for} \;\; x(0) > s\,,$$

$$\tau_{opt} = 0\,, \quad \text{for} \;\; x(0) \leq s\,.$$

- The optimal control is given in feedback, Markovian form

$$u_{opt}(t) = \mathbf{u}^*(x(t))\,, \quad 0 \leq t < \infty\,,$$

where $\mathbf{u}^*(\cdot)$ is the function

$$\mathbf{u}^*(x) = \xi\big(-V'(x)\big)\,, \quad x \in \mathbb{R}$$

we encountered before.

CHANGE OF PROBABILITY MEASURE

Let us write

$$\mathcal{L}\, V \ := \ \frac{1}{2}\, V'' + \lambda\big(V'\big) \ = \ \frac{1}{2}\, V'' + V' \cdot \vartheta\big(V'\big)\,,$$

$$\vartheta(z) \ := \ \frac{\lambda(z)}{z}\,, \quad z \neq 0 \qquad \textit{and} \qquad \vartheta(z) \ := \ 0\,, \quad z = 0\,.$$

It is helpful to view the factor $\vartheta(V')$ in $\mathcal{L}V$ as a new constant drift, in the following manner: Let $b(\cdot)$ be standard Brownian motion on the probability space, fix $\mathrm{x} \in \mathbb{R}$, introduce the time

$$\sigma \ := \ \inf\big\{\, t \geq 0 \ : \ \mathrm{x} + b(t) \leq s \,\big\}\,,$$

and let $M(\cdot)$ be the martingale defined by the usual Cameron-Martin-Girsanov functional as the solution of the SDE

$$\mathrm{d}M(t) \ = \ M(t)\ \vartheta\big(V'(\mathrm{x} + b(t))\big)\,\mathrm{d}b(t)\,, \qquad M(0) = 1\,.$$

Then we write

$$\mathrm{d}b(t) \;=\; \vartheta\big(k'(s)\big)\,\mathrm{d}t + \mathrm{d}w(t)\,,$$

so that by the GIRSANOV theorem the stopped process $w(\cdot \wedge \sigma)$ is Brownian motion under the new measure $\mathrm{d}\widetilde{\mathbb{P}} = M(\sigma)\,\mathrm{d}\mathbb{P}$.

- The idea is, that this change of measure makes the process

$$x(\cdot) \;:=\; \mathrm{x} + b(\cdot)$$

a solution of the equation

$$\mathrm{d}x(t) \;=\; \vartheta\big(V'(x(t))\,\mathrm{d}t + \mathrm{d}w(t)\,, \qquad x(0) = \mathrm{x}$$

with $w(\cdot)$ a $\widetilde{\mathbb{P}}-$Brownian motion, as befits the meaning of $\mathcal{L}$, on the stochastic interval $[0, \sigma]$; and it makes σ a surrogate of τ.

However under the new measure, and until it hits the point $s > 0$, the process $x(\cdot) = \mathrm{x} + b(\cdot)$ is Brownian motion with constant drift $\vartheta\big(k'(s)\big) < 0$, so the expected time until this happens is given as

$$\mathbb{E}^{\widetilde{\mathbb{P}}}(\sigma) \;=\; \mathbb{E}\big(M(\sigma)\,\sigma\big) \;=\; -\frac{\mathrm{x} - s}{\vartheta\big(k'(s)\big)}\,, \qquad \mathrm{x} > s\,.$$

Then on the strength of ITÔ's rule and of

$$\mathrm{d}b(t) = \vartheta\big(V'(x(t))\,\mathrm{d}t + \mathrm{d}w(t)\,, \quad \mathcal{L}\,V = \frac{1}{2}\,V'' + V'\cdot\,\vartheta\big(V'\big)\,,$$

we have

$$\begin{aligned} k(s) \;&=\; V\big(\mathrm{x} + b(\sigma)\big) \;=\; V\big(x(\sigma)\big) \;= \\ &=\; V(\mathrm{x}) + \int_0^\sigma \left(V'(x(t))\,\mathrm{d}b(t) + \frac{1}{2}V''(x(t))\,\mathrm{d}t \right) \\ &=\; V(\mathrm{x}) + \int_0^\sigma \Big(V'(x(t))\,\mathrm{d}w(t) + \mathcal{L}V(x(t))\,\mathrm{d}t \Big) \\ &=\; V(\mathrm{x}) + \int_0^\sigma V'(x(t))\,\mathrm{d}w(t) - c\,\sigma\,. \end{aligned}$$

Taking expectations with respect to the new measure $\widetilde{\mathbb{P}}$ and recalling

$$-c \,=\, \lambda\big(k'(s)\big) \,=\, k'(s)\cdot\vartheta\big(k'(s)\big)\,,$$

we find that for $x > s$ we have

$$\begin{aligned} V(x) \,=\, k(s) - c\,\mathbb{E}\big(M(\sigma)\,\sigma\big) \,&=\, k(s) - c\,(x-s)\,/\,\vartheta(k'(s)) \\ &=\, k(s) + k\,'(s)\,(x-s)\,. \end{aligned}$$

Nota bene: The same observation and solution hold if the stochastic DE for $x(\cdot)$ is of the form

$$\mathrm{d}x(t) \,=\, u(t)\,\mathrm{d}t + g\big(x(t)\big)\,\mathrm{d}w(t)\,, \qquad x(0) = \mathrm{x}$$

Then the result can be described and understood by a time-change argument using the quadratic variation of the martingale term.

And what happens if the scalar equation

$$\lambda(k'(s)) + c \;=\; 0$$

has no solution on $(0, \infty)$?

Then, as it turns out, *the best thing to do is stop at once.* Indeed, let us recall that we have $c > 0$, $\lambda(0) = 0$, and

$$\lambda'(z) \;=\; \xi(-z) < 0\,, \quad \lambda(z) < 0 \qquad \text{for} \;\; z > 0\,,$$

so $\lambda(k'(z)) + c > 0$ holds for all $z > 0$.

- Let now $u(\cdot)$ be an admissible control process, let $x(\cdot)$ be the corresponding solution to the Stochastic Differential Equation

$$\mathrm{d}x(t) \;=\; u(t)\,\mathrm{d}t + \mathrm{d}w(t)\,,$$

where $\mathrm{x} = x(0)$ is a given real number, and consider an arbitrary stopping time σ of the filtration of $x(\cdot)$.

The cost of using the strategy (u, σ) starting at x, is

$$k(x(\sigma)) + \int_0^\sigma \psi\big(u(t)\big)\,\mathrm{d}t + c\,\sigma \;\geq$$

$$\geq\; k(\mathrm{x}) + \int_0^\sigma \left(\frac{1}{2}k''(x(t)) + u(t)k'(x(t)) + \psi\big(u(t)\big) + c\right)\mathrm{d}t$$

$$+ \int_0^\sigma k'(x(t))\mathrm{d}w(t)$$

$$\geq\; k(\mathrm{x}) + \int_0^\sigma \left(\frac{1}{2}k''(x(t)) + \lambda\big(k'(x(t))\big) + c\right)\mathrm{d}t + \int_0^\sigma k'(x(t))\mathrm{d}w(t)$$

$$\geq\; k(\mathrm{x}) \;+\; \int_0^\sigma k'(x(t))\mathrm{d}w(t)\,,$$

thanks to the convexity of $k(\cdot)$. The last term is a stochastic integral whose expectation is equal to zero, so we conclude

$$\mathbb{E}\left(k(x(\sigma)) + \int_0^\sigma \psi\big(u(t)\big)\,\mathrm{d}t + c\,\sigma\right) \;\geq\; k(\mathrm{x})\,.$$

In other words, if there is no solution to

$$\lambda(k'(s)) + c \;=\; 0\,, \qquad s > 0\,,$$

the best thing to do at x is to do no control, stop at once, and pay $k(\mathrm{x})$.

NO, THESE IDEAS DO NOT GENERALIZE
EASILY TO SEVERAL DIMENSIONS.

Happy Birthday, Yannis

Takk, díky vřelé, e spasibo.