Equivalence of simplicial and singular homology

\[ X : \Delta \text{-Complex} \quad i : \Delta_n(X) \rightarrow C_n(X) \]

Prop \[ i_* : H_n(X) \rightarrow H_n(X) \] is an isomorphism.

- \[ A \subset X \quad \Delta_n(X, A) = \frac{\Delta_n(X)}{\Delta_n(A)} \]
  \[ H_n(X, A) \], exact sequence:

\[
\cdots \rightarrow H_n(X, A) \xrightarrow{i_*} H_n(X) \rightarrow H_n(X, A) \xrightarrow{j_*} H_{n-1}(A) \rightarrow \cdots
\]

Suppose \( X \) is a finite \( \Delta \)-complex. We prove the Prop by induction.

\[ X^K \subset X \quad \text{K-Skeleton} \]

For \( K = 0 \) \[ i_* : H_n(X^0) \cong H_n(X) \].

Suppose \( i_* : H_n(X^K) \cong H_n(X^K) \). Consider \( (X^{K+1}, X^K) \):

\[
\cdots \rightarrow H_n(X^{K+1}, X^K) \rightarrow H_n(X^K) \rightarrow H_n(X^{K+1}) \rightarrow H_n(X^{K+1}, X^K) \rightarrow \cdots
\]

Lem (Five Lemma) Suppose the diagram

\[
\begin{array}{cccccc}
A & \rightarrow & B & \rightarrow & C & \rightarrow & D & \rightarrow & E \\
\alpha & \downarrow & \beta & \downarrow & \gamma & \downarrow & \delta & \downarrow & \epsilon \\
A' & \rightarrow & B' & \rightarrow & C' & \rightarrow & D' & \rightarrow & E'
\end{array}
\]

is commutative, consisting of abelian groups, and the first and the second rows are exact. If \( \alpha, \beta, \delta, \epsilon \) are isomorphisms, then \( \gamma \) is an isomorphism too.

Proof Exercise!

Five Lemma \[ \Rightarrow \] It's enough to prove the following lemma.
Lemma: \( i_*: H^A_n(X^{k+1}, X^K) \to H_n(X^{k+1}, X^K) \) is an isomorphism for all \( n \).

**Step 1:**

\[ \Delta_n(X^{k+1}, X^K) = \Delta_n(X^{k+1}) \bigg/ \Delta_n(X^K) \cong \bigoplus_{\alpha} \mathbb{Z} \quad \text{one } \mathbb{Z} \text{ for each } (k+1) \text{-simplex} \]

\[ \Rightarrow \quad \cdots \to 0 \to \Delta^{k+1}(X^{k+1}, X^K) \to 0 \to 0 \cdots \]

\[ \Rightarrow H^A_n(X^{k+1}, X^K) \cong \bigoplus_{\alpha} \mathbb{Z} \quad \text{for } n \neq k+1 \]

\[ n = k+1 \]

**Step 2**

\[ H_n(X^{k+1}, X^K) \]

\[ X^{k+1} = \Delta^{k+1}, \; X^K = \partial \Delta^{k+1} \quad \sigma = \Pi: \Delta^{k+1} \to \Delta^{k+1} \quad \text{and } 1 \in C_{k+1}(\Delta^{k+1}, \partial \Delta^{k+1}) \]

\[ H_n(\coprod_{\alpha} \Delta^{k+1}_{\alpha}, \coprod_{\alpha} \partial \Delta^{k+1}_{\alpha}) \xrightarrow{\Phi_*} H_n(X^{k+1}, X^K) \]

\[ \Rightarrow i_* \text{ is an isomorphism.} \]

Suppose \( X \) is infinite dimensional. For every \( K \) \( i_*: H^A_n(X^K) \to H_n(X^K) \) is an isomorphism. Consider an \( n \)-cycle \( Z \in C_n(X) \). \( Z = \sum_{i=1}^m n_i \cdot \epsilon_i \)

- Every compact set in \( X \) meets only finitely many open simplices of \( X \).

\[ \Rightarrow Z \in C_n(X^K) \text{ for some } K \Rightarrow [Z] \in H_n(X^K) \xrightarrow{i_*} H^A_n(X^K) \Rightarrow [Z] \in \text{im}(i_*) \]

\[ \Rightarrow i_*: \text{surjective. Show that } i_* \text{ is injective.} \]
Mayer-Vietoris Sequence

Let \( A, B \subseteq X \) s.t. \( X = \text{int}(A) \cup \text{int}(B) \). Then there exists an exact seq of the form:

\[
\cdots \xrightarrow{\partial} H_n(A \cap B) \xrightarrow{\Phi} H_n(A) \oplus H_n(B) \xrightarrow{\Psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \rightarrow \cdots
\]

where:

\[
i : A \cap B \hookrightarrow A, \quad j : A \cap B \hookrightarrow B
\]

\[
\Phi(x) = (i_*(x), -j_*(x))
\]

\[
i' : A \hookrightarrow X, \quad j' : B \hookrightarrow X
\]

\[
\Psi(x, y) = i'_*(x) + j'_*(y)
\]

\[
[\Psi \circ \Phi = 0]
\]

\textbf{Proof}

Let \( C_n(A+B) \) be the free abelian group gen. by \( n \)-simplices in \( A \) and \( n \)-simplices in \( B \):

\[
\begin{array}{rcl}
2 & : & C_n(A+B) \hookrightarrow C_n(X)
\end{array}
\]

Chain complex

\[
\cdots \rightarrow C_n(A+B) \xrightarrow{\partial} C_{n-1}(A+B) \xrightarrow{\partial} C_{n-2}(A+B) \rightarrow \cdots
\]

Let

\[
\begin{array}{c}
0 \rightarrow C_n(A \cap B) \xrightarrow{\Phi} C_n(A) \oplus C_n(B) \xrightarrow{\Psi} C_n(A+B) \rightarrow 0
\end{array}
\]

\[
\Phi(\sigma) = (\sigma, -\sigma), \quad \Psi(\sigma_1, \sigma_2) = \sigma_1 + \sigma_2
\]

This is a short exact seq. of chain complexes.

1. \( \Phi \) : injective \( \checkmark \)
2. \( \ker(\Psi) = \{ (\sigma_1, \sigma_2) \mid \sigma_1 + \sigma_2 = 0 \Rightarrow \sigma_2 = -\sigma_1 \} = \text{Im}(\Phi) \) \( \checkmark \)
3. \( \Psi \) : surjective \( \checkmark \)

\[
\Rightarrow \text{Exact seq:} \quad \cdots \xrightarrow{\partial} H_n(A \cap B) \xrightarrow{\Phi} H_n(A) \oplus H_n(B) \xrightarrow{\Psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \rightarrow \cdots
\]

\[
\text{Exact sequence:}
\]

\[
(B, A \cap B) : \quad \begin{array}{c}
H_n(A \cap B) \xrightarrow{\Phi} H_n(B) \xrightarrow{\Phi} H_n(A, A \cap B) \rightarrow H_{n-1}(A \cap B) \rightarrow \cdots
\end{array}
\]

\[
\begin{array}{cc}
\downarrow j_* & \downarrow i_* \\
\downarrow \Phi & \downarrow \Phi
\end{array}
\]

\[
(X, A) \rightarrow H_n(X) \xrightarrow{\Phi} H_n(A) \xrightarrow{\Phi} H_n(X, A) \rightarrow H_{n-1}(A) \rightarrow \cdots
\]

3
Short exact sequence

\[ 0 \rightarrow C_n(A) \rightarrow C_n(X) \rightarrow C_n(X, A) \rightarrow 0 \]

\[ \forall i, j \in \mathbb{Z} \quad C_n(A) \rightarrow C_n(A + B) \rightarrow C_n(A + B, A) \rightarrow 0 \]

\[ C_n(A + B) \cong C_n(B) / C_n(A) \]

\[ \cdots \rightarrow H_{n+1}(X, A) \rightarrow H_n(A) \rightarrow H_n(X) \rightarrow H_n(X, A) \rightarrow H_{n-1}(A) \rightarrow \cdots \]

\[ \forall i \in \mathbb{Z} \quad H_{n+1}(A) \cong H_n(A) \rightarrow H_n(A + B) \rightarrow H_n(A + B, A) \rightarrow H_{n-1}(A) \rightarrow \cdots \]

\[ H_{n+1}(B, A \cap B) \]

\[ Z = X \setminus \{ \text{points of } B \} \Rightarrow B = X \setminus Z = \text{int}(B) = X \setminus \overline{Z} \Rightarrow \overline{Z} \subset \text{int}(A). \]

\[ \text{int}(A) \cap \text{int}(B) = X \]

\[ \text{Excursion Theorem} \quad i_* : H_n(B, A \setminus Z) \rightarrow H_n(X, A) \quad \text{for all } n \]

\[ \Rightarrow \text{Five Lemma} : i_* : H_n(A + B) \rightarrow H_n(X) \text{ is an isomorphism.} \]

Ex: \( X = S^n, A = S^n \setminus \{p\}, B = S^n \setminus \{q\} \)

Proof by induction: Suppose \( \tilde{H}_i(S^{n-1}) = \begin{cases} \mathbb{Z} & \text{if } i = n-1 \\ 0 & \text{otherwise} \end{cases} \)

\[ \cdots \rightarrow \tilde{H}_i(A \cap B) \rightarrow \tilde{H}_i(A) \oplus \tilde{H}_i(B) \rightarrow \tilde{H}_i(S^n) \rightarrow \tilde{H}_{i-1}(A \cap B) \rightarrow \tilde{H}_{i-1}(A) \oplus \tilde{H}_{i-1}(B) \rightarrow \cdots \]

\[ \Rightarrow \tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1}) = \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{otherwise} \end{cases} \]