Invariant measures and the soliton resolution conjecture

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  - Choose a function $f : \mathbb{R}^d \to \mathbb{C}$ uniformly from the set of all functions v that are piecewise constant in these small cubes and zero outside the box $[-L, L]^d$, and satisfy $M(v) = m$.
  - This is a probabilistically sensible question; the resulting f approaches zero in the L^∞ norm in the "infinite volume continuum limit".

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- Before answering this question, let us first connect it to the study of the nonlinear Schrödinger equation (NLS).
- $M(v)$ is called the mass of v and $H(v)$ is called the energy of v in the context of the NLS.

The focusing nonlinear Schrödinger equation

- A complex-valued function u of two variables x and t, where $x \in \mathbb{R}^d$ is the space variable and $t \in \mathbb{R}$ is the time variable, is said to satisfy a d-dimensional focusing nonlinear Schrödinger equation (NLS) with nonlinearity parameter p if

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- The mass and energy defined before are conserved quantities for this flow.

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- These are solutions of the form $u(x,t) = v(x)e^{\mathrm{i}\omega t}$, where ω is a positive constant and the function v is a solution of the soliton equation

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- Often, the function v is also called a soliton.

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- The only case where it is partially solved is when $d = 1$ and $p = 3$, where the NLS is completely integrable. In higher dimensions, some progress in recent years.
- It is generally believed that proving a precise statement is "far out of the reach of current technology". See e.g. Terry Tao's blog entry on this topic, or Avy Soffer's ICM lecture notes.

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  - In statistical physics parlance, this is the Grand Canonical Ensemble.

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- Other important contributions from Bourgain, McKean, Vaninsky, Zhidkov, Rider, Brydges, Slade,....
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- However, all in all, not much is known in $d \geq 3$. In fact, it is possible that the idea does not work at all in $d \geq 3$.
- More importantly, no one has analyzed the behavior of random functions picked from these measures. Such behavior would reflect the long-term behavior of NLS flows.

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- Instead of considering the Grand Canonical Ensemble of Lebowitz, Rose & Speer, one may alternatively consider the Microcanonical Ensemble.
- The microcanonical ensemble, in this context, is the restriction of our fictitious Lebesgue measure on function space to the manifold of functions satisfying $M(v) = m$ and $H(v) = E$, where m and E are given.

The microcanonical ensemble contd.

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- I tried to make sense of the microcanonical ensemble in some simpler settings before, one on my own and one with Kay Kirkpatrick. Could not pass to the continuum limit.
- The main goal of this talk is to show that it is indeed possible to take the discretized microcanonical ensemble to a continuum limit in such a way that very conclusive results can drawn about it in all dimensions.

Equivalence classes

- If v satisfies $M(v) = m$ and $H(v) = E$, so does the function

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- Thus, it is reasonable to first quotient the function space by the equivalence relation $\sim$, where $u \sim v$ means that u and v are related in the above manner.
- We will generally talk about functions and equivalence classes as the same thing.

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Then each Q_λ is also a soliton.
  - For each $m > 0$, there is a unique $\lambda(m) > 0$ such that $Q_{\lambda(m)}$ is the ground state soliton of mass m.

Main result

Theorem (C., 2012; rough statement)

Suppose that $p < 1 + 4/d$, and that E is a real number bigger than the ground state energy at a given mass m. If we attempt to choose a function uniformly at random from all functions satisfying $M(v) = m$ and $H(v) = E$, by first discretizing the problem and then passing to the infinite volume continuum limit, then the resulting sequence of discrete random functions (equivalence classes) converges in the L^∞ norm to the ground state soliton of mass m.

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- Actually, this is a theorem about microcanonical invariant measures of the discrete NLS. I do not construct an invariant measure for the continuum NLS.

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Main result

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- What about multi-soliton solutions? Will discuss later.

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- The main theorem says that the equivalence class corresponding to this random function $\tilde{f}$ converges to the ground state soliton of mass m if (ϵ, h, nh) is taken to $(0, 0, \infty)$ in an appropriate manner.

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- How is this compatible with multi-soliton solutions in the continuum case? May be the recession of the solitons "outruns" the convergence to equilibrium.

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- On the other hand

$$\mathrm{Vol}(\mathbb{R}^d \setminus V_\delta) \le \frac{1}{\delta^2} \int_{\mathbb{R}^d \setminus V_\delta} |f(x)|^2 dx \le \frac{m}{\delta^2}.$$

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- The visible part, being supported on a "small" set, can be analyzed directly.
- For the invisible part, one has to develop joint large deviations for the mass and the gradient. (There is no nonlinear term!)
- The large deviation analysis throws up the following key conclusion: If the visible part has mass m', then with high probability, the energy of the visible part must be close to the lowest possible energy at mass m'.

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- Use these smoothness estimates, together with the stability of the ground state soliton, to prove convergence of discrete solitons to continuum solitons.