Lecture 11

Wednesday, February 27, 2019

Deek transformations

Def: For a covering space \( p: \tilde{X} \rightarrow X \) the covering space isomorphism \( \tilde{X} \rightarrow \tilde{X} \) are called deek transformations.

\[ \tilde{X} \xrightarrow{p} \tilde{X}, \quad i \cdot p = p \]

deck trans. + composition: group denoted by \( G(\tilde{X}) \).

Suppose \( \tilde{X} \) is path connected, every isom. of \( \tilde{X} \) is determined by where it sends one point.

Special case: the only isomorphism

\[ p: \tilde{X} \rightarrow \tilde{X}, \quad p(\tilde{x}_1) = p(\tilde{x}_2) \]

that preserves one point is identity.

\[ p(\tilde{x}_1) = p(\tilde{x}_2) \]

\[ x_1 \xrightarrow{y} y \xrightarrow{f(y)} \]

Ex 1: \( IR \xrightarrow{p} S^1 \)

\[ t \xrightarrow{\sin(2\pi t), \cos(2\pi t)} \]

\[ \Rightarrow G(IR) \cong \mathbb{Z} \]

Ex 2: \( IR^2 \rightarrow T^2 \)

\[ f_{(n,m)}(x,y) = (x+n, y+m) \]

isomorphism

\[ \Rightarrow G(IR^2) \cong \mathbb{Z} \]

Thm: Let \( X \) be a path connected and locally path connected topological space.
If \( \tilde{X} \xrightarrow{p} X \) is a path connected covering space, then \( G(\tilde{x}) \cong N(H) \)

where \( H = p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \) and \( N(H) \) is the normalizer of \( H \) in \( \pi_1(X, x_0) \).

**Proof**  For \( \tilde{x}_j \in p^{-1}(x_0) \), there exists an isomorphism \( (\tilde{x}, \tilde{x}_j) \mapsto (\tilde{x}, \tilde{x}_i) \)

iff \( p_*(\pi_1(\tilde{x}, \tilde{x}_j)) = p_*(\pi_1(\tilde{x}, \tilde{x}_i)) \)

\[
\begin{array}{c}
\tilde{x}_j
\end{array}
\]

\[
\begin{array}{c}
\pi_1(\tilde{x}, \tilde{x}_i) \xrightarrow{\tilde{Y}} \pi_1(\tilde{x}, \tilde{x}_0)
\end{array}
\]

\[
\begin{array}{c}
[\pi_1(\tilde{x}, \tilde{x}_i)] \xrightarrow{[\gamma]} \pi_1(\tilde{x}, \tilde{x}_0)
\end{array}
\]

\[
P_*(\pi_1(\tilde{x}, \tilde{x}_0)) = [\gamma] \cdot P_*(\pi_1(\tilde{x}, \tilde{x}_i)) \cdot [\gamma]^{-1}
\]

\[
= [\gamma] \in N(H)
\]

\[\varphi: N(H) \rightarrow G(\tilde{X})\]

isomorphism which maps \( \tilde{x}_0 \) to \( \tilde{x}_1 = \tilde{\gamma}(1) \)

\[\varphi([\gamma]) = \tilde{\gamma} \]

\[\varphi: \text{surjective} \]

\[\varphi: \text{homomorphism} \]

\[\varphi([\gamma], [\gamma']) = \tilde{\gamma}(1) = \tilde{\gamma} \circ \tilde{\gamma}' \]

\[\text{Let } \tau = \varphi([\gamma]), \tau' = \varphi([\gamma']) \Rightarrow \tilde{\gamma}' = \tilde{\gamma} \circ \tau(\tilde{\gamma}') \]

\[\varphi([\gamma]) \in \ker(\varphi) \Rightarrow \varphi([\gamma]) = 1 \Rightarrow \tilde{\gamma}(1) = \tilde{x}_0 \Leftrightarrow [\gamma] \in H \]

\[\Rightarrow \ker(\varphi) = H \]

**Cor**  If \( \tilde{X} \xrightarrow{p} X \) is the universal cover, then \( G(\tilde{X}) = \pi_1(X, x_0) \).
**Cor** If $H \triangleleft \Pi(X,x_0)$, then $G(\tilde{x}) = \frac{\Pi(X,x_0)}{H}$.

**Def** A covering space $p: \tilde{X} \rightarrow X$ is called **normal** if for every $x \in X$ and $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x)$ there exists an isomorphism $(\tilde{x}, \tilde{x}_1) \rightarrow (\tilde{x}, \tilde{x}_2)$.

**Prop** Assume $X$ is path connected and locally path connected. A path connected covering space $p: \tilde{X} \rightarrow X$ is **normal** iff $P_*(\Pi(X,\tilde{x}_0)) \triangleleft \Pi_*(X,x_0)$.

**Proof**

For $\tilde{x}_0 \in \tilde{X}$, $p^{-1}(x_0)$.

For $\tilde{y}, \tilde{y}' \in p^{-1}(y)$, choose a path $\tilde{\eta}$ from $\tilde{y}$ to $\tilde{x}_0$. Let $\eta = p\tilde{\eta}$ and $\tilde{\eta}'$ be the lift of $\eta$ starting at $\tilde{y}'$. $\tilde{\eta}'(1) = \tilde{x}_0 \in p^{-1}(x_0)$. Then $\tilde{\eta}' = \tilde{z}\tilde{\eta}$ and $\tilde{y}' = \tilde{z}\tilde{y}$ where $\tilde{z}$ is the deck tran. that takes $\tilde{x}_0$ to $\tilde{x}_0'$.

**Ex** Suppose $G$ acts on $X$ s.t. any $x \in X$ has a nbd $x \in U$ where $g_1U \cup g_2U = \emptyset$ for every $g_1, g_2 \in G$. Then $p: X \rightarrow X/G$ is a normal covering space.

Point of $X/G$ : $Gx = \{ gx \ | \ g \in G \}$.

$q(x_1) = q(x_2) \iff x_2 = gx_1 \Rightarrow g: \text{isom. that maps } x_1 \text{ to } x_2$.

**Cor** With the above notation fixed, if $X$ is path connected then $G(X) \cong G$.

Every deck tran. is determined by where it sends one point. If $f(x_1) = x_2$ then for some $g \in G$, $gx_1 = x_2 \Rightarrow f = g$.

**Cor** If $X = \tilde{X}/G$ where $\tilde{X}$ is simply connected and action of $G$ on $\tilde{X}$
Shift the above condition then $\pi_1(X) \cong G$.

\[ \begin{align*}
\text{Ex} & \quad \mathbb{R} \longrightarrow S^1 = \mathbb{R} / \mathbb{Z} \quad t \rightarrow t + n \\
\Rightarrow & \quad \pi_1(S^1) \cong \mathbb{Z} .
\end{align*} \]

\[ \begin{align*}
\text{Ex} & \quad \mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2 \quad (x, y) \longrightarrow (x + n, y + m) \\
\Rightarrow & \quad \pi_1(\mathbb{T}^2) \cong \mathbb{Z}^2 .
\end{align*} \]

\[ \begin{align*}
\text{Ex} & \quad \mathbb{R}P^n = S^n / \mathbb{Z}_2 \quad x \longrightarrow -x \quad \Rightarrow \quad \pi_4(\mathbb{R}P^2) \cong \mathbb{Z}_2 .
\end{align*} \]