Simulating the Greeks in Finance

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Outline

- Background on Greeks, problems with barrier options
- Representation of Greeks as an expected value
- Capriotti's method for determining a good importance sampling distribution
- Combining variance reduction methods for a specific problem

Background on the Greeks

Option prices can be expressed as the expected value of the discounted option payoff under the risk neutral measure, $p = E(G(\mathbf{S}))$

The derivatives of option prices with respect to various parameters, e.g. $\frac{\partial p}{\partial S_0}, \frac{\partial p}{\partial \sigma}, -\frac{\partial p}{\partial T}, \frac{\partial^2 p}{\partial S_0^2}$, play an important role in understanding:

- the sensitivity of prices to relevant parameters,
- in constructing a hedging portfolio,
- in approximating the loss distribution for risk management,

Both option prices and their derivatives often must be calculated numerically, often using Monte Carlo simulation. For payoff functions that are discontinuous (for example barrier options), straightforward procedures such as numerical differentiation may perform poorly.

Example: Knock-in Digital Option

Consider an example (4.6.4, p264-267 in Glasserman's text):

$\{S_t,\ t \geq 0\}$ is a GBM with parameters μ, σ, so

$$dS_t = rS_t dt + \sigma dW_t,$$

under the risk neutral measure.

A continuous European knock-in digital option has a payoff function given by

$$G(\mathbf{S}) = ce^{-rT} I_{\{\min_{0 \leq t \leq T} S_t\}} \cdot I_{\{S_T \geq K\}}.$$

The discrete version would replace the event $\{\min_{0 \leq t \leq T} S_t \leq H\}$ with $\{\min_{0 \leq i \leq N} S_{ih} \leq H\}$.

The prices and Greeks are different but will converge as N increases.

Estimating Delta for the Knock-in Digital Option

The most obvious way to estimate a derivative is to set up a numerical difference quotient, e.g.

$$
\begin{aligned}
\hat{\delta} &\approx \frac{\hat{p}(S_0 + h) - \hat{p}(S_0 - h)}{2h} \\
&= (\sum_{i=1}^{n} I(x+h) - \sum_{i=1}^{n} I(x-h))/2nh \\
&= (\sum_{i=1}^{n} (I(x+h) - I(x-h)))/2nh
\end{aligned}
$$

The summands will be 0, +1 or -1, and the probability of ± 1 is $O(h * \delta)$.

The estimate is biased and the variance is $O((nh)^{-1})$. Consequently, a very large sample size is needed to overcome the h in the denominator.

Numerical derivatives for the Glasserman Example

Consider 1,000,000 paths for $S_0 = 95$, $H = 90$, $K = 96$

- 619,351 did not reach H
- 337,826 reached H but not K
- 42,819 both x-h and x+h paid off, but they cancelled each other
- 4 made a contribution to the numerator

Representing the Greeks as an Expected Value

In 1996, Broadie and Glasserman developed two methods to represent Greeks as expected values as opposed to numerical derivatives: pathwise and log-likelihood derivatives. For $S_0 = x$ and $\mathbf{s} = (s_1, \ldots, s_N)$, price(x) = $p(x) = \int \ldots \int G(\mathbf{s}) f(\mathbf{s}|x) d\mathbf{s}$.

$$
\begin{aligned}
\frac{\partial p(x)}{\partial x} &= \frac{\partial}{\partial x} \int \ldots \int G(\mathbf{s}) f(\mathbf{s}|x) d\mathbf{s} \\
&= \int \ldots \int \frac{\partial}{\partial x} G(\mathbf{s}) f(\mathbf{s}|x) d\mathbf{s} \\
&= \int \ldots \int G(\mathbf{s}) \frac{\partial}{\partial x} f(\mathbf{s}|x) d\mathbf{s} \\
&= \int \ldots \int G(\mathbf{s}) (\frac{\partial}{\partial x} \log f(\mathbf{s}|x)) f(\mathbf{s}|x) d\mathbf{s} \\
&= E(G(\mathbf{S}) \frac{\partial}{\partial x} \log f(\mathbf{S}|x)).
\end{aligned}
$$

Formalization Using Malliavin Calculus

The two methods of Broadie and Glasserman express the Greeks (delta in the case at hand) as an expectation which can be unbiasedly estimated by Monte Carlo simulation.

The log-likelihood approach required determination of $\frac{\partial}{\partial x} \log f(\mathbf{S}|x)$, which is easy for GBM, but requires an Euler discretization in general, for which the transitions are conditionally normal.

A general method was developed by Fournie, Lasry, Lebuchoux, Lions and (Touzi) in 1999 and 2001 based on an integration by parts formula from the Malliavin Calculus.

Fournie et al Results for Delta

Assume

$$dX_t = b(X(t))dt + \sigma(X(t))dW(t),$$

and the discounted payoff, G, is determined by a set of times $0 \leq t_1 < \ldots < t_N = T$, so

$$p(x) = E(G(X_{t_1}, \ldots, X_{t_N})|X_0 = x).$$

$Y_t, t \geq 0$, $Y(t) = dX_t/dx$ is the first variation process or pathwise derivative satisfies.

$$dY_t = b'(X_t)Y_t dt + \sigma^{'}(X_t)Y_t dW_t,$$

then (FLLLT I, Proposition 3.2, p399)

$$p^{'}(x) = E(G(X_{t_1}, \ldots, X_{t_N})\frac{1}{T}\int_0^T \frac{Y_t}{\sigma(X_t)}dW_t).$$

Malliavin-Based Greeks

The expression for δ and other Greeks does not require knowledge of the likelihood function as required by the Broadie and Glasserman approach. In general, one needs to discretize X and Y.

In 2007, Chen and Glasserman began with the likelihood approach and introduced an Euler discretization for the underlying X process. The transition densities are conditionally normal, and the log-likelihood is easily written.

They showed that as the time increments converge to 0, the log-likelihood-based estimates converge to the Malliavin calculus-based estimates, so either approach can be used.

Partial List of work on the Greeks

- Broadie and Glasserman, 1996 “Estimating security price derivatives using simulation,” *Management Science*.
- Fournie, Lasry, Lebuchoux, Lions and Touzi, 1999, “Applications of Malliavin calculus to Monte Carlo Method in Finance, I” *Finance and stochastics*.
- Fournie, Lasry, Lebuchoux, and Lions, 2001, “Applications of Malliavin calculus to Monte Carlo Method in Finance, II” *Finance and Stochastics*
- Benhamou 2003 “Optimal Malliavin weighting function for the computation of Greeks,” *Mathematical Finance*

Partial List of Work on the Greeks

- Gobet and Kohatsu-Higa, 2003 “Computation of Greeks for barrier and lookback options using Malliavin calculus” *Electronic Communications in Probability*.
- Cvitanic, Ma and Zhang, 2003 “Efficient computation of hedging portfolios for options with discontinuous payoffs,” *Mathematical Finance*.
- Davis and Johansson, 2006 “Malliavin Monte Carlo Greeks for jump diffusions,” *Stochastic Processes and their Applications*.x
- Chen and Glasserman, 2007 “Malliavin Greeks without Malliavin calculus”, *Stochastic Processes and Their Application*.

The Knock-in Barrier Option Issues: 1

In considering Glasserman's example, there are a number of issues and approaches that arise:

- Numerical derivatives will be biased and require huge sample sizes for sufficient accuracy.
- The Likelihood method or the Malliavin derivative approach will recast the Greek to be an expected value, i.e. $\delta = E(G(\mathbf{S}) \cdot \text{weight function})$
- The likelihood method is still vulnerable to failure of the knock-in condition to be satisfied, especially if S_0, K, and H are somewhat far apart. This problem can be solved by importance sampling. Will discuss Capriotti's method to determine an optimal importance sampling distribution.

The Knock-in Barrier Option Issues: 2

- The variance can be further reduced by combining conditional Monte Carlo methods. For delta under GBM, once the knock-in condition is satisfied, if ever, the price of the Greek can be determined analytically. Using this can reduce the variance significantly.
- Extensions to other Greeks and to more general price processes possibly using Euler discretization.

Importance Sampling: Capriotti's Method 1

In 2008, L. Capriotti (*Quantitative Finance*, **8**, 485-497) developed a method to find a good importance sampling distribution based on a non-linear least squares approach.

Suppose $\mathcal{P}_0$ corresponds to the risk-neutral measure, and suppose there is a family of possible importance sampling distributions indexed by θ, $\{\mathcal{P}_\theta,\ \theta \in \Theta\}$. The goal is to find θ to minimize the variance of the importance sampled estimator.

For example, $\Theta = \{\theta = (\mu, \eta);\ \mu \in (-\infty, \infty), \eta > 0\}$ corresponding to changes from $N(0,1)$ to $N(\mu, \eta^2)$.

The above example assumes that all random variables used to construct a path have the same distribution. But this can be generalized to allow different changes for different parts of the path; for example allowing the mean to vary with time to price an Asian option.

Importance Sampling: Capriotti's Method 2

A preliminary simulation-based non-linear least squares analysis is conducted to determine a good θ using a sample of size M. This is followed by a relatively large study with sample size $n >> M$ using importance sampling to reach the desired low-variance estimate.

Let

$$W_\theta = \frac{d\mathcal{P}_0}{d\mathcal{P}_\theta}, \ \forall\theta \in \Theta.$$

Note

$$E_0(G(\mathbf{S})) = E_\theta(G(\mathbf{S})W_\theta(\mathbf{S})), \ \theta \in \Theta$$

where G is the discounted payoff function.

Importance Sampling: Capriotti's Method 3

Using importance sampling, one would generate a random sample of paths, $\mathbf{S}_1, \ldots, \mathbf{S}_M$ from P_θ and form the estimator of $E_\theta(G(\mathbf{S})W_\theta(\mathbf{S}))$, namely $\frac{1}{M}\sum_{i=1}^{M} G(\mathbf{S}_i)W_\theta(\mathbf{S}_i)$.

We want to choose $\theta \in \Theta$ to minimize the variance of this estimator. Since the mean is the same, independent of θ, we can, instead, focus on minimizing the second moment.

Using the second moment, this means that we want to choose θ to minimize $E_\theta((G(\mathbf{S})W_\theta(\mathbf{S}))^2)$.

Importance Sampling: Capriotti's Method 4

$$
\begin{aligned}
E_\theta((G(\mathbf{S})W_\theta(\mathbf{S}))^2) &= \int (G(\mathbf{S})W_\theta(\mathbf{S}))^2 d\mathcal{P}_\theta(\mathbf{S}) \\
&= \int G^2(\mathbf{S})W_\theta(\mathbf{S})W_\theta(\mathbf{S}) d\mathcal{P}_\theta(\mathbf{S}) \\
&= \int G^2(\mathbf{S})W_\theta(\mathbf{S}) d\mathcal{P}_0(\mathbf{S}) \\
&= E_0(G^2(\mathbf{S})W_\theta(\mathbf{S})).
\end{aligned}
$$

It follows that we want to minimize the final quantity, $E_0(G^2(\mathbf{S})W_\theta(\mathbf{S}))$, where the random paths, $\mathbf{S}$, now come from the original probability distribution, P_0.

Importance Sampling: Capriotti's Method 5

Now, $E_0(G^2(\mathbf{S})W_\theta(\mathbf{S}))$ can be estimated from the paths $\mathbf{S}_1, \ldots, \mathbf{S}_M$ by $\frac{1}{M}\sum_{i=1}^M G^2(\mathbf{S}_i)W_\theta(\mathbf{S}_i)$. Ignoring the constant $\frac{1}{M}$, we want to minimize $\sum_{i=1}^M G^2(\mathbf{S}_i)W_\theta(\mathbf{S}_i)$.

To solve this optimization problem, consider the statistical model

$$Y = G(\mathbf{X})\sqrt{W_\theta(\mathbf{X})} + \epsilon.$$

This is a non-linear model. Suppose we have values $\mathbf{X}_1, \ldots, \mathbf{X}_M$, and we use those values to create model data given by $\{(G(\mathbf{X}_i)\sqrt{W_\theta(\mathbf{X}_i)}, 0),\ 1 \leq i \leq M\}$ (so the dependent variable values are all 0.

Importance Sampling: Capriotti's Method 6

The least squares solution given these data will find the θ that minimizes the squared residuals, i.e. $\sum_{i=1}^{M}(Y_i - G(\mathbf{X}_i)\sqrt{W_\theta(\mathbf{X}_i)})^2$.

However, since $Y_i = 0$, this minimizes $\sum_{i=1}^{M} G^2(\mathbf{X}_i)W_\theta(\mathbf{X}_i)$, the second moment expression we sought to minimize.

Thus, approximating the optimal importance sampling distribution is equivalent to finding the non-linear least squares estimator in the above statistical model with $Y = 0$. The resulting $\hat{\theta}$ is approximately optimal, since it is, in fact, a random variable derived from the initial $\mathbf{X}$ data.

Capriotti suggest using the Levenberg-Marquandt non-linear least squares algorithm. This is implemented in Matlab by nlinfit.

Results for Glasserman Example: Delta

For $T = .25, N = 50, r = .05, \sigma = .15, S_0 = 95$ and payoff = \$10,000, $P_\theta = N(\theta, 1)$.

H	K	Delta	Var Ratio 1	Var Ratio 2	θ
94	96	-544.7	2.62	3.32	-.24
90	96	-136.68	6.65	22.21	-.36
85	96	-2.86	52.71	1714	-.56
90	106	-6.51	120.64	774.9	-.52

Var Ratio 1 = Standard MC versus Conditional MC

Var Ratio 2 = Standard MC versus Cond. MC and Imp. Sampling