(ω_t) "nice" (uniformly integrable)
under $P^* \in \mathcal{Q}^*_{UI}$:

$$\omega_t = E_{P^*}[\omega_\infty \mid \mathcal{F}_t]$$

hence

$$S_t = E_{P^*}[D_\infty - D_t \mid \mathcal{F}_t]$$

= potential generated by (D_t)

= "fundamental" price as seen under P^*

"not nice" under $P^* \in \mathcal{Q}^*_{NUI}$:

$$\Rightarrow \quad \omega_t > E_{P^*}[\omega_\infty \mid \mathcal{F}_t]$$

hence

$$S_t > E_{P^*}[D_\infty - D_t \mid \mathcal{F}_t]$$

– both views may coexist
(cf. Examples in Jarrow–Protter)

"<u>Bubble</u>" as seen under P^*:

$$\beta^*_t := S_t - \underbrace{E^*[D_\infty - D_t \mid \mathcal{F}_t]}_{\text{fundamental value as seen under } P^*}$$

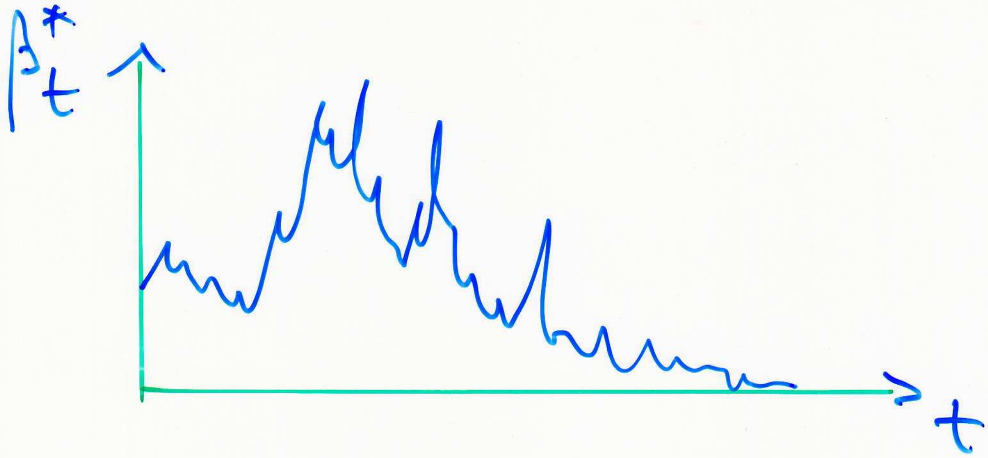
$$= \omega_t - E^*[\omega_\infty \mid \mathcal{F}_t]$$

$$\begin{cases} = 0 & \text{if } P^* \in \mathcal{P}^*_{UI} \\ > 0, & \text{(local) martingale, } \to 0 \\ & \text{not nice (not unif. integrable)} \\ & \text{if } P^* \in \mathcal{P}^*_{NUI} \end{cases}$$

In general:

$$S_t = \underbrace{\beta^*_t}_{\text{local martingale}} + \underbrace{E^*[D_\infty - D_t \mid \mathcal{F}_t]}_{\text{potential of claim } (D)}$$

<u>Riesz decomposition</u>



$\exists$ Bubble ?

depends on choice of

i) market-consistent valuation $P \in \mathcal{P}^*$ to determine the fundamental value

ii) information structure (cf. F., Protter 2011)

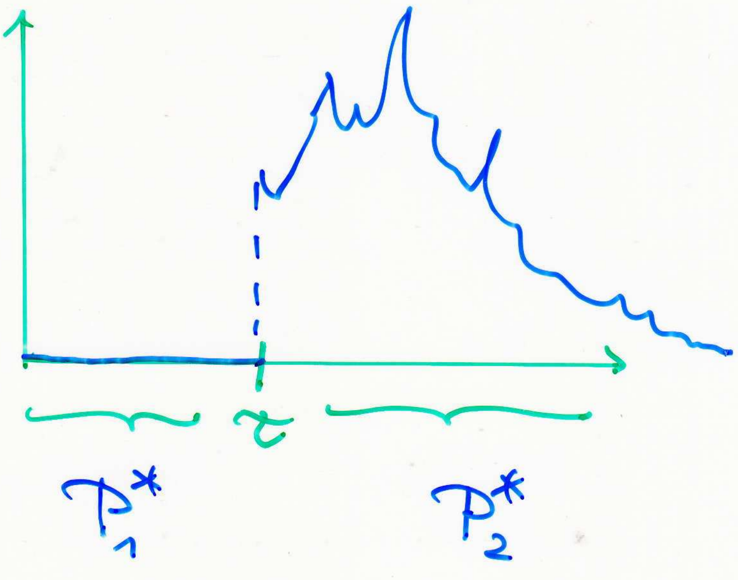
Birth of a Bubble ?

$\sim$ dynamics in $\mathcal{P}^*$

Fix $\mathcal{P}_1^* \in \mathcal{Q}^*_{UI}$ "nice"

$\mathcal{P}_2^* \in \mathcal{Q}^*_{NUI}$ "not nice"

Jarrow-Protter: sudden birth of a bubble by sudden regime change:



Biagini, F., Nedelcu:

slow birth by slow shift from $\mathcal{P}_1^*$ to $\mathcal{P}_2^*$:

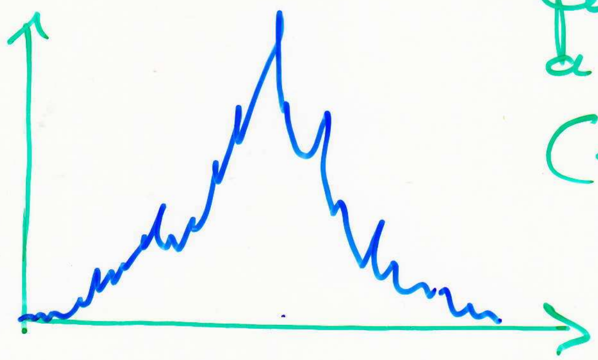
optimistic ("exuberant") view

pessimistic ("realistic"?) view

$$P_t^*[\cdot\,|\mathcal{F}_t] = \lambda_t P_2^*[\cdot\,|\mathcal{F}_t] + (1-\lambda_t)\, P_1^*[\cdot\,|\mathcal{F}_t]$$

$0 \le \lambda_t \le 1, \quad \lambda_0 = 0, \quad \lambda_t \nearrow 1$

$$\beta_t := S_t - \underbrace{E_t^*[D_\infty - D_t\,|\mathcal{F}_t]}_{\text{fundamental value as seen at } t \text{ (shifting view)}}$$



Theorem: (β_t) is a submartingale in the "moving frame" (P_t^*) up to $\tau := \inf\{t>0 \,/\, \lambda_t = 1\}$, then a local martingale (supermartingale).

— and under P_2^*, under an additional condition

The Turner Review:

A regulatory response to the global banking crisis

(FSA = Financial Services Authority March 2009)

1.1. (iv), p. 22:

"Misplaced reliance on sophisticated maths"

1.4 (iii), p. 44-45:

"mathematically modellable risk"

vs.

"Knightian uncertainty"

?

Frank Knight:

"Risk, Uncertainty, and Profit"

(Dissertation, Yale, 1916 / 21)

Risk =

"known unknowns"

$\sim \mathcal{P}$

(Knightian)
Uncertainty =

"unknown unknowns"

$\sim$ model ambiguity

$\underbrace{\mathcal{P}}_{\text{a whole class of "plausible" probabilistic models}}$

Knightian Uncertainty:

a rich source of new mathematical problems!

Possible approaches:

1. Replace $\mathbb{P}$ by its "support"

2. Replace $\mathbb{P}$ by $\mathcal{P}$

"plain vanilla" options:

via "Calcul d'Itô sans probabilités"

(H.F., 2001, 1st ECM)

"exotic" options:

R. Cont et al. (2009, 2011)

via strictly pathwise Malliavin calculus

In the same spirit:

Mark Davis, Jan Obłój

in particular:

Itô–Tanaka via strictly pathwise construction of local time *

* Footnote:
The mystery person is a woman, and her name is

Monika Würmli

Diplomarbeit ETHZ (1983)
"Lokalzeiten für stetige Martingale"

II. Bubbles in the dynamic of convex risk measures

B. Acciaio, A.F., I. Penner
Finance and Stochastics (2012)

A.F., I. Penner
Int. J. Theor. Appl. Finance (2011)

Convex / coherent risk measures

Artzner, Delbaen, Eber, Heath (1999)
Fritelli, Rosazza-Gianin (2002)
F., Schied (2002)

Monetary risk as a
capital requirement:

(regulatory perspective, cf.
Basel II, III)

$$\rho(X) = \inf\{m \mid X+m \in \mathcal{A}\}$$

= "acceptable positions"

$\mathcal{A}$ convex (diversification is not penalized)

$\Downarrow$ Fenchel-Moreau + "monetary"

$$\rho(X) = \sup_{Q \in \mathcal{Q}} \Big(\underbrace{E_Q[-X]}_{\substack{\text{expected loss}\\ \text{in model } Q}} - \alpha(Q) \Big)$$

where

$\mathcal{Q}$:= a class of probability measures on $(\Omega, \mathcal{F})$

$$\alpha(Q) := \sup_{X \in \mathcal{A}} E_Q[-X]$$

— an explicit formalization of

model uncertainty

("Knightian" ")

"robust" view:

no probability measure
is given a priori,
but:

probability measures
do come in as

"<u>stress tests</u>" !

Risk measure / monetary valuation

ρ $\qquad\qquad$ $U = -\rho$

applied to (discounted) cash flows (in discrete time)

$$C_t \;, \quad t = 0,1,\dots \qquad \text{adapted}$$

on $(\Omega, \mathcal{F})$, $(\mathcal{F}_t)_{t=0,1,\dots}$

(under Knightian uncertainty)

such that

$$X_t := \sum_{k=0}^{t} C_k \;, \quad t = 0,1,\dots$$

is a bounded measurable function on

$$\overline{\Omega} := \Omega \times \{0,1,\dots\}$$

$$\overline{\mathcal{F}} := \sigma(\text{adapted processes})$$

"optional" σ-field

i.e. $\overline{X} \in \overline{\mathcal{X}}$, even $\in \overline{\mathcal{X}}_\infty$

$$\rho : \bar{\mathcal{X}} \to \mathbb{R}^1$$

convex risk measure

$$\Downarrow$$

$$\rho(\bar{X}) = \sup_{\bar{Q} \in \mathcal{M}_{1,f}(\bar{\Omega}, \bar{\mathcal{F}})} \left(E_{\bar{Q}}[-\bar{X}] - \alpha(\bar{Q}) \right)$$

$$= \, ?$$

(cf. Banach limits etc.)

ρ "regular" $:\Leftrightarrow$

$$\rho(\bar{X}) = \lim_n \rho(\bar{X}_n)$$

if $\bar{X}_n \nearrow \bar{X}$ uniformly in t

("local" continuity from below)

$\Downarrow$

For any "relevant" $\bar{Q} \in \mathcal{M}_{1,f}(\bar{\Omega}, \bar{\mathcal{F}})$

with $\alpha(\bar{Q}) < \infty$

$\exists$ σ-additive probability measure

$\hat{Q}$ on extended space

$\hat{\Omega} := \Omega \times \{0, \dots, \infty\}$ included

$\hat{\mathcal{F}} :=$ optional

such that

$$E_{\bar{Q}}[\bar{X}] = E_{\hat{Q}}[\hat{X}]$$

$\forall \bar{X} \in \bar{\mathcal{X}}_\infty$ $\quad \hat{X}(\infty) := \lim_{t \uparrow \infty} X_t(\omega)$

via Itô-Watanabe factorization:

$$\tilde{Q} = Q \otimes \gamma$$

Q = probability measure on $(\Omega, \mathcal{F})$

γ = optional random measure on $\{0, \dots, \infty\}$, ≥ 0

$$\sum_{t=0}^{\infty} \gamma_t(\omega) + \gamma_\infty(\omega) = 1 \quad Q\text{-a.s.}$$

$\Downarrow$

$$\rho(\bar{X}) = \sup_{Q} \sup_{\gamma} E_Q\Big[\sum_{t=0}^{\infty} (-X_t)\gamma_t - X_\infty \gamma_\infty\Big] - \alpha(Q, \gamma)$$

(cf. also Kardaras, Ann. Appl. Prob.) for a related decomposition

or
for (via integration by parts)

$$C_t := X_t - X_{t-1}\ , \quad \underbrace{D_t := \sum_{s\in\{t,\dots,\infty\}} \gamma_s}_{\text{predictable discounting}} \searrow$$

$$\rho(\bar{C}) = \sup_Q \sup_D \left(E_Q\left[\sum_{t=0}^{\infty} (-C_t) D_t \right] - \alpha(Q,D) \right)$$

— combines model ambiguity (Q)
with discounting ambiguity (D)
under Knightian uncertainty

translated into

monetary valuation $\quad U := -\rho$

dynamic version:

$$U_t(C_{t+1}, C_{t+2}, \ldots) =$$

$$\operatorname*{ess.\,inf}_{Q,D} \left(E_Q\left[\sum_{s=t+1}^{\infty} \frac{D_s}{D_t} C_s \,\middle|\, \mathcal{F}_t \right] + \alpha_t(Q,D) \right)$$

Time consistency $\iff$

$\forall\, Q, D$ with $\alpha_0(Q,D) < \infty$,

$\alpha_t := \alpha_t(Q,D)$ satisfies

$$D_t \alpha_t = D_t \underbrace{\alpha_{t,t+1}}_{\text{one-step penalties} \geq 0} + E_Q\left[D_{t+1} \alpha_{t+1} \,\middle|\, \mathcal{F}_t \right]$$

i.e. Q - supermartingale

+ explicit Doob decomposition

$\Rightarrow$ Riesz decomposition into "potential" + "bubble"

Asymptotic safety (or even precision)

$:\Leftrightarrow \quad \forall Q, D: \quad \alpha_0(Q,D) < \infty$

$U_t := U_t(C_{t+1}, \dots), \quad \alpha_t := \alpha_t(Q,D)$

satisfy

$$\underbrace{D_t U_t}_{\text{assessment of future cash flow}} + \sum_{s=0}^{t} D_s C_s \xrightarrow{Q\text{-a.s., in } L^1(Q)} \underset{(=)}{\leq} \underbrace{\sum_{s=0}^{\infty} D_s C_s}_{\text{actual final outcome}}$$

Theorem: Asymptotic safety holds iff, $\forall Q, D$ with $\alpha_0(Q,D) < \infty$

$D_t \alpha_t$ is a Q-potential

i.e. $\nexists$ Bubble in the penalization

i.e.

no excessive neglect of relevant models!

In contrast to Turner Review:

"Knightian Uncertainty"

is not orthogonal to "sophisticated maths", but a

rich source

of mathematical problems!

for example

Ioannis Karatzas
Daniel Fernholz :

"Optimal Arbitrage
under model uncertainty"
(2011)

Back to

Ioannis

60

cf. Financial Times ≈ 3 weeks ago

I am curious,

and I wish you
(to the benefit of all of us!)

many happy returns
and

Alles Gute,
Ioannis!