\[ f(x) = x \cos x \]
\[ f'(x) = \cos x - x \sin x \]
\[ f''(x) = -\sin x - \sin x - x \cos x = -2 \sin x - x \cos x \]
\[ f'''(x) = -2 \cos x - C_3 \xi + x \sin x = -3 \cos x + x \cos x - x \sin x \]
\[ f^{(n)}(x) = 3 \sin x + C_n \xi + x \cos x = 4 \sin x + x \sin x \]

Given:
\[ f^{(n)}(x) = (-1)^n \frac{d^n}{dx^n} \sin x + (-1)^n x \cos x \]

Thus:
\[ f^{(2n)}(0) = 0 \]
\[ f^{(2n+1)}(0) = (-1)^n (2n+1) \]

\[ x \cos x = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \ldots \]

\[ = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \]

\[ = (2n+1)! = (2n)! (2n+1) \]
Solution 1

\[ f(x) = x^6 - x^4 + 2 \]
\[ f'(x) = 6x^5 - 4x^3 \]
\[ f''(x) = 30x^4 - 12x^2 \]
\[ f'''(x) = 120x^3 - 24x \]
\[ f^{(4)}(x) = 360x^2 - 24 \]
\[ f^{(5)}(x) = 720x \]
\[ f^{(6)}(x) = 720 \]

\[ f(-2) = 64 - 16 + 2 = 50 \]
\[ f'(-2) = -192 + 32 = -160 \]
\[ f''(-2) = 480 - 48 = 432 \]
\[ f'''(-2) = -960 + 48 = -912 \]
\[ f^{(4)}(-2) = 1440 - 24 = 1416 \]
\[ f^{(5)}(-2) = -1440 \]
\[ f^{(6)}(-2) = 720 \]

\[ \Rightarrow f(x) = 50 - 160(x+2) + \frac{432}{2}(x+2)^2 - \frac{912}{3!}(x+2)^3 + \frac{1416}{4!}(x+2)^4 - \frac{1440}{5!}(x+2)^5 + \frac{720}{6!}(x+2)^6 \]

\[ = 50 - 160(x+2) + 216(x+2)^2 - 152(x+2)^3 + 59(x+2)^4 - 12(x+2)^5 \]

It's a polynomial and has finitely many terms. So \( R = \infty \), i.e., it's convergent for any real number \( x \).

Solution 2

Let \( t = x + 2 \) \( \Rightarrow x = t - 2 \)

\[ f(t-2)^6 - (t-2)^4 + 2 = \left( t^6 - 12t^5 + 60t^4 - 160t^3 + 240t^2 - 192t + 64 \right) \]

find Taylor series at \( x = -2 \) by substitution from Macaulin series

\[ f(t-2)^6 - (t-2)^4 + 2 = \left( t^6 - 12t^5 + 59t^4 - 152t^3 + 216t^2 - 160t + 50 \right) \]

\((*)\) is a polynomial, therefore its Macaulin series is equal to itself i.e.

\[ \text{Macaulin Series} = 50 - 160t + 59t^2 - 152t^3 + 216t^2 - 160t + 50 \]

\[ R = \infty \]

\[ \text{Taylor Series} \]

\[ +(x+2)^6 \]

\[ R = \infty \]
Solution 1

\[
\begin{align*}
    f(x) &= C \exp x \\
    f'(x) &= -8 \sin x \\
    f''(x) &= -C \cos x \\
    f'''(x) &= 8 \sin x \\
    f''''(x) &= -C \cos x
\end{align*}
\]

\[
\begin{align*}
    f(\frac{\pi}{2}) &= 0 \\
    f'(\frac{\pi}{2}) &= -1 \\
    f''(\frac{\pi}{2}) &= 0 \\
    f'''(x) &= 1
\end{align*}
\]

\[
\sum_{n=0}^{\infty} \left( \frac{x - \frac{\pi}{2}}{3} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left( \frac{x - \frac{\pi}{2}}{3} \right)^{2n+1}
\]

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n+3}}{(2n+3)!} \cdot \frac{(2n+3)!}{(2n+1)!} \right| = 0 < 1 \quad \forall \quad R = \infty
\]

Solution 2

\[
\begin{align*}
    x &= \frac{\pi}{2} \\
    \sin x &= -8 \sin t = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left( \frac{x - \frac{\pi}{2}}{3} \right)^{2n+1}
\end{align*}
\]

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left( \frac{x - \frac{\pi}{2}}{3} \right)^{2n+1} \quad R = \infty
\]

\[
\begin{align*}
    f(x) &= 3 \sqrt{8 + x} = (8 + x)^{1/3} = 2 \left( 1 + \frac{x}{8} \right)^{1/3} \\
    2 \left( 1 + \frac{x}{8} \right)^{1/3} &= 2 \sum_{n=0}^{\infty} \binom{1/3}{n} \left( \frac{x}{8} \right)^n = \sum_{n=0}^{\infty} \binom{1/3}{n} \frac{2}{8^n} x^n
\end{align*}
\]

\[
\begin{align*}
    1 + \frac{x}{8} &< 1 \\
    |x| &< 8 \quad \forall \quad R = 8
\end{align*}
\]
\[ f(x) = \sin \left( \frac{\pi x}{4} \right) = \sum_{n=0}^{\infty} (-1)^n \frac{\left( \frac{\pi x}{4} \right)^{2n+1}}{(2n+1)!} \]

[Note that: \( \sqrt{1+x} = (1+x)^{1/2} = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right) x^n = 1 + \frac{1}{2} x - \frac{1}{8} x^2 + \frac{1}{16} x^3 + \cdots \)]

\[ \frac{1+x}{x^2} = \lim_{x \to 0} \left( 1 + \frac{1}{2} x - \frac{1}{8} x^2 + \frac{1}{16} x^3 + \cdots \right) - \frac{1}{2} x = \lim_{x \to 0} \frac{-\frac{1}{8} x^2 + \frac{1}{16} x^3 + \cdots}{x^2} = -\frac{1}{8} \]

\[ \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1} (2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \left( \frac{\pi}{4} \right)^{2n+1} = 8 \sin \left( \frac{\pi}{4} \right) = \frac{\sqrt{2}}{2} \]

\[ \left( \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right) \]

\[ P \text{ is a } n \text{th degree polynomial, thus } P^{(i)} = 0 \text{ for any } i \geq n+1 \]

Therefore Taylor series of \( P \) at any \( a = x \) is a degree \( n \) polynomial.

\[ P(y) = \sum_{i=0}^{n} \frac{P^{(i)}(x)}{i!} (y-x)^i \iff P(x) = \sum_{i=0}^{n} \frac{P^{(i)}(x)}{i!} (y-x)^i \]

Taylor series at \( a = x \)
\[ f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases} \]

We show that for any \( n \), \( \int_{0}^{\infty} t^n f(t) \, dt = 0 \).

**First**

\[
\lim_{x \to 0} e^{-\frac{1}{x^2}} = \lim_{x \to 0} \frac{1}{e^{\frac{1}{x^2}}} = \lim_{x \to 0} \frac{1}{e^{t^2}} = 0 \quad \text{if } t = \frac{1}{x},
\]

\[
f'(0) = \lim_{x \to 0} e^{-\frac{1}{x^2}} = \lim_{x \to 0} \frac{1}{xe^{\frac{1}{x^2}}} = \lim_{x \to 0} \frac{1}{t} e^{\frac{1}{t^2}} = \lim_{t \to \infty} \frac{t}{2t e^{t^2}} = 0
\]

prove by induction on \( n \), \( \lim_{x \to 0} \frac{e^{-\frac{1}{x^2}}}{x^n} = 0 \).

Assume \( \lim_{x \to 0} \frac{e^{-\frac{1}{x^2}}}{x^n} = 0 \).

\[
\lim_{x \to 0} \frac{e^{-\frac{1}{x^2}}}{x^{n+1}} = \lim_{x \to 0} \frac{1}{t^{n+1}} e^{\frac{1}{t^2}} = \lim_{t \to \infty} \frac{t^{n+1}}{t^{n+1} 2te^{t^2}} = \lim_{t \to \infty} \frac{(n+1)t^{n-1}}{2e^{t^2}} = 0
\]

Second

By induction on \( n \) we prove \( \frac{d^n e^{-\frac{1}{x^2}}}{dx^n} = f_n(x) e^{-\frac{1}{x^2}} \)

where \( f_n \) is a polynomial \( x \neq 0 \).

\[
\frac{(n+1)!}{x^{n+1}} \frac{d e^{-\frac{1}{x^2}}}{dx} = \frac{d}{dx} \left( f_n(x) e^{-\frac{1}{x^2}} \right) = \frac{d}{dx} \left( f_n(x) \right) e^{-\frac{1}{x^2}} + f_n(x) \frac{d}{dx} \left( e^{-\frac{1}{x^2}} \right) = \left( -\frac{1}{x^2} f_n(x) + \frac{2}{x^3} f_n'(x) \right) e^{-\frac{1}{x^2}}
\]

\[f_n(x) = -x^2 f_n'(x) + 2x^3 f_n(x)\]
Therefore, \( f^{(n+1)}(0) = \lim_{x \to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x} \)

\[ = \lim_{x \to 0} \frac{P_n \left( \frac{1}{x} \right) e^{-\frac{1}{x^2}}}{x} \]

\[ = \lim_{x \to 0} \left( \frac{1}{x} P_n \left( \frac{1}{x} \right) \right) e^{-\frac{1}{x^2}} = 0 \]

since \( \lim_{x \to 0} \left( \frac{1}{x^2} e^{-\frac{1}{x^2}} \right) = 0 \)

By induction on \( n \) we have \( f^{(n)}(0) = 0 \)

Thus, MacLaurin Series of \( f(x) \) is equal to zero, but the function is non-zero.

The function \( f(x) \) is very flat at origin, all of its derivatives at origin is zero.

\[
g(x) = \sum_{n=0}^{\infty} \left( \begin{array}{c} k \end{array} \right) x^n \implies g'(x) = \sum_{n=0}^{\infty} \left( \begin{array}{c} k \end{array} \right) n x^{n-1}, \quad |x| < 1
\]

\[
g'(x) = \sum_{n=0}^{\infty} \frac{k(k-1) \cdots (k-n+1)}{n!} n x^{n-1}
\]

\[ = \sum_{n=1}^{\infty} \frac{k(k-1) \cdots (k-n+1)}{(n-1)!} x^{n-1} = \sum_{n=1}^{\infty} \frac{k}{n-1} \left( \begin{array}{c} k-1 \end{array} \right) x^{n-1}
\]

\[
(1+x)g'(x) = k \left[ \sum_{n=1}^{\infty} \left( \begin{array}{c} k-1 \end{array} \right) x^n + \sum_{n=1}^{\infty} \left( \begin{array}{c} k-1 \end{array} \right) x^{n-1} \right]
\]

\[ = k \left[ 1 + \sum_{n=1}^{\infty} \left( \begin{array}{c} k-1 \end{array} \right) + \left( \begin{array}{c} k-1 \end{array} \right) x^n \right]
\]
\[
\binom{k-1}{n-1} + \binom{k-1}{n} = \frac{(k-1)(k-2)\ldots(k-n+1)}{(n-1)!} + \frac{(k-1)(k-2)\ldots(k-n)}{n!}
\]
\[
= \frac{(k-1)\ldots(k-n+1)n + (k-1)(k-2)\ldots(k-n)}{n!}
\]
\[
= \frac{(k-1)\ldots(k-n+1)(n+k-n)}{n!} = \frac{k(k-1)\ldots(k-n+1)}{n!}\binom{k}{n}
\]

\[
\sum_{n=0}^{\infty} (1+x)^n g(x) = K \sum_{n=0}^{\infty} \binom{k}{n} x^n = K g(x)
\]

(b) \( h(x) = (1+x)^{-k} g(x) \implies h'(x) = -K (1+x)^{-k-1} g(x) + (1+x)^{-k} g'(x) \)

\[
= -\frac{K g(x)}{(1+x)^{k+1}} + \frac{g'(x)}{(1+x)^k} = \frac{g'(x)(1+x)-kg(x)}{(1+x)^{k+1}}
\]
\[
= 0
\]

(c) \( h'(x) = 0 \implies h(x) = C \quad \text{constant} \)

\[
\implies \frac{g(x)}{(1+x)^k} = C \implies g(x) = C (1+x)^k
\]
\[
g(0) = 1 \implies C = 1 \implies g(x) = (1+x)^k
\]