Trivariate Density Revisited

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Outline

Rank-based diffusion

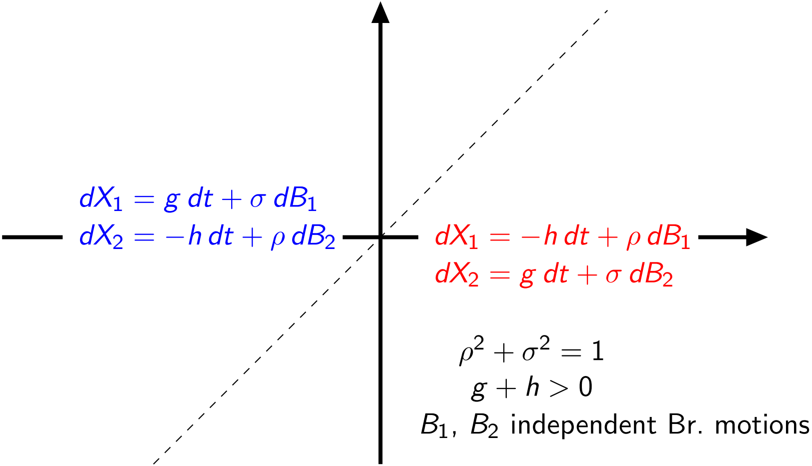
Bang-bang control

Trivariate density

Personal reflections

Rank-based diffusion

E. R. Fernholz, T. Ichiba, I. Karatzas & V. Prokaj, Planar diffusions with rank-based characteristics and perturbed Tanaka equation, *Probability Theory and Related Fields*, to appear.



Remarks

Karatzas, et. al. examine:

- Weak and strong existence and uniqueness in law;
- Properties of $X_1 \vee X_2$ and $X_1 \wedge X_2$;
- The reversed dynamics of (X_1, X_2);

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Karatzas, et. al. examine:

- Weak and strong existence and uniqueness in law;
- Properties of $X_1 \vee X_2$ and $X_1 \wedge X_2$;
- The reversed dynamics of (X_1, X_2);
- Transition probabilities (X_1, X_2).

The difference process

Define

$$Y(t) = X_1(t) - X_2(t).$$

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Then

$$\begin{aligned} dY &= I_{\{Y\leq 0\}}(g+h)\,dt - I_{\{Y>0\}}(g+h)\,dt \\ &\quad + I_{\{Y\leq 0\}}\sigma\,dB_1 - I_{\{Y\leq 0\}}\rho\,dB_2 + I_{\{Y>0\}}\rho\,dB_1 - I_{\{Y>0\}}\sigma\,dB_2 \end{aligned}$$

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where

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where

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Summary

$$\begin{aligned} X_1(t) + X_2(t) &= Z(t) \\ &= X_1(0) + X_2(0) + \nu t + V(t), \end{aligned}$$

$$\begin{aligned} X_1(t) - X_2(t) &= Y(t) \\ &= X_1(0) + X_2(0) - \lambda \int_0^t \operatorname{sgn}\big(Y(s)\big)\, ds + W(t), \end{aligned}$$

where V and W are correlated Brownian motions.

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- To determine transition probabilities for (X_1, X_2), it suffices to determine transition probabilites for (Z, Y).
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- (Z, W) is a Gaussian process.
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$$dY(t) = -\lambda \operatorname{sgn}\big(Y(t)\big)\, dt + dW(t).$$

- It suffices to determine the distribution of $(W(t), Y(t))$.

Bang-bang control

Minimize

$$\mathbb{E}\int_0^\infty e^{-t}X_t^2\,dt,$$

Subject to

$$X_t = x + \int_0^t u_s\,ds + W_t, \quad a \le u_t \le b, \quad t \ge 0.$$

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Beneš, Shepp & Witsenhausen (1980): Solution is $u_t = f(X_t)$, where

$$f(x) = \begin{cases} b, & \text{if } x < \delta, \\ a, & \text{if } x \ge \delta, \end{cases}$$

and $\delta = (\sqrt{b^2 + 2} + b)^{-1} - (\sqrt{a^2 + 2} - a)^{-1}$.

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What is the transition density for the controlled process X?

(Beneš, et. al. computed its Laplace transform.)

Transition density by Girsanov

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$$W_t = X_t - x - \int_0^t f(X_s)\,ds.$$

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Change to a probability measure $\mathbb{P}$ under which W is a Brownian motion. Compute the transition density for X under $\mathbb{P}$.

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$$\mathbb{P}\{X_t \in B\} = \mathbb{E}^X \left[I_{\{X_t \in B\}} \left. \frac{d\mathbb{P}}{d\mathbb{P}^X} \right|_{\mathcal{F}_t} \right],$$

where

$$\left. \frac{d\mathbb{P}}{d\mathbb{P}^X} \right|_{\mathcal{F}_t} = \exp\left[\int_0^t f(X_s)\,dX_s - \frac{1}{2}\int_0^t f^2(X_s)\,ds \right].$$

Transition density by Girsanov (continued)

To compute $\mathbb{P}\{X_t \in B\}$, we need the joint distribution of

$$X_t, \quad \int_0^t f(X_s)\, dX_s, \quad \int_0^t f^2(X_s)\, ds.$$

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Assume without loss of generality that $\delta = 0$. Define

$$F(x) = \int_0^x f(\xi)\,d\xi = \begin{cases} bx, & \text{if } x \leq 0, \\ ax, & \text{if } x \geq 0. \end{cases}$$

Tanaka's formula implies

$$F(X_t) = \int_0^t f(X_s)\,dX_s + \frac{1}{2}(a-b)L_t^X$$

so

$$\int_0^t f(X_s)\,dX_s = F(X_t) - \frac{1}{2}(a-b)L_t^X.$$

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$$X_t, \quad L_t^X, \quad \int_0^t f^2(X_s)\,ds.$$

Transition density by Girsanov (continued)

Define

$$\Gamma_+(t) = \int_0^t I_{(0,\infty)}\big(X(s)\big)\,ds,$$

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Then

$$\int_0^t f^2(X_s)\, ds = b^2\Gamma_-(t) + a^2\Gamma_+(t) = b^2 t + (a^2 - b^2)\Gamma_+(t).$$

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We need the joint distribution of

$$X_t, \quad L_t^X, \quad \Gamma_+(t) = \int_0^t I_{(0,\infty)}(X_s)\,ds,$$

where X is a Brownian motion.

Trivariate density

Theorem (Karatzas & Shreve (1984))

Let W be a Brownian motion, let L be its local time at zero, and let

$$\Gamma_+(t) \triangleq \int_0^t I_{(0,\infty)}(W_s)\, ds$$

denote the occupation time of the right half-line. Then for $a \leq 0$, $b \geq 0$, and $0 < t < T$,

$$\mathbb{P}\{W_T \in da, L_T \in db, \Gamma_+(T) \in dt\} = \frac{b(b-a)}{\pi\sqrt{t^3(T-t)^3}} \exp\left[-\frac{b^2}{2t} - \frac{(b-a)^2}{2(T-t)}\right] da\, db\, dt.$$

Remark

Because the occupation time of the left half-line is

$$\Gamma_-(T) \triangleq \int_0^T I_{(-\infty,0)}(W_t)\,dt = T - \Gamma_+(T),$$

we also know $\mathbb{P}\{W_T \in da, L_T \in db, \Gamma_-(T) \in dt\}$ *for* $a \le 0,\ b \ge 0,$ *and* $0 < t < T$. *Applying this to* $-W$, *we obtain a formula for*

$$\mathbb{P}\{W_T \in da, L_T \in db, \Gamma_+(T) \in dt\}, \quad a \ge 0,\ b \ge 0,\ 0 < t < T.$$

Classic trivariate density

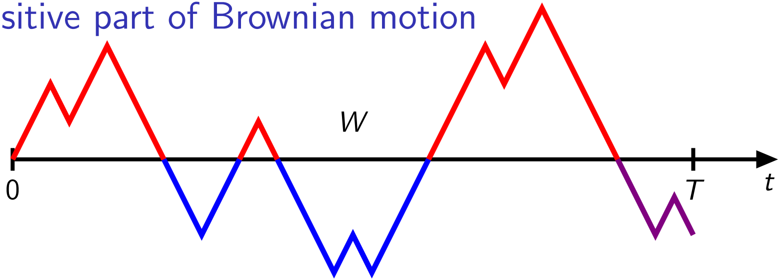
Theorem (Lévy (1948))

Let W be a Brownian motion, let $M_T = \max_{0 \le t \le T} W_t$, and let θ_T be the (almost surely unique) time when W attains its maximum on $[0, T]$. Then for $a \in \mathbb{R}$, $b \ge \max\{a, 0\}$, and $0 < t < T$,

$$\begin{aligned} &\mathbb{P}\{W_T \in da, M_T \in db, \theta_T \in dt\} \\ &\qquad = \frac{b(b-a)}{\pi\sqrt{t^3(T-t)^3}} \exp\left[-\frac{b^2}{2t} - \frac{(b-a)^2}{2(T-t)}\right] da\, db\, dt. \end{aligned}$$

The elementary proof uses the reflection principle and the Markov property.

Positive part of Brownian motion



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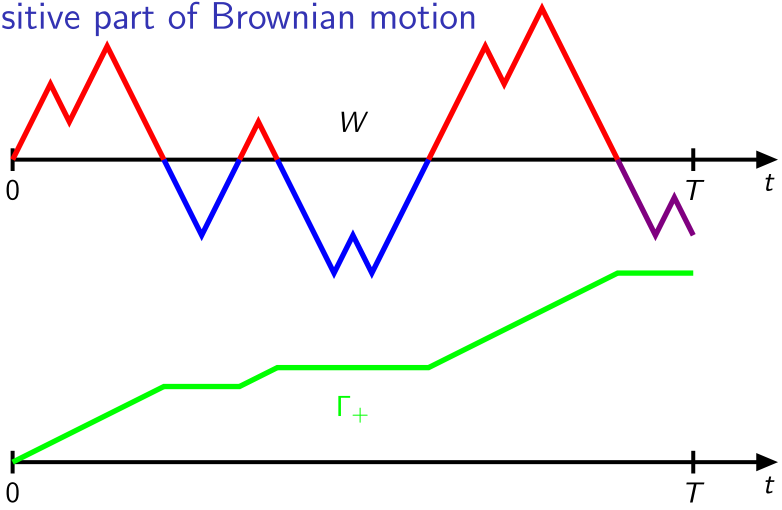

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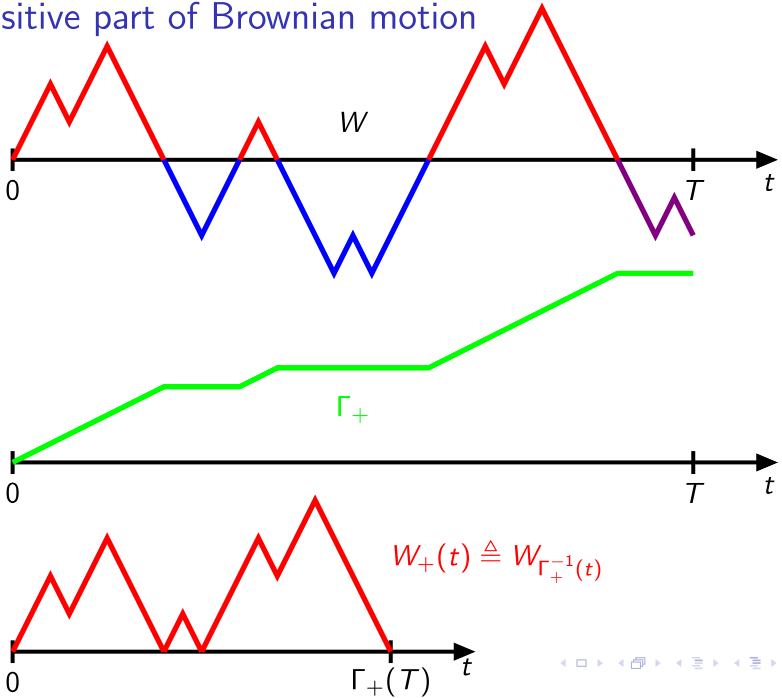
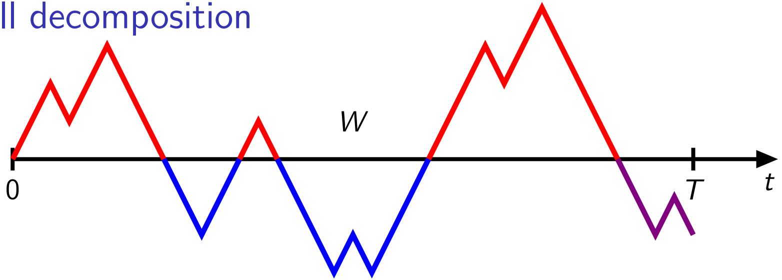


Full decomposition



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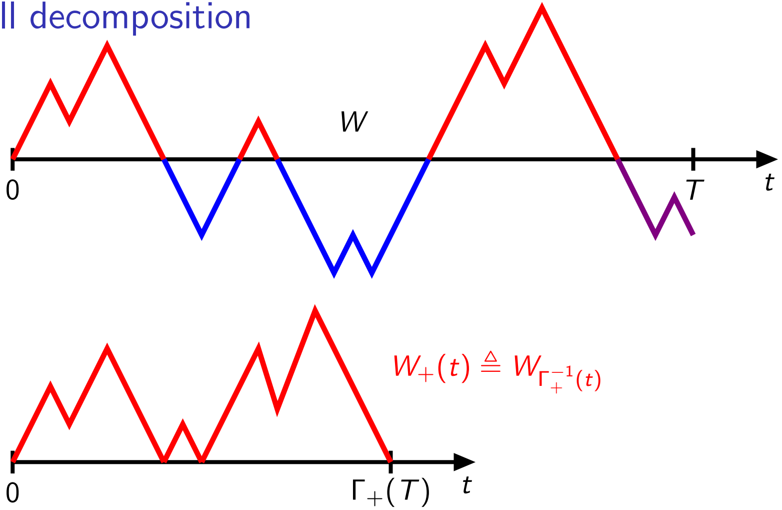

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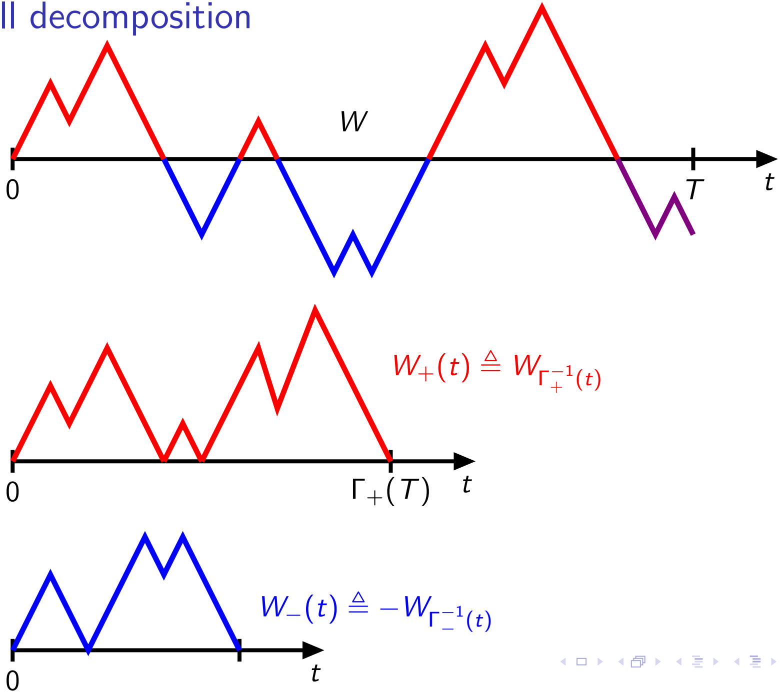

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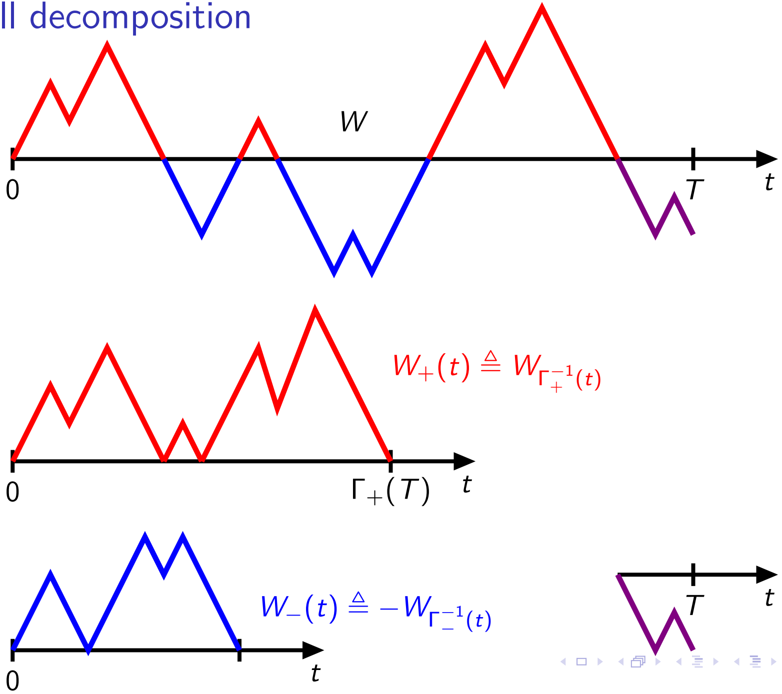

Local time

Local time at zero of W:

$$L_t = \frac{1}{2}\int_0^t \delta_0(W_s)\,ds \qquad = \lim_{\epsilon\downarrow 0}\frac{1}{4\epsilon}\int_0^t I_{(-\epsilon,\epsilon)}(W_s)\,ds$$

$$= \lim_{\epsilon\downarrow 0}\frac{1}{2\epsilon}\int_0^t I_{(0,\epsilon)}(W_s)\,ds = \lim_{\epsilon\downarrow 0}\frac{1}{2\epsilon}\int_0^t I_{(-\epsilon,0)}(W_s)\,ds.$$

Tanaka's formula

Tanaka formula:

$$\max\{W_t, 0\} = \int_0^t I_{(0,\infty)}(W_s)\, dW_s + \frac{1}{2}\int_0^t \delta_0(W_s)\, ds$$

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Time-changed Tanaka formula:

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where $L_+(t) = L_{\Gamma_+^{-1}(t)}$.

Conclusion: W_+ is a reflected Brownian motion. It is the Brownian motion $-B_+$ plus the nondecreasing process L_+ that grows only when W_+ is at zero.

Skorohod representation

Time-changed Tanaka formula:

$$W_+(t) = -B_+(t) + L_+(t).$$

Skorohod representation:
The nondecreasing process added to $-B_+$ that grows only when $W_+ = 0$ is

$$L_+(t) = \max_{0 \le s \le t} B_+(s).$$

In particular,

$$B_+(t) = -W_+(t) + L_+(t) = -W_+(t) + \max_{0 \le s \le t} B_+(s).$$

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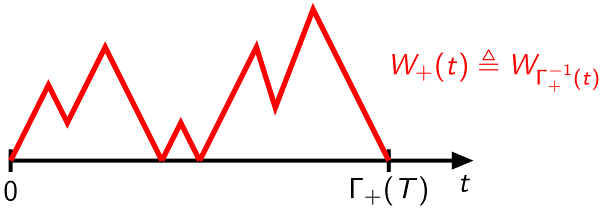

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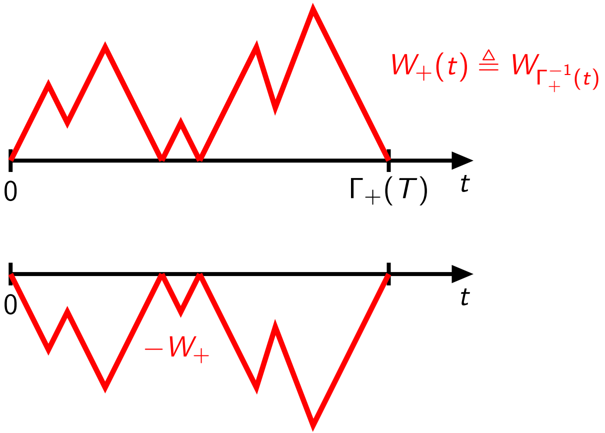

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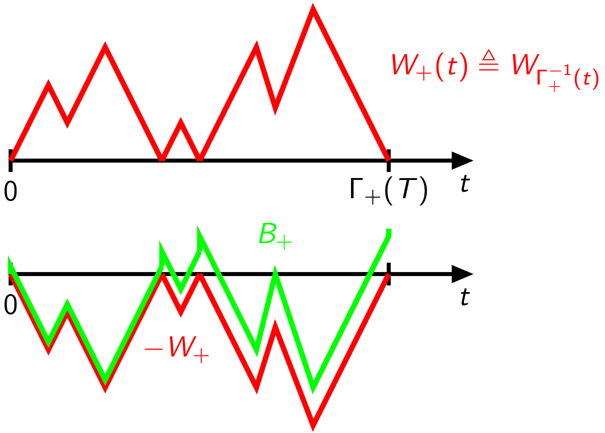

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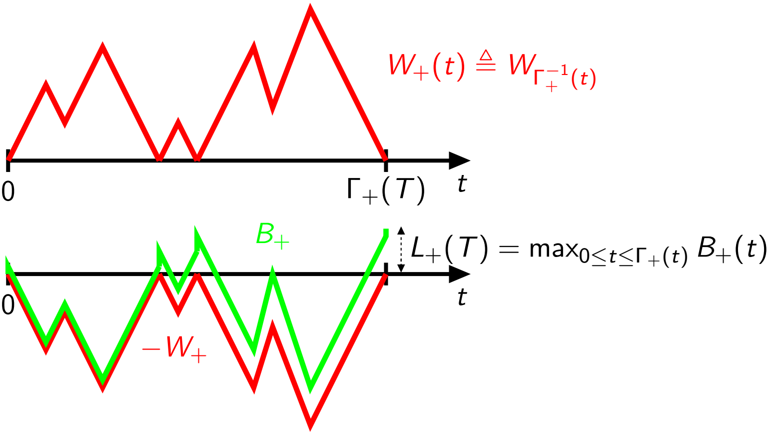

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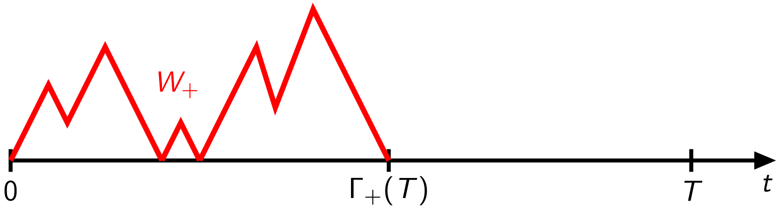

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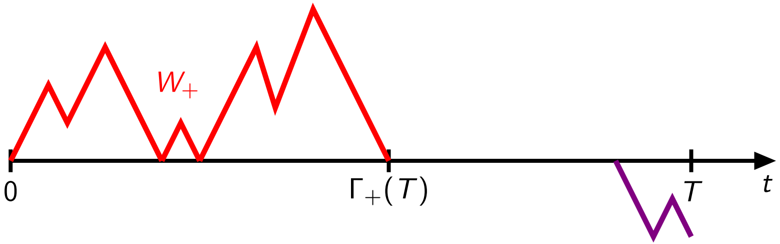

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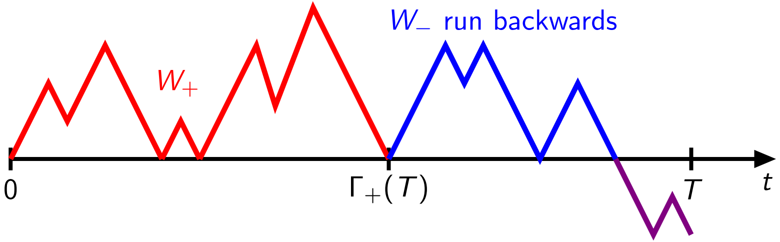

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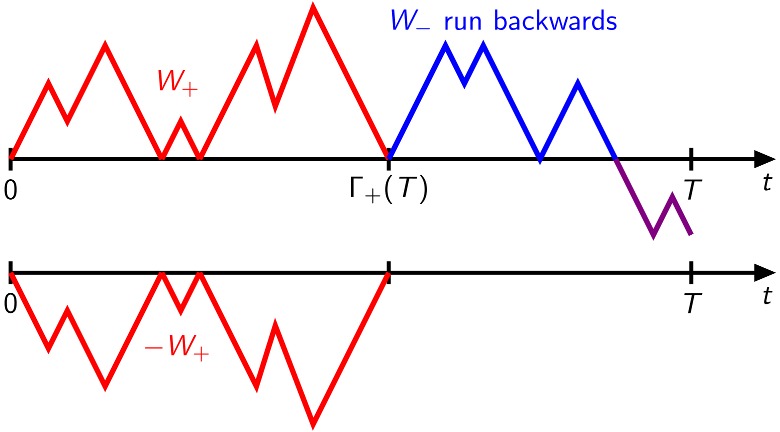

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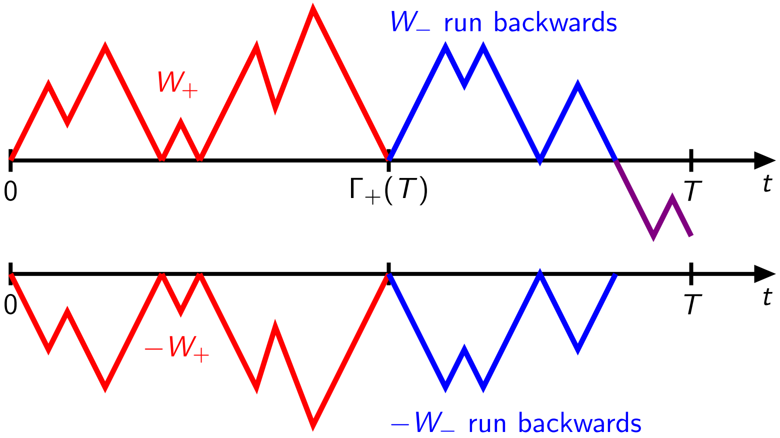

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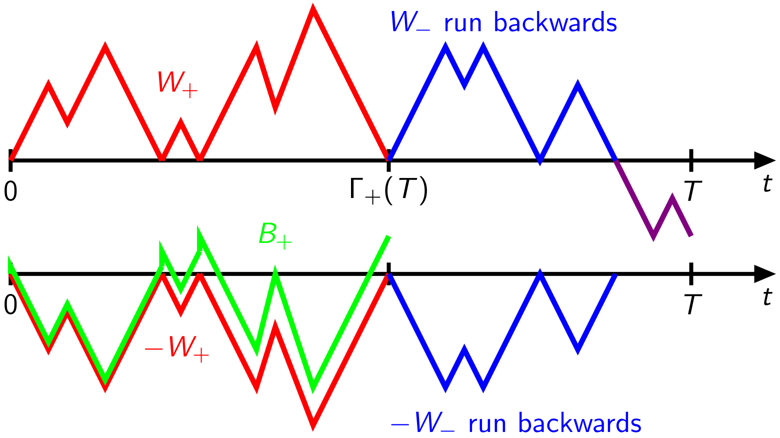

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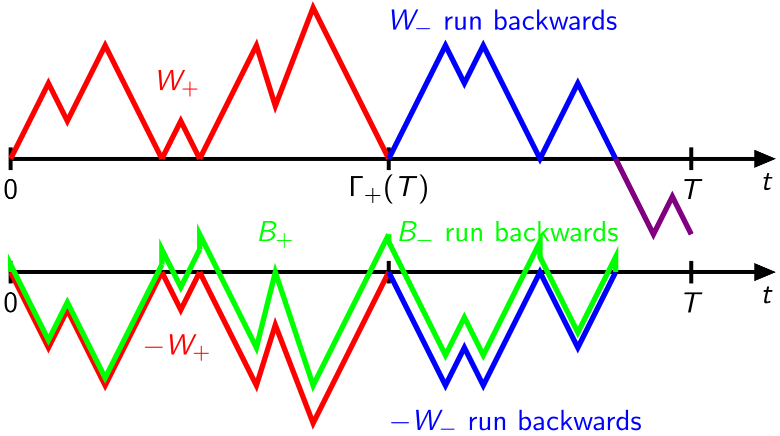

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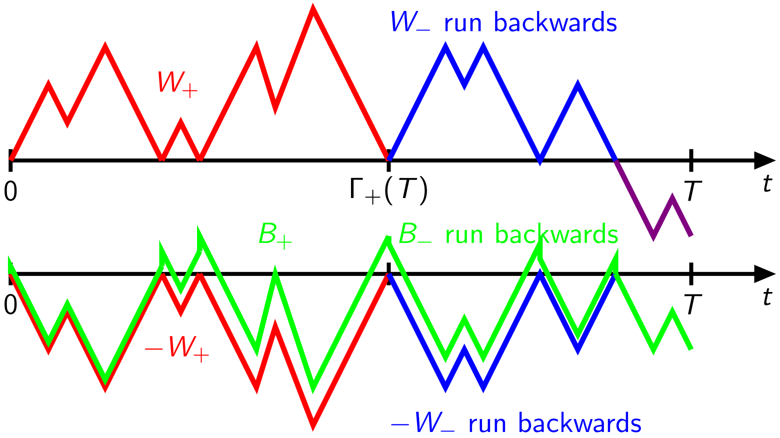

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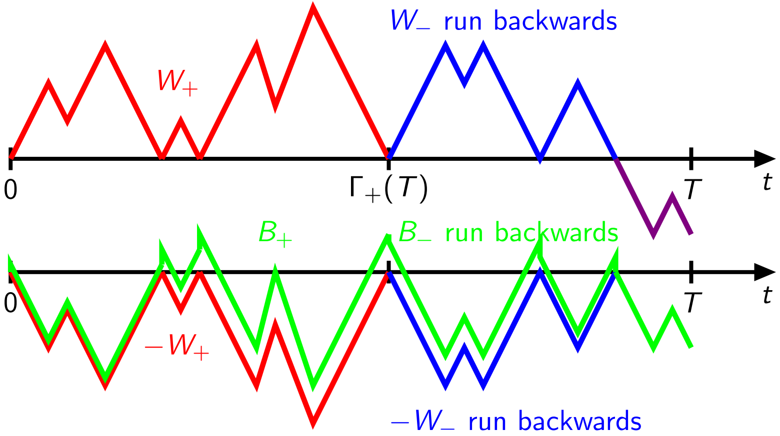

Local time $L_+(T) = L_T$ time has become the maximum.
$\Gamma_+(T)$ has become the time of the maximum.

Personal reflections

In the beginning....

S. Shreve, Reflected Brownian motion in the “bang-bang” control of Browian drift, *SIAM J. Control Optimization* **19**, 469–478, (1981).

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- Work on stochastic control (monotone follower, bounded variation follower, finite-fuel, ...)
- Work on optimal investment, consumption and duality with John Lehoczky, Suresh Sethi, Gan-Lin Xu, Jaksa Cvitanič ....

....and then THE BOOK,

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Columbia University in the City of New York | *New York, N.Y. 10027*

DEPARTMENT OF STATISTICS

618 Mathematics
(212) 280-3652

25 July 1985

Dear Steve:

I just received in the mail the Dirichlet section for Chapter 4, as well as your introduction and changes (for Doob). I plan to read there things as soon as I can, and get back to you with my comments. It looks like you have put a lot of work into the Dirichlet thing!

This morning I went downtown and delivered to Kaufmann-Bühler his copy of Chapters 1-3 and a copy of the contract. He was "all smiles" - he seems to be very pleased with us so far, and he liked our deadline of Sept. 1, 1986. His only advice was a warning against "too much perfectionism". Do we suffer from that? Perhaps, I do not know. How can one be too careful with a project like this?

You won't hear from me for some time, because I am still on my extended vacation. Please let me know, though, how the Las Vegas meeting goes.

Greetings to Dot and to the kiddies.

Yours,

Yannis

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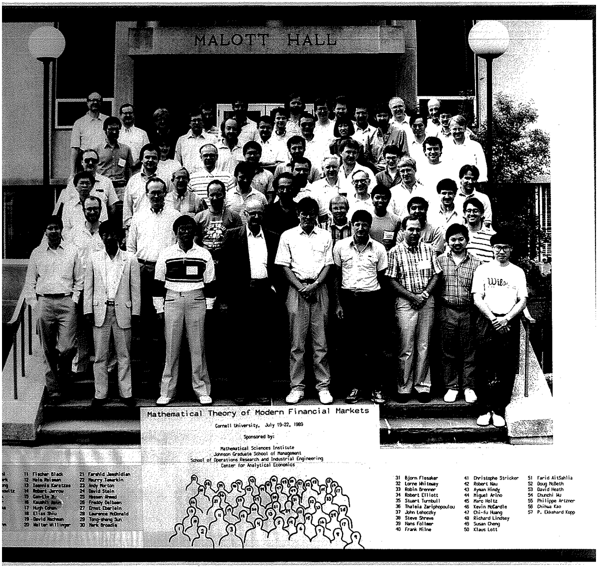

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Number one

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Thank you, Yannis,

for all you have done

- for your students,
- for your colleagues,
- for science,
- and for me personally.