Max-plus Stochastic Processes and Control

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Historical Background

a) Optimal deterministic control

Pontryagin's principle, Bellman's dynamic programming principle (1950s)

b) Two-player, zero-sum differential games

Isaacs pursuit-evasion games (1950s)

c) Stochastic control

Deterministic control theory ignores time varying disturbances in dynamics

Stochastic differential equations models

Dynamic programming/PDE methods (1960s)

Changes of probability measure-Girsanov

d) Freidlin-Wentzell large deviations theory

Small random perturbations, rare events (late 1960s)

e) H-infinity control theory (1980s) Disturbances not modeled as stochastic processes, min-max viewpoint

Stochastic vs deterministic views of uncertainty

$v \in \Omega$ an "uncertainty"

$J(v)$ a "criterion" or "cost"

Stochastic view: J a random variable on $(\Omega, \mathcal{F}, P)$

Evaluate $E[(F(J)]$

Nonstochastic view: Evaluate $\max_v J(v)$

Less conservative viewpoint: evaluate

$$\max_v [q(v) + J(v)] = E^+(J)$$

$q(v)$ "likelihood" of v

$$q(v) \leq 0, \quad q(v_0) = 0$$

Connection between stochastic and nonstochastic views

$$F(J) = F_\theta(J) = e^{\theta J},$$

θ a risk sensitivity parameter

$p_\theta(v)$ probability of v

$$p_\theta(v) \sim e^{-\theta q(v)}$$

$$\lim_{\theta \to \infty} \theta^{-1} \log E\left[e^{\theta J}\right] = E^{+}(J)$$

2. **Max-plus expectations**

Max-plus addition and multiplication

$$
\begin{aligned}
-\infty \; &\leq \; a, b < \infty \\
a \oplus b \; &= \; \max(a, b) \\
a \otimes b \; &= \; a + b
\end{aligned}
$$

Maslov idempotent probability calculus

$$Q(A) = \sup_{v \in A} q(v)$$

max-plus probability of $A \subset \Omega$

$$E^{+}(J) = \oplus_v [q(v) \otimes J(v)]$$

max-plus expectation of J

Max-plus linearity

$$\begin{aligned} E^{+}(J_1 \oplus J_2) &= E^{+}(J_1) \oplus E^{+}(J_2) \\ E^{+}(c \otimes J) &= c \otimes E^{+}(J) \end{aligned}$$

3. **Max-plus stochastic differential equations and large deviations**

Fleming Applied Math. Optimiz. 2004

$x(s) \in \mathbb{R}^n$ solution to the ODE

$$\begin{aligned} dx(s) &= f(x(s))ds + g(x(s))v(s)ds, t \le s \le T \\ x(t) &= x, \quad v(s) \in \mathbb{R}^d \end{aligned}$$

$v(\cdot)$ a disturbance control function

$$
\begin{aligned}
v(\cdot) \in \Omega &= L^2([t,T];\mathbb{R}^d) \\
q(v) &= -\frac{1}{2}\int_t^T |v(s)|^2 ds \\
J(v) &= \mathcal{J}(x(\cdot)) \\
E^+[\mathcal{J}(x(\cdot))] &= \sup_{v(\cdot)} \left[\mathcal{J}(x(\cdot)) - \frac{1}{2}\int_t^T |v(s)|^2 ds\right]
\end{aligned}
$$

Example 1: $\mathcal{J}(x(\cdot)) = \ell(x(T))$ terminal cost

Example 2: $\mathcal{J}(x(\cdot)) = \max_{[t,T]} \ell(x(s))$ max-plus additive running cost

Assumptions: $f, g, \ell \in C^1$

$f_x, g, g_x, \ell, \ell_x$ bounded

Connection with large deviations

$X_\theta(s)$ solution to the SDE

$$
\begin{aligned}
dX_\theta(s) &= f(X_\theta(s))ds + \theta^{-\frac{1}{2}} g(X_\theta(s))dw(s), \\
& \quad t \le s \le T \\
X_\theta(t) &= x
\end{aligned}
$$

$w(s)$ d-dimension Brownian motion

In Example 1

$$\lim_{\theta\to\infty} \theta^{-1} \log E\left[e^{\theta\ell(X_\theta(T))}\right] = E^+[\ell(x(T))]$$

In Example 2

$$\lim_{\theta\to\infty} \theta^{-1} \log E \int_t^T e^{\theta\ell(X_\theta(s))} ds = E^+[\max_{[t,T]} \ell(x(s))]$$

If $L = e^\ell$, then $L^\theta = e^{\theta\ell}$.

4. **Max-plus martingales and differential rule**

Conditional likelihood of v, given $A \subset \Omega$

$$
\begin{aligned}
q(v|A) &= q(v) - \sup_{\omega \in A} q(\omega), \text{ if } v \in A \\
&= -\infty \text{ if } v \notin A
\end{aligned}
$$

$$v^\tau = v|_{[t,\tau]}$$

$$q(v|v^\tau) = -\frac{1}{2}\int_\tau^T |v(s)|^2 ds$$

$M(s) = M(s, v^s)$ is a max-plus martingale if

$$E^+[M(s)|v^\tau] = M(\tau),\ t \le \tau < s \le T$$

Max-plus differential rule

$$H(x,p) = f(x)\cdot p + \frac{1}{2}|pg(x)|^2,\ x,p \in \mathbb{R}^n$$

If $\phi \in C_b^1([0,T]\times\mathbb{R}^n)$, $x(s)$ a solution to the ODE on $[t,T]$ with $t \geq 0$

$$d\phi(s,x(s)) \;=\; [\phi_t(s,x(s)) + H(x(s),\phi_x(s,x(s))] \\ ds + dM(s)$$

$$M(s) = \int_t^s \left[\zeta(r)\cdot v(r) - \frac{1}{2}|\zeta(r)|^2\right] dr$$

$$\zeta(r) = \phi_x(r,x(r))g(x(r))$$

$M(s)$ is a max-plus martingale

Backward PDE

$$\phi_t + H(x, \phi_x) = 0$$

If ϕ satisfies the backward PDE, $M(s) = \phi(s, x(s))$ is a max-plus martingale.

Taking $\tau = t, s = T$

$$\phi(t, x) = E^{+}_{tx}[\phi(T, x(T)] = E^{+}_{tx}[\ell(x(T))]$$

5. **Dynamic programming PDEs and variational inequalities**

A) Terminal cost problem: value function

$$W(t,x) = E^{+}_{tx}[\ell(x(T))]$$

Dynamic programming principle

$$W(\tau, x(\tau)) = \sup_{v(\cdot)} \left[-\frac{1}{2} \int_{\tau}^{s} |v(r)|^2 dr + W(s, x(s)) \right]$$

is equivalent to $W(s, x(s)$ a max-plus martingale

W is Lipschitz continuous and satisfies the backward PDE almost everywhere and in the viscosity sense

$$0 = W_t + H(x, W_x),\ 0 \leq t \leq T, x \in \mathbb{R}^n$$

$$W(T, x) = \ell(x)$$

B) Max-plus additive running cost value function

$$\begin{aligned} V(t,x) &= E^{+}_{tx}\left[\oplus \int_t^T \ell(x(s)ds\right] \\ &= E^{+}_{tx}\left[\max_{[t,T]} \ell(x(s))\right] \end{aligned}$$

Since E^{+}_{tx} is max-plus linear

$$V(t,x) = \max_{[t,T]} E^{+}_{tx}[\ell(x(s))]$$

Dynamic programming principle

$$V(t,x) = E^{+}_{tx}\left[(\oplus \int_t^s \ell(x(r))dr) \oplus V(s,x(s))\right]$$

V is Lipschitz continuous and satisfies almost everywhere and in viscosity sense

$$
\begin{aligned}
0 &= \max[\ell(x) - V(t,x), V_t + H(x, V_x)], \\
& 0 \le t \le T, x \in \mathbb{R}^n
\end{aligned}
$$

$$V(T,x) = \ell(x)$$

Idea of proof: Both terms on right are ≤ 0

Two cases:

$\ell(x) \, = \, V(t,x)$ OK

$\ell(x) \, < \, V(t,x)$ standard control argument

Infinite time horizon bounds

Take $t = 0$, T large

$$W(x) \in C^1, \ell(x) \le W(x), H(x, W_x(x)) \le 0$$

$$\Rightarrow V(0, x; T) \le W(x)$$

Equivalently: For $0 \le s \le T$, $x = x(0)$

$$\ell(x(s)) \le \frac{1}{2}\int_0^s |v(r)|^2 dr + W(x)$$

A nonlinear H-infinity control inequality

Example

$$
\begin{aligned}
f(0) &= 0,\ x \cdot f(x) \leq -c|x|^2, c > 0 \\
0 &\leq \ell(x) \leq M|x|^2, \\
W(x) &= K|x|^2, M \leq K, \| g \|^2 K \leq c
\end{aligned}
$$

6. **Max-plus stochastic control I**: **terminal cost**

Fleming-Kaise-Sheu Applied Math Optimiz. 2010

$x(s) \in \mathbb{R}^n$ state

$u(s) \in \mathcal{U}$ control ($\mathcal{U}$ compact)

$v(s) \in \mathbb{R}^d$ disturbance control

$$\begin{aligned} dx(s) &= f(x(s), u(s))ds + g(x(s), u(s))v(s)ds, \\ &\quad t \le s \le T \\ x(t) &= x \end{aligned}$$

Control $u(s)$ chosen "depending on $v(\cdot)$ past up to s"

Terminal cost criterion: minimize $E^+_{tx}[\ell(x(T))]$

Corresponding risk sensitive stochastic control problem: choose a progressively measurable control to minimize

$$E_{tx}\left[e^{\theta\ell(X_\theta(T))}\right]$$

As $\theta \to \infty$, obtain a two player differential game.

Minimizing player chooses $u(s)$

Maximizing player chooses $v(s)$

Game payoff

$$P(t, x; u, v) = -\frac{1}{2}\int_t^T |v(s)|^2 ds + \ell(x(T))$$

Want the upper differential game value (not the lower value).

Illustrative example (Merton terminal wealth problem)

$x(s) > 0$ wealth at time s

$u(s)$ fraction of wealth in risky asset

$1 - u(s)$ fraction of wealth in riskless asset

Riskless interest rate $= 0$

$$\begin{aligned} \frac{dx(s)}{ds} &= x(s)u(s)[\mu + \nu v(s)], t \leq s \leq T \\ x(t) &= x \\ f(x,u) &= \mu x u \\ g(x,u) &= \nu x u \end{aligned}$$

Usual terminal wealth problem, parameter θ: choose $u(s)$ to minimize

$$E_{tx}\left[e^{\theta\ell(X_\theta(T))}\right]$$

Take HARA utility, parameter $-\theta \ll 0$.

$$\begin{aligned}
\ell(x) &= -\log x, \quad x^{-\theta} = e^{-\theta \log x} \\
\log x(s) &= \log x + \int_t^s u(r)[\mu + \nu v(r)]dr \\
P(t, x; u, v) &= -\log x + \int_t^T \tilde{P}(u(r), v(r))dr \\
\tilde{P}(u, v) &= -u(\mu + \nu v) - \frac{1}{2}v^2
\end{aligned}$$

$$\min_u \max_v \tilde{P}(u,v) = \min_u \left[-\mu u + \frac{1}{2}\nu^2 u^2\right]$$
$$= -\frac{\mu^2}{2\nu^2}$$

Minimum when $u = u^* = \frac{\mu}{\nu^2}$

The optimal control is $u(s) = u^*$ for all s.

$$E^+\left[-\log x^*(T)\right] = -\log x - \Lambda(T-t)$$

$\Lambda = \mu^2/2\nu^2$ is the max-plus optimal growth rate

Elliott-Kalton upper and lower differential game values

Elliott-Kalton strategy α for minimizer (progressive strategy)

$$\begin{aligned} u(s) &= \alpha[v](s) \\ v(r) &= \tilde{v}(r) \text{ a.e. in } [t,s] \Rightarrow \\ \alpha[v](r) &= \alpha[\tilde{v}](r) \text{ a.e. in } [t,s] \\ \Gamma_{EK} &= \{EK \text{ strategies } \alpha\} \end{aligned}$$

The lower game value is

$$\inf_{\alpha \in \Gamma_{EK}} E_{tx}^{+}[\ell(x(T))] = \inf_{\alpha \in \Gamma_{EK}} \sup_{v(\cdot)} P(t,x;\alpha[v],v)$$

We want the upper game value

$\Gamma = \{EK$ strategies : $\alpha[v](s)$ is left continuous with limits on right $\}$

$$W(t,x) = \inf_{\alpha \in \Gamma} E_{tx}^{+}[\ell(x(T)]$$

is the upper EK value. It is Lipschitz continuous and satisfies (viscosity sense) the Isaacs PDE

$$
\begin{aligned}
0 &= W_t + \min_{u \in U} H^u(x, W_x), t \leq T \\
& W(t,x) = \ell(x) \\
H^u(x,p) &= f(x,u) \cdot p + \frac{1}{2}|pg(x,u)|^2 \\
&= f(x,u) \cdot p + \max_{v \in \mathbb{R}^d} \left[pg(x,u)v - \frac{1}{2}|v|^2 \right]
\end{aligned}
$$

Recipe for optimal control policy

$$
\underline{u}^*(s, x(s)) \in \arg\min_{u \in U} H^u(x(s), W_x(s, x(s)))
$$

Merton terminal wealth problem with non-HARA utility

$$H^u(x,p) = \mu x u p + \frac{\nu^2}{2} x^2 u^2 p^2$$

$$\min_u H^u(x,p) = -\frac{\mu^2}{2\nu^2} = -\Lambda$$

$$\begin{aligned} W(t,x) &= \ell(x) - \Lambda(T-t) \\ u^*(x) &= -\frac{\mu}{\nu^2 \ell_x(x)} \end{aligned}$$

Example: Exponential utility $\ell(x) = -x$

$$x u^*(x) = \frac{\mu}{\nu^2}$$

7. **Max-plus stochastic control II**

Max-plus additive running cost function $\ell(x,u)$

$$
\begin{aligned}
P(t,x;u,v) &= -\frac{1}{2}\int_0^T |v(s)|^2 ds + \max_{[t,T]} \ell(x(s),u(s)) \\
V(t,x) &= \inf_{\alpha\in\Gamma} E_{tx}^{+}\left[\bigoplus \int_t^T \ell(x(s),\alpha[v](s))ds\right] \\
&= \inf_{\alpha\in\Gamma} \sup_{v(\cdot)} P(t,x;\alpha[v],v)
\end{aligned}
$$

Assumptions on f,g,ℓ as before, $u\in\mathcal{U}$ compact

Isaacs variational inequality

$$0 = \min_{u \in \mathcal{U}} \max\{\ell(x,u) - V(t,x), V_t + H^u(x, V_x)\},$$
$$t \le T, x \in \mathbb{R}^n$$

$$V(T,x) = \min_{u \in \mathcal{U}} \ell(x,u)$$

V is the unique bounded, Lipschitz viscosity solution

Equivalent to a nonlinear PDE with discontinuous Hamiltonian $\mathcal{H}$

$$0 = V_t + \mathcal{H}(x, V, V_x)$$

$$\mathcal{H}(x, V, p) = \min_{u \in A(x,V)} H^u(x, p)$$

$$A(x, V) = \{u \in \mathcal{U} \colon \ell(x, u) \leq V\}$$

8. **Merton optimal consumption problem**

$$\frac{dx(s)}{ds} = x(s)u(s)[\mu + \nu v(s)] - C(s)$$

$$C(s) \geq 0 \text{ consumption rate}$$

Two controls $u(s), c(s) = C(s)/x(s)$

$$\ell(x, c) = L(cx) = L(C)$$

$L(C)$ decreasing function of C

$$\max_{[t,T]} L(C(s)) = L\left[\min_{[t,T]} C(s)\right]$$

depends on minimum consumption

$$H^{u,c}(x,p) = \mu x u p + \frac{\nu^2}{2} x^2 u^2 p^2 - cxp$$

$$\min_u H^{u,c}(x,p) = -\Lambda - cxp, \ \Lambda = \frac{\mu^2}{2\nu^2}$$

Isaacs VI becomes

$$0 = \min_{c>0} \max\{L(cx) - V(t,x), V_t - \Lambda - cxV_x\}$$

For HARA utility

$$
\begin{aligned}
L(cx) &= -\log c - \log x \\
V(t,x) &= -\log x + B(t)
\end{aligned}
$$

$$0 = \min_{c>0} \max\{-\log c - B(t), \dot{B}(t) - \Lambda + c\}$$

$$
\begin{aligned}
c^*(t) &= e^{-B(t)} \\
\dot{B}(t) &= \Lambda - c^*(t), B(T) = 0 \\
c^*(t) &= \Lambda\left(1 - e^{-\Lambda(T-t)}\right)^{-1}
\end{aligned}
$$

$c^*(t)$ tends to Λ as $T - t \to \infty$

Λ is the optimal growth rate in the Merton model without consumption (max-plus version)

Balance between growth and consumption

Fleming-Hernandez-Hernandez, Appl. Math. Optim. 2005

For non-HARA utility

$$L(c^* x) = V(t, x)$$

$$V_t - \Lambda - c^* x V_x = 0$$

Nonlinear PDE for $V(t, x)$

$$V_t - \Lambda - L^{-1}(V) V_x = 0, \; t \leq T$$