Lecture 9  
Thursday, February 21, 2019  9:13 AM

Covering space  
\[ P: (\tilde{X}, \tilde{x}_0) \to (X, x_0) \]  
\[ \pi_1(P(X, x_0)) \subset \pi_1(X, x_0) \]

**Lemma** If covering spaces \((\tilde{X}_1, P_1)\) and \((\tilde{X}_2, P_2)\) are isomorphic by \(f: \tilde{X}_1 \to \tilde{X}_2\) that \(f(\tilde{x}_1) = \tilde{x}_2\) then  
\[ P_1^* (\pi_1(\tilde{X}_1, \tilde{x}_1)) = P_2^* (\pi_1(\tilde{X}_2, \tilde{x}_2)) \]

**Proof** \[ P_2 \circ f = P_1 \Rightarrow P_2^* \circ f^* = P_1^* \]

\[ \Rightarrow \text{Im } P_{2*} = \text{Im } P_{1*} \]

**Example**  
\[ \pi_1(\tilde{X}_1, \tilde{x}_1) \]

\[ P_1 \]

\[ a, b, a^2, b^2, ab, a^{-1}b \]

\[ \text{Im } P_{1*} \neq \text{Im } P_{2*} \text{ because if } a \in \text{Im } P_{2*} \Rightarrow b \in \text{Im } P_{2*} \Rightarrow \text{Im } P_{2*} = \langle a, b \rangle \cdot \tilde{x}_1 \]

**Example**  
\[ \chi = \mathbb{RP}^2 \lor \mathbb{RP}^2 \]

\[ \text{2-sheeted cover } S^2 \to \mathbb{RP}^2 \]

\[ \tilde{X}_1 \to [x] \]

\[ \pi_1(\mathbb{RP}^2 \lor \mathbb{RP}^2) \sim \mathbb{Z}_2 \times \mathbb{Z}_2 \]

\[ p_1^* (\pi_1(\tilde{X}, \tilde{x}_1)) \text{ is the subgroup } \mathbb{Z}_2 \times \{0\} \]
Thm: Suppose $X$ is path connected and locally path connected. Any two covering spaces $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_1)$ of $X$ with $P_1(p_1(\tilde{x}_1)) = P_1(p_1(\tilde{x}_2))$ are isomorphic by $f: \tilde{X}_1 \to \tilde{X}_2$ s.t. $f(\tilde{x}_1) = \tilde{x}_2$ ($p_2 \circ f = p_1$).

Prop (Lifting Property): Let $(\tilde{X}, p_1)$ be a covering space of $X$. For a path connected and locally path connected space $Y$, a map $f: (Y, y_0) \to (X, x_0)$ has a lift $\tilde{f}$ s.t. $\tilde{f}(y_0) = \tilde{x}_0$ iff $f_*(\pi_1(Y, y_0)) \subseteq P_1(\pi_1(\tilde{X}, \tilde{x}_0))$.

Proof: If such a lift exists, it is unique.

Proof of Thm: $X$ locally path connected, then $\tilde{X}_1$ and $\tilde{X}_2$ are locally path connected. The lifts of $P_2$ are unique:

Prop (Lifting Property) Part 1: $(\Rightarrow) P_1 \tilde{f} = \tilde{f} \Rightarrow P_1 \tilde{f}_* = \tilde{f}_*$.

Diagram: $P_1 \tilde{f}_* = \tilde{f}_*$ for $f_*(\pi_1(Y, y_0)) = P_1(\tilde{f}_*(\pi_1(Y, y_0)))$.

Proof: $P_1 \tilde{f}_* = \tilde{f}_*$ for $f_*(\pi_1(Y, y_0)) = P_1(\tilde{f}_*(\pi_1(Y, y_0)))$.

Diagram: $P_1 \tilde{f}_* = \tilde{f}_*$ for $f_*(\pi_1(Y, y_0)) = P_1(\tilde{f}_*(\pi_1(Y, y_0)))$. $\tilde{f}_* \tilde{f} = \tilde{f}$ uniqe, $\tilde{1} = \tilde{P}_2 \tilde{P}_1 = \tilde{P}_1 \tilde{P}_2$. Similarly, $\tilde{P}_1 \tilde{P}_2 = \tilde{1}$.
$(\Leftarrow)$ Construct $\tilde{f} : \tilde{f}(y_0) = \tilde{z}_0$.

$\tilde{z}_0 \xrightarrow{\tilde{f}} \tilde{f}(y) \xrightarrow{p} f(y)$

**Def** $\tilde{f}(y) = \tilde{f}_Y(1)$

**Well-defined:** $Y, \tilde{Y}'$ loop based at $y_0$

$\Rightarrow \tilde{f}_* \left( [Y, \tilde{Y}'] \right) = [\tilde{f}_Y, \tilde{f}_Y'] \in \pi_1(\tilde{X}, \tilde{z}_0)$

$\Rightarrow$ lift of $\tilde{f}_Y, \tilde{f}_Y'$ starting at $\tilde{z}_0$ is a loop.

$\Rightarrow \tilde{f}_Y(1) = \tilde{f}_Y'(1)$

**Continuous:** Let $U$ be an evenly covered open nbhd of $f(y)$ i.e. $p^{-1}(U) = \bigcup U_\alpha$

where $p : \tilde{U}_\alpha \to U$ is a homeo. for all $\alpha$. Let $\tilde{U}$ be the lift of $U$ st. $\tilde{f}(y) \subseteq \tilde{U}$. Consider a path connected nbhd $y \in V$ st. $f(v) \subseteq U$. For any $y' \in V$, $\exists$ a path $\eta$ from $y$ to $y'$ in $V$. Then $f\eta \subseteq U$. Lift of $f\eta$ starting at $\tilde{f}(y)$ is $(p|_U)^{-1} f\eta$. Thus $\tilde{f}(y') \subseteq \tilde{U} = \tilde{f}(v) \subseteq \tilde{U}$.
Part 2 (Uniqueness) Let \( \tilde{f}_1 \) and \( \tilde{f}_2 \) be lifts of \( f \), and suppose
\[
\tilde{f}_1(y) \neq \tilde{f}_2(y).
\]
Take an evenly covered nbhd \( U \) of \( f(y) \).

Suppose \( \tilde{f}_1(y) \in \tilde{U}_1 \) and \( \tilde{f}_2(y) \in \tilde{U}_2 \) be lifts of

\[
\tilde{f}(y) \in \tilde{U}_1 \quad \text{and} \quad \tilde{f}(y) \in \tilde{U}_2.
\]

\[ u \text{ s.t. } p: \tilde{U}_1 \to U \text{ and } p: \tilde{U}_2 \to U \text{ is a homeo.} \]

\[ \exists \text{ a nbhd } y \in V \text{ s.t. } \tilde{f}_1(v) \subset \tilde{U}_1 \text{ and } \tilde{f}_2(v) \subset \tilde{U}_2. \]

i.e. for \( v \in V \), \( \tilde{f}_1(v) \neq \tilde{f}_2(v) \) \( \Rightarrow \) The set of pts that \( \tilde{f}_1(v) = \tilde{f}_2(v) \)

is open.

Suppose \( \tilde{f}_1(y) = \tilde{f}_2(y) \). Take an evenly covered nbhd \( U \) of \( f(y) \) and

a lift \( \tilde{U} \) of \( U \) containing \( \tilde{f}_1(y) = \tilde{f}_2(y) \). Consider a nbhd \( V \)

s.t.

\[
\tilde{f}_1(v), \tilde{f}_2(v) \subset \tilde{U}.
\]

\[
f = p_{\tilde{f}_1} = p_{\tilde{f}_2}, \quad p: \tilde{U} \to U \quad \Rightarrow \quad \tilde{f}_1|_V = \tilde{f}_2|_V = (p|_U)^{-1}f
\]

\( \Rightarrow \) The set of pts that \( \tilde{f}_1 = \tilde{f}_2 \) is open.

Since \( Y \) is connected, \( \tilde{f}_1(y_0) = \tilde{f}_2(y_0) = \tilde{x}_0 \), we have \( \tilde{f}_1 = \tilde{f}_2 \).

Q Suppose \( X \) is path-connected and locally path-connected. Given a

Subgroup \( H \subset \pi_1(X, x_0) \), do there exist a path-connected covering space

\( p: (\tilde{X}, \tilde{x}_0) \to (X, x_0) \text{ s.t. } \pi_1(\tilde{X}, \tilde{x}_0) = H \)?

Special case \( H = \{e\} \Rightarrow \pi_1(\tilde{X}, \tilde{x}_0) = \text{trivial}. \)

Def A simply connected covering space \( p: \tilde{X} \to X \) is called a

universal cover of \( X \).