Time Consistency and Calculation of Risk Measures in Markets with Transaction Costs

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Outline

1. Dynamic set-valued risk measures
2. Time consistency
3. Examples and calculation of risk measures
    1. Superhedging under transaction costs
    2. AV@R
4. Multi-portfolio time consistency by composition

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$$X + u \in A_t$$

  for some set $A_t \subseteq L^p_d(\mathcal{F}_T)$ of acceptable positions.

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- **dynamic risk measure**: sequence $(R_t)_{t=0}^T$ of conditional risk measures

Primal Representation

Risk measures and acceptance sets are one-to-one via

$$R_t(X) = \{u \in M_t : X + u \in A_t\}$$

and
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	R_t	A_t
finite at zero	$\emptyset \neq R_t(0) \neq M_t$	$M_t 1\!\!1 \cap A_t \neq \emptyset$ $M_t 1\!\!1 \cap (L^p_d \backslash A_t) \neq \emptyset$
monotone	$Y - X \in L^p_d(\mathcal{F}_T)_+$ $\Rightarrow R_t(Y) \supseteq R_t(X)$	$A_t + L^p_d(\mathcal{F}_T)_+ \subseteq A_t$
	convex	convex
	positively homogeneous	cone
	subadditive	$A_t + A_t \subseteq A_t$
	closed images	directionally closed
	lsc	closed
market compatible	$R_t(X) = R_t(X) + K_t^{M_t}$	$A_t + L^p_d(K_t^{M_t}) \subseteq A_t$

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Dual Representation, $1 \leq p \leq \infty$

A function $R_t : L_d^p(\mathcal{F}_T) \to \mathcal{G}((M_t)_+)$ is a closed **coherent conditional risk measure** if and only if there is a nonempty set $\mathcal{W}_{t,R_t}^q \subseteq \mathcal{W}_t^q$ such that

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- $w \in \left((M_t)_+\right)^+$
- $G_t(w) = \{v \in L^p_d(\mathcal{F}_t) : E[w^T v] \geq 0\}$.

$$\mathcal{W}_t^q \;=\; \Big\{(\mathbb{Q}, w) \in \mathcal{M}_{1,d}^{\mathbb{P}} \times \Big(\big((M_t)_+\big)^+ \setminus (M_t)^\perp\Big) : \\ \operatorname{diag}(w) \operatorname{diag}\left(E_t\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right]\right)^{-1} \frac{d\mathbb{Q}}{d\mathbb{P}} \in L_d^p(\mathcal{F}_T)_+\Big\}.$$

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static set-valued risk measures: ▷ JOUINI, TOUZI, MEDDEB (2004), HAMEL, RUDLOFF (2008), HAMEL, HEYDE (2010), HAMEL, HEYDE, RUDLOFF (2011)

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In higher dimensions: multi-portfolio time consistency implies time consistency.

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Examples

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Set of superhedging portfolios for $X \in L_d^p(\mathcal{F}_T)$

$$SHP_t(X) := \{u \in L_d^p(\mathcal{F}_t) : -X + u \in -C_{t,T}\}.$$

3.1 Superhedging

(Kabanov 99, Schachermayer 04, Pennanen, Penner 08,...)

- $(V_t)_{t=0}^T$ self-financing portfolio process if

$$V_t - V_{t-1} \in -K_t \quad P-a.s. \quad \forall t \in \{0, ..., T\} \quad (V_{-1} \equiv 0)$$

- $L_d^p(\mathcal{F}_T)$-attainable claims (from zero cost at time t)

$$C_{t,T} = \sum_{s=t}^{T} -L_d^p(\mathcal{F}_s; K_s)$$

Set of superhedging portfolios for $X \in L_d^p(\mathcal{F}_T)$

$$SHP_t(X) := \{u \in L_d^p(\mathcal{F}_t) : -X + u \in -C_{t,T}\}.$$

Under robust no arbitrage condition (NA^{r}):
$R_t(X) := SHP_t(-X)$ is a closed market-compatible **coherent dynamic risk measure** on $L_d^p(\mathcal{F}_T)$ that is **multi-portfolio time consistent**.

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Loehne, Rudloff 12 (submitted), Hamel, Loehne, Rudloff 12 (working paper)

European Call Option

Asset 0: riskless bond, $r = 10\%$, no transaction cost

Asset 1: stock, CRR, constant transaction cost $\lambda = 0.125\%$

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$\lambda = 0.125\%$ for all t			
n	6	250	1800
vert $SubHP_0(X)$	$\begin{pmatrix} -74.434 \\ 0.953 \end{pmatrix}$	$\begin{pmatrix} -76.348 \\ 0.969 \end{pmatrix}$	$\begin{pmatrix} -79.049 \\ 0.992 \end{pmatrix}$
$\pi^b(X)$	27.552	27.381	27.191
vert $SHP_0(X)$	$\begin{pmatrix} -73.814 \\ 0.948 \end{pmatrix}$	$\begin{pmatrix} -72.856 \\ 0.941 \end{pmatrix}$	$\begin{pmatrix} -70.209 \\ 0.921 \end{pmatrix}$
$\pi^a(X)$	27.854	27.994	28.370

$\lambda = 0.125\%$ for $t = 1, \ldots, T$, but no transaction cost at $t = 0$			
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$\pi^b(X)$	27.671	27.502	27.315
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Note: small intervals despite Kusuoka (95) result!

Multiple vertices

$-SHP_0(-X)$, $\lambda = 2\%$, $K = 110$, $n = 52$: 8 vertices

$$\begin{pmatrix} -34.743 & -48.097 & -79.757 & -88.323 & -91.778 & -84.331 & -54.520 & -41.461 \\ 0.322 & 0.445 & 0.732 & 0.809 & 0.840 & 0.774 & 0.504 & 0.384 \end{pmatrix}$$

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$-SHP_0(-X)$, $\lambda = 2\%$, $K = 110$, $n = 250$: 3 vertices

$$\begin{pmatrix} 2.370 & -107.125 & -110.107 \\ -0.036 & 0.973 & 1.001 \end{pmatrix}$$

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Tree approximating $(d-1)$-dim Black-Scholes-Model by Korn, Müller (09)

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Example: Exchange Option

physical delivery

$$X = (X_1, X_2, X_3)^T = \left(0, I_{\{S_T^{a,1} \geq S_T^{a,2}\}}, -I_{\{S_T^{a,1} \geq S_T^{a,2}\}}\right)^T.$$

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(cash delivery: $X = ((S_T^1 - S_T^2)^+, 0, 0)^T$)

Exchange Option, $n = 4$, includes transaction costs for bond

$r = 5\%,\ \lambda = (1\%, 2\%, 4\%)^T$ vertex of $SHP_0(X)$	$\begin{pmatrix} 13.341 & 0.000 & -7.760 \\ 0.347 & 0.498 & 0.584 \\ -0.446 & -0.331 & -0.260 \end{pmatrix}$
$\pi_0^a(X)$ (in bonds) $\pi^a(X)$ (in cash)	7.418 6.988
$r = 5\%,\ \lambda = (0.2\%, 0.4\%, 0.1\%)^T$ vertex of $SHP_0(X)$	$\begin{pmatrix} 12.403 & 8.230 & 0.000 & -6.236 & -4.237 \\ 0.308 & 0.353 & 0.441 & 0.507 & 0.486 \\ -0.433 & -0.394 & -0.317 & -0.257 & -0.276 \end{pmatrix}$
$\pi_0^a(X)$ (in bonds) $\pi^a(X)$ (in cash)	4.310 4.109

Definition: set-valued AV@R (static case): HAMEL, RUDLOFF, YANKOVA 12

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Let $\alpha \in (0,1]^d$ and $X \in L^1_d$.

$$AV@R^{reg}_{\alpha}(X) = \Big\{ \operatorname{diag}(\alpha)^{-1} \mathbb{E}[Z] - z : \\ Z \in \left(L^1_d\right)_+,\ X + Z - z\mathbb{1} \in \left(L^1_d\right)_+,\ z \in \mathbb{R}^d \Big\} \cap M.$$

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Remark: If $m = d = 1$: conditions $Z \in \left(L_1^1\right)_+$ and $X + Z - z\mathbb{1} \in \left(L_1^1\right)_+$ are equivalent to $Z \geq (-X + z\mathbb{1})^+$ with $X^+ = \max\{0, X\}$.

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Thus, $AV@R^{reg}_{\alpha}(X) = AV@R^{sca}_{\alpha}(X) + \mathbb{R}_+$

3.2 AV@R

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Thus, $AV@R^{reg}_{\alpha}(X) = AV@R^{sca}_{\alpha}(X) + \mathbb{R}_+$ with

$$AV@R^{sca}_{\alpha}(X) = \inf_{z \in \mathbb{R}} \left\{ \frac{1}{\alpha} E\left[(-X + z\mathbb{1})^+\right] - z \right\}$$

which is optimized certainty equivalent representation of the AV@R by Rockafellar and Uryasev '00.

Good-deal bounds

The market extension R^{mar} of a risk measure R satisfies

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$$AV@R_\alpha^{mar}(X) = \bigcup \left\{ AV@R^{reg}(X-Y) : Y \in \sum_{s=0}^{T} L_d^1(K_s) \right\}$$
$$= \Big\{ \operatorname{diag}(\alpha)^{-1} \mathbb{E}[Z] - z :$$
$$Z \in \left(L_d^1\right)_+,\ X + Z - z\mathbb{1} \in \sum_{s=0}^{T} L_d^1(K_s),\ z \in \mathbb{R}^d \Big\} \cap M$$

is a again set-valued coherent risk measure.

Let Ω be finite.

3.2 AV@R

Let Ω be finite. Then, $AV@R_{\alpha}^{reg}(X)$ and $AV@R_{\alpha}^{mar}(X)$ can be calculated by solving a **linear vector optimization problem** (using Benson's algorithm)

$$\text{minimize } P(x) \text{ with respect to } \leq_{M_+} \text{ subject to } Bx \geq b.$$

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Furthermore,

$$AV@R_\alpha^{reg}(X) = \bigcap_{(Q,w)\in\mathcal{W}^\alpha} \left(E^Q[-X] + G(w)\right) \cap M,$$

where

$$\mathcal{W}^\alpha = \left\{(Q,w) \in \mathcal{W} \colon \operatorname{diag}(w)\left(\operatorname{diag}(\alpha)^{-1}\mathbf{e}1\!\!1 - \frac{dQ}{dP}\right) \in (L_d^\infty)_+\right\},$$

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$\mathbf{e} = (1, ..., 1)^T \in \mathbb{R}^d$. Recall, $G(w) = \left\{x \in \mathbb{R}^d \colon 0 \leq w^T x\right\}$ and

$$\mathcal{W} = \left\{(Q,w) \in \mathcal{M}_{1,d}^{P} \times \mathbb{R}^d_+ \backslash M^{\perp} \colon \operatorname{diag}(w)\frac{dQ}{dP} \in (L_d^{\infty})_+\right\}.$$

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- can construct a multi-portfolio time consistent version of $(AV@R_\alpha)_t\,(X)$ by composition

Construction of multi-portfolio time consistent risk measures

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4. Construction of mptc risk measures

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Feinstein, Rudloff (12): Set-valued dynamic risk measures. Submitted for publication.

Thank you!