36. \[ \int_{-5}^{5} (x - \sqrt{25 - x^2}) \, dx = \int_{-5}^{5} x \, dx - \int_{-5}^{5} \sqrt{25 - x^2} \, dx = -\frac{25\pi}{2} \]

First we divide \([0,1]\) into \(n\) subintervals of equal width \(\Delta x = \frac{1}{n}\).

\[
0 \quad \frac{1}{n} \quad \frac{2}{n} \quad \frac{3}{n} \quad \cdots \quad \frac{n-1}{n} \quad 1 = \frac{n}{n}
\]

In any interval \([\frac{i-1}{n}, \frac{i}{n}]\) pick a point \(x_i^*\). Then, if rational

\[
\sum_{i=1}^{n} f(x_i^*) \Delta x = \sum_{i=1}^{n} 0 \cdot \frac{1}{n} = 0
\]

because all \(x_i^*\)'s are rational.

Therefore \(\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x = 0\) when we pick \(x_i^*\)'s to be rational.
If in all subintervals \([\frac{i-1}{n}, \frac{i}{n}]\), we pick an irrational point \(x_{i}^{*}\), then

\[
\sum_{i=1}^{n} f(x_{i}^{*}) \Delta x = \sum_{i=1}^{n} 1 \cdot \frac{1}{n} = 1
\]

Therefore, when all \(x_i^{*}\)s are irrational \(\lim_{n \to \infty} \sum_{i=1}^{n} f(x_{i}^{*}) \Delta x = 1 \).

The limit depends on the possible choices of sample points. Therefore, \(f\) is not integrable.

\[ \text{Divide } [0,1] \text{ into } n \text{ subinterval with equal width. Pick } x_{i}^{*} = \frac{1}{n^2} \text{ and } x_{i}^{*} \]

\[
0 \quad \frac{1}{n} \quad \frac{2}{n} \quad \frac{3}{n} \quad \ldots \quad \frac{n-1}{n} \quad 1 \frac{1}{n}
\]

an arbitrary pt inside \([\frac{i-1}{n}, \frac{i}{n}]\) for any \(2 \leq i \leq n\). Then

\[
\sum_{i=1}^{n} f(x_{i}^{*}) \Delta x = \sum_{i=1}^{n} \frac{1}{x_{i}^{*}} \cdot \frac{1}{n} = \frac{1}{n} \cdot \frac{1}{n^2} + \sum_{i=2}^{n} \frac{1}{x_{i}^{*}} \cdot \frac{1}{n}
\]

because \(x_{i}^{*} = \frac{1}{n^2}\)

\[
= n + \sum_{i=2}^{n} \frac{1}{x_{i}^{*}} \cdot \frac{1}{n} \geq n
\]

\(\Rightarrow \sum_{i=1}^{n} f(x_{i}^{*}) \Delta x \geq n\).
Section 5.3

\[ g(x) = \int_{1-2x}^{1+2x} t \sin t \, dt = \int_{1-2x}^1 t \sin t \, dt + \int_1^{1+2x} t \sin t \, dt \]

\[ = \int_{1-2x}^{1+2x} t \sin t \, dt - \int_1^{1-2x} t \sin t \, dt \]

Let \( h(x) = \int_1^x t \sin t \, dt \). Therefore \( h'(x) = x \sin x \). Now,

\[ \int_{1-2x}^{1+2x} t \sin t \, dt = h((1+2x)) \implies \frac{d}{dx} \left( \int_{1}^{2(1+2x)} t \sin t \, dt \right) = \frac{d}{dx} (h(1+2x)) \]

\[ = h'(1+2x) \cdot \frac{2}{dx} (1+2x) = 2(1+2x) \sin (1+2x) \]

Similarly,
\[ \int_1^{1-2x} t \sin t \, dt = -2(1-2x) \sin (1-2x) \]

\[ \Rightarrow g'(x) = 2(1+2x) \sin (1+2x) + 2(1-2x) \sin (1-2x) \]

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\[ x = 2 \implies F(2) = \int_2^2 e^{t^2} \, dt = 0 \implies \text{line passes through the pt } (2,0). \]

Line is tangent to the curve \( y = F(x) \). Therefore, its slope is equal to \( F'(2) \).

\[ F'(x) = e^{x^2} \implies F'(2) = e^4 \implies \text{equ. of the line: } y = e^4(x-2) = e^4x - 2e^4 \]

Fundamental Thm of Calc.
\[ 6 + \int_{a}^{x} \frac{f(t)}{t^2} \, dt = 2\sqrt{x} \quad \text{for all } x > 0 \]

*Take the derivative of both sides:

\[ \frac{f(x)}{x^2} = \frac{1}{\sqrt{x}} \quad \rightarrow \quad f(x) = \frac{x^2}{\sqrt{x}} = x\sqrt{x} \]

Substitute \( x = a \) in * \( \rightarrow \) \( 6 = 2\sqrt{a} \) \( \rightarrow \) \( \sqrt{a} = 3 \) \( \rightarrow \) \( a = 9 \)

Section 5.4

(49) \[ \int_{0}^{2} (2y - y^2) \, dy = 2 \left[ y^2 - \frac{1}{3} y^3 \right]_{0}^{2} = 2 - \frac{1}{3} \cdot 8 = 4 - \frac{8}{3} = \frac{4}{3} \]

Section 5.5

(31) \[ \int \frac{(\arctan x)^2}{x^2 + 1} \, dx = \int u^2 \, du = \frac{u^3}{3} + C = \frac{(\arctan x)^3}{3} + C \]

\( u = \arctan x \quad \rightarrow \quad du = \frac{1}{x^2 + 1} \, dx \)

(43) \[ \int \frac{dx}{\sqrt{1 - x^2}} \, \sin^{-1} x \, dx = \int \frac{du}{u} = \ln |u| + C = \ln |\sin^{-1} x| + C \]

\( u = \sin^{-1} x \quad \rightarrow \quad du = \frac{1}{\sqrt{1 - x^2}} \, dx \)

(68) \[ \int_{0}^{4} \frac{x}{\sqrt{1 + 2x}} \, dx = \int_{u(0) = 1}^{u(4) = 9} \frac{u^4}{4 + \sqrt{u}} \, du = 2 \left[ \frac{\sqrt{u - 1/2}}{4} \right]_{1}^{9} = \left( \frac{27}{6} - \frac{1}{6} \right) - \left( \frac{1}{6} - \frac{1}{2} \right) = \frac{10}{3} \]
\( f(-x) = \sin^{-3/2}x = -\sin^{-3/2}x = -f(x) \Rightarrow f \text{ is odd} \)

\[
\int_{-2}^{3} \sin^{3/2}x \, dx = \int_{-2}^{1} \sin^{3/2}x \, dx + \int_{1}^{3} \sin^{3/2}x \, dx
\]

For any \( 2 \leq x \leq 3 \) we have \( 0 \leq 3/2 \leq x \leq \pi \approx 3.14 \)

\[
0 \leq \sin^{3/2}x \leq 1
\]

\[
0.3 \leq \int_{2}^{3} \sin^{3/2}x \, dx \leq 1 (3 - 2)
\]

\[
\int_{0}^{1} x^a (1-x)^b \, dx = -\int_{0}^{1} (1-u)^a u^b \, du = \int_{0}^{1} (u-x)^a x^b \, dx
\]

These two integrals are equal

\[
2 \int_{0}^{\pi} x f(\sin x) \, dx = \pi \int_{0}^{\pi} f(\sin x) \, dx = \frac{\pi}{2} \int_{0}^{\pi} f(\sin x) \, dx
\]
\( \int_{0}^{\pi} \frac{x \sin x}{1 + \cos^2 \frac{x}{2}} \, dx = \int_{0}^{\pi} \frac{\sin x}{2 - \sin^2 x} \, dx = \int_{0}^{\pi} \frac{\sin x}{2 - \sin^2 x} \, dx \)

Substitute \( u = \sin x \Rightarrow du = \cos x \, dx \)

\[ = \frac{\pi^2}{4} \]

94) \[ \int_{0}^{\pi/2} f(\cos x) \, dx = \int_{0}^{\pi/2} f(\cos(\pi/2 - u)) \, du = \int_{0}^{\pi/2} f(\sin u) \, du \]

Substitute \( u = \pi/2 - x \)

\[ \Rightarrow du = -dx \]

Note that \( \cos(\pi/2 - u) = \sin u \)

\[ \int_{0}^{\pi/2} f(\cos(\pi/2 - u)) \, du = \int_{0}^{\pi/2} f(\sin u) \, du = \int_{0}^{\pi/2} f(\sin x) \, dx \]

9b) \[ \int_{0}^{\pi/2} \cos^2 x \, dx = \int_{0}^{\pi/2} \sin^2 x \, dx \]

Because of Part @

Moreover, \( \sin^2 x + \cos^2 x = 1 \)

\[ \int_{0}^{\pi/2} (\sin^2 x + \cos^2 x) \, dx = \int_{0}^{\pi/2} 1 \, dx = \frac{\pi}{2} \]

\[ \int_{0}^{\pi/2} \sin^2 x \, dx + \int_{0}^{\pi/2} \cos^2 x \, dx = \frac{\pi}{2} \Rightarrow \int_{0}^{\pi/2} \sin^2 x \, dx = \left\lfloor \frac{\pi}{4} \right\rfloor \]
Problem Plus (Chapter 5)

3) \[ \int_0^4 x e^{(x-2)^4} \, dx = \int_0^4 (x-2)^4 e^{(x-2)^4} \, dx + \int_2^4 2e^{(x-2)^4} \, dx \]

\[ = \int_0^4 (x-2)^4 e^{(x-2)^4} \, dx + 2k \]

\[ \int_0^4 (x-2)^4 e^{(x-2)^4} \, dx = \int_{-2}^2 u e^{u^4} \, du = 0 \quad \text{because } u e^{u^4} \text{ is an odd function.} \]

\[ \Rightarrow \int_0^4 x e^{(x-2)^4} \, dx = 0 + 2k = 2k \]

9) \[ 2 + x - x^2 = 0 \implies x = 1, x = 2 \quad \text{moderate, } 2 + x - x^2 > 0 \text{ for any } -1 < x < 2 \]

\[ 2 + x - x^2 < 0 \text{ for any } x < 1 \text{ and } x > 2 \]

Therefore, \( \int_0^b 2 + x - x^2 \) is maximum if \( a = 1, b = 2 \).

10) \( f(x) = \lfloor x \rfloor \) is the largest integer not greater than \( x \). Its graph is as follows.

\[ \int_0^a [x] \, dx = \int_0^1 [x] \, dx + \int_1^2 [x] \, dx + \int_2^3 [x] \, dx + \cdots + \int_a^{a-1} [x] \, dx \]

\[ + \int_{a-1}^a [x] \, dx = 1 + 2 + 3 + \cdots + (a-1) \]

\[ = \frac{(a-1)a}{2} \]
\[ \int_a^b \lfloor x \rfloor \, dx = \int_a^{[a]+1} \frac{\lfloor x \rfloor \, dx}{[a]} + \int_{[a]+1}^{[a]+2} \frac{\lfloor x \rfloor \, dx}{[a]+1} + \cdots + \int_{[b]-1}^b \frac{\lfloor x \rfloor \, dx}{[b]-1} + \int_b^b \frac{\lfloor x \rfloor \, dx}{[b]} \]

\[ = \lfloor a \rfloor \left( ([a]+1-a) + ([a]+1) + ([a]+2) + \cdots + ([b]-1) + [b] \left( b - [b] \right) \right) \]

\[ = \lfloor a \rfloor^2 + \lfloor a \rfloor - a \lfloor a \rfloor + \left( \lfloor a \rfloor + [b] \right) \left( [b] - [a] - 1 \right) + b[b] - [b]^2 \]

\[ = \frac{1}{2} \lfloor a \rfloor^2 + \lfloor a \rfloor - a \lfloor a \rfloor - \frac{1}{2} [b]^2 - \frac{1}{2} [b] + b[b] \]
The area between the curves \( y = 2x^2 \), \( y = \frac{x^2}{4} \) and \( x+y=3 \) is equal to the sum of:

the area between \( y = \frac{x^2}{4} \) and \( y = 2x^2 \) from 0 to 1 (area of the domain A),

and the area between \( y = \frac{x^2}{4} \) and \( x+y=3 \) from 1 to 2 (area of the domain B).

Therefore,

\[
\int_0^1 \left( 2x^2 - \frac{x^2}{4} \right) \, dx + \int_1^2 \left( (3-x) - \frac{x^2}{4} \right) \, dx = \left( \frac{2}{3} x^3 - \frac{x^3}{12} \right) \bigg|_0^1 + \left( 3x - \frac{x^2}{2} - \frac{x^3}{12} \right) \bigg|_1^2
\]

\[= \left( \frac{2}{3} - \frac{1}{12} \right) + \left( 6 - 2 - \frac{5}{12} - 3 + \frac{1}{2} + \frac{1}{12} \right) = \frac{18}{12} = \frac{3}{2} \]

46. \( x - 2y^2 \geq 0 \) sketches the graph of \( x = 2y^2 \). Since \( x - 2y^2 \) is negative at point \((0,1)\), then the area region of \( x - 2y^2 \geq 0 \) is on the right-hand side of the graph \( x = 2y^2 \).
\[ |x| - |y| = 0 \implies x = |y| \]

At \((0,0)\) we have \(|x| - |y| = 0\), therefore, the region \(|x| - |y| > 0\) is the left-hand side of the graph \(|x| - |y| = 0\).

Intersection pts of the graphs:

\[ x = 2y^2 = |y| \]
\[ \implies 2y^2 + |y| - 1 = 0 \]
\[ \implies (2|y|-1)(|y|+1) = 0 \]
\[ \implies |y| = \frac{1}{2} \]
\[ \implies y = \pm \frac{1}{2} \pm \frac{1}{2} \]
\[ \implies (\frac{1}{2}, \frac{1}{2}) , (\frac{1}{2}, -\frac{1}{2}) \]

Area:

\[
\begin{align*}
&= \int_{0}^{\frac{1}{2}} \left[ \frac{\sqrt{\frac{3}{2}}}{\sqrt{\frac{3}{2}}} - \left( -\sqrt{\frac{3}{2}} \right) \right] \, dx + \int_{\frac{1}{2}}^{1} \left[ \frac{1}{2} - (x-1) \right] \, dx \\
&= \frac{\sqrt{2}}{3} \left[ \frac{1}{2} + \left( \frac{3}{2} - x^2 \right) \right] \bigg|_{0}^{\frac{1}{2}}
&= \frac{1}{3} + \left( 1 - \frac{3}{4} \right) = \frac{7}{12}
\end{align*}
\]

\[
\begin{align*}
&= \frac{\pi}{4} \left[ \sqrt{3}^2 - (3 - \sqrt{x})^2 \right] \, dx \\
&= \int_{1}^{4} \left[ \sqrt{3}^2 - (3 - \sqrt{x})^2 \right] \, dx
\end{align*}
\]

You can describe the solid and its volume in different ways:

For example, the solid constructed by revolving the region between

\[ y = 0, \ y = -\sqrt{x}, \ x = 1, \ x = 4 \]
about the \( y = -3 \) line.

Or, the solid constructed by revolving the region between

\[ y = 3, \ y = 3 - \sqrt{x}, \ x = 1, \ x = 4 \]
about \( y = 0 \).
at any \(-r \leq x \leq r\), the cross section is a square.

\[ \text{area} = (2 \cdot \sqrt{r^2 - x^2})^2 \]

\[ = 4 \cdot (r^2 - x^2) \]

\[ \int_{-r}^{r} A(x) \, dx = \int_{-r}^{r} 4(r^2 - x^2) \, dx = 4\pi r^3 - 4\frac{\pi}{3} r^3 \]

\[ = (4\pi r^3 - 4\frac{\pi}{3} r^3) - (4\pi r^3 + 4\frac{\pi}{3} r^3) = \frac{16}{3} \pi r^3 \]

radius of the quarter-circle at \(y\) is equal to: \(2 \cdot \sqrt{2-y}\)

\[ \text{area} = \frac{1}{4} \pi (2 \cdot \sqrt{2-y})^2 \]

\[ = \frac{\pi}{4} \cdot 4(2-y) = \pi (2-y) \]

\[ \int_{0}^{2} \pi (2-y) \, dy = 2\pi y - \pi \frac{y^2}{2} \bigg|_{0}^{2} \]

\[ = 4\pi - 2\pi = 2\pi \]

The torus is obtained by revolving a disk of radius \(r\) with center at \((R, 0)\) around the \(y\)-axis. at each point \(y\), the slice has the shape of a washer as below.

\[ (x-R)^2 + y^2 = r^2 \]

\[ x = R \pm \sqrt{r^2 - y^2} \]

\[ R_{\text{out}} = R + \sqrt{r^2 - y^2} \]

\[ R_{\text{in}} = R - \sqrt{r^2 - y^2} \]

\[ A(y) = \pi (R + \sqrt{r^2 - y^2})^2 - \pi (R - \sqrt{r^2 - y^2})^2 = 4\pi R \sqrt{r^2 - y^2} \]}
Volume = \int_{-r}^{r} 4\pi R \sqrt{r^2 - y^2} \, dy

\begin{align*}
\int_{-r}^{r} \sqrt{r^2 - x^2} \, dx &= \text{area (A)} = \frac{\pi r^2}{2} \\
\int_{-r}^{r} 4\pi R \sqrt{r^2 - y^2} \, dy &= 4\pi R \int_{-r}^{r} \sqrt{r^2 - y^2} \, dy = 2\pi^2 r^2 R
\end{align*}

Put the axe of the first cylinder over the x-axis such that its mid pt. is at origin. The pts (x, y, z) inside this cylinder satisfy the following relations:

\(-r \leq x \leq r\) and \(y^2 + z^2 \leq r^2\)

The pts inside a disk with radius \(r\) with center at origin in \(yz\)-plane.

Similarly, we can put the axe of the second cylinder over the y-axis such that its mid pt is at origin. The coordinates of the pts inside this cylinder satisfy the following relations:

\(-r \leq y \leq r\) and \(x^2 + z^2 \leq r^2\)

Therefore, their intersection consists of the pts \((x, y, z)\) whose coordinates satisfy the following condition:

\[-r \leq x \leq r \quad y^2 + z^2 \leq r^2 \]

\[-r \leq y \leq r \quad x^2 + z^2 \leq r^2\]

If we slice this solid, with planes perpendicular to z-axis the intersection consists of the pts \((x, y, z)\) where

\[y^2, x^2 \leq r^2 - z^2 \quad \text{and} \quad -\sqrt{r^2 - z^2} \leq xy \leq \sqrt{r^2 - z^2}\]
Therefore, it's a square, with edge length \(2\sqrt{r^2-z^2}\).

\[ A(z) = \frac{1}{2} \left( 2\sqrt{r^2-z^2} \right)^2 = 4(r^2-z^2) \]

\[ \text{Vol} = \int_{-r}^{r} A(z) \, dz = \int_{-r}^{r} \left( 4r^2-4z^2 \right) \, dz = 4r^2z - \frac{4z^3}{3} \bigg|_{-r}^{r} = \frac{16r^3}{3} \]

---

planes perpendicular to z axis cut the
intersection region in squares!
20) \[ x = y^2 + 1 \text{ vertex at } (1, 0), \text{ passing through } (2, 1), (2, -1) \]
\[ x = 2y^2 \text{ ~a parabola with vertex at } (0, 0) \text{ passing through } (2, 1), (2, -1) \]
\[ h = y^2 + 1 - 2y^2 = -y^2 + 1 \]
\[ r = \frac{y + 2}{y - (-2)} \]

\[ \text{Vol} = \int_{-1}^{1} 2\pi \left( \frac{y + 2}{-y^2 + y + 2} \right) dy = \int_{-1}^{1} 2\pi \left( -y^3 - 2y^2 + y + 2 \right) dy \]
\[ = 2\pi \left( -\frac{y^4}{4} - \frac{2y^3 + 1}{2} y^2 + 2y \right) \bigg|_{-1}^{1} = 2\pi \left( -\frac{1}{4} - \frac{2}{3} + \frac{1}{2} + 2 - \left(-\frac{1}{4} + \frac{2}{3} + 1 - 2\right) \right) \]
\[ = 2\pi \left( \frac{8}{3} \right) = \frac{16}{3}\pi \]

(31) \[ 2\pi \int_{1}^{4} \frac{y+2}{y^2} dy \]

Compute the volume of the solid obtained by revolving the region between \(x=f(y)\) and \(y\)-axis from \(a\) to \(b\) around \(y=c\).

Using cylindrical shells method we get:
\[ \text{Vol} = \int_{a}^{b} 2\pi (y-c) f(y) dy \]

Therefore, \(2\pi \int_{1}^{4} \frac{y+2}{y^2} dy\) represents the volume of the solid obtained by revolving the region between \(x=\frac{1}{y^2}\), \(y\)-axis, \(y=1\) and \(y=4\) about the line \(y=-2\).
(42) \( x = (y-3)^2, \ x = 4 \) around \( y = 1 \)

\[
\begin{align*}
(4,5) \quad (4,1) \quad \text{h} = \sqrt{4 - (y-3)^2} \\
\text{r} = y-1
\end{align*}
\]

Using cylindrical shell method:

\[
\int_{1}^{5} 2\pi (y-1) \left(4 - (y-3)^2\right) dy = \int_{1}^{5} 2\pi \left(-y^3 + 6y^2 - 5y - y^2 - 6y + 5\right) dy
\]

\[
= 2\pi \left(-\frac{y^4}{4} + \frac{7y^3}{3} - \frac{11y^2}{2} + 5y\right)
\]

\[
= 2\pi \left[-\frac{5^4}{4} + \frac{7\cdot5^3}{3} - \frac{11\cdot5^2}{2} + 5\cdot5\right]
\]

\[
= 2\pi \left[-\frac{625}{4} + \frac{875}{3} - \frac{275}{2} + 25 - \frac{19}{12}\right]
\]

\[
= 2\pi \left[-\frac{1875}{12} + \frac{3500}{12} - \frac{3275}{12} + \frac{300}{12} - \frac{19}{12}\right]
\]

\[
= 2\pi \cdot \frac{256}{12} = \frac{128\pi}{3}
\]

(43) \( x = (y-1)^2, \ x = y - 1 \) about \( x = -1 \)

Intersection pts:

\( (y-1)^2 = y - 1 \)

\( y^2 - 2y + 1 = y - 1 \)

\( y^2 - 3y = 0 \)

\( y = 0 \) or \( y = 3 \)

\( (0,1,0) \) and \( (4,3) \)
Slicing method: For any $0 < y < 3$, the cross sectional slice is a washer with outer radius $y$ and inner radius $(y+1)^2 - 1 = y^2$, and inner radius $(y-1)^2 - 1 = y^2 - 2y + 1$.

Therefore: $A(y) = \pi (y+2)^2 - \pi (y+1)^2 - \pi (y^2 - 2y + 1) = \pi \left[ y^2 + 4y + 4 - (y^2 - 4y^3 + 8y^2 - 8y + 4) \right] = \pi \left[ -y^4 + 4y^3 - 7y^2 + 12y \right]$

\[ \text{Vol.} = \int_0^3 \pi \left[ -y^4 + 4y^3 - 7y^2 + 12y \right] \, dy = \pi \left[ -\frac{y^5}{5} + y^4 - \frac{7y^3}{3} + 6y \right]_0^3 = \pi \left[ -\frac{243}{5} + 81 - \frac{63}{3} + 6 \right] = \pi \frac{117}{5} \]