Lecture 11
Sunday, February 26, 2017  8:12 PM

- **Part 1**: Let $H < \pi_1(X, x_0)$, construct a covering space $p_H : (\tilde{X}_H, \tilde{x}_0) \rightarrow (X, x_0)$ such that $p_{H*} \left( \pi_1(\tilde{X}_H, \tilde{x}_0) \right) = H$

- **Part 2**: Symmetries of a covering space (Deck transformation group)

**Recall** Define an action of $H$ on $\tilde{X}$ such that $p_H \circ \tilde{h} = p_H \circ \tilde{x}$ for all $x \in X$. Universal Cover $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$

- $H \ni h = [f]$, $\tilde{x} = [\gamma] \rightarrow h \cdot \tilde{x} = [f \cdot \gamma]$

- $p(\tilde{x}) = \gamma(1) = f \cdot \gamma(1) = p(h \cdot \tilde{x})$

**Equivalence relation** $\tilde{x}_1 \sim \tilde{x}_2$ if for $h \in H$ we have $h \cdot \tilde{x}_1 = \tilde{x}_2$.

Let $\tilde{X}_H = \tilde{X} / \sim$, $\tilde{X}_H = [\tilde{x}_0]$, $p$ induces a map $p_H : (\tilde{X}_H, \tilde{x}_0) \rightarrow (X, x_0)$.

\[ (P_H \circ q = p) \]

1. $p_H$ is a covering space map. For any $x \in X$, let $U$ be a path connected open nbd s.t. $\pi_1(U) \rightarrow \pi_1(X)$ is trivial.

$$
\Rightarrow p^{-1}(U) = \bigsqcup_{\gamma(0) = x_0} U_{\gamma(1)}
$$

If for $[\gamma_1, \eta_1] \in U_{[\gamma_1]}$ and $[\gamma_2, \eta_2] \in U_{[\gamma_2]}$ we have $[\gamma_1, \eta_1] \sim [\gamma_2, \eta_2]$

$$
\Rightarrow [\gamma_1, \eta_1] \cdot [\gamma_2, \eta_2] \in H \Rightarrow [\gamma_1 \cdot \gamma_2, \eta_1 \cdot \eta_2] \in H \Rightarrow \gamma_2 \cdot \gamma_1 = \gamma_1 \cdot \gamma_2
$$

and $[\gamma_1, \eta_1] \sim [\gamma_2, \eta_2] \Rightarrow$ equivalence relation identifies $U_{[\gamma_1]}$ and $U_{[\gamma_2]}$ iff $[\gamma_1] \sim [\gamma_2] \Rightarrow p_H$ is covering map.

$$
\Rightarrow q : (\tilde{X}, \tilde{x}_0) \rightarrow (\tilde{X}_H, \tilde{x}_0) \text{ is covering map.}
$$

2. $p_H \circ (\pi_1(\tilde{X}_H, \tilde{x}_0)) \circ P_H = P$. For a loop $f$ based at $x_0$, lift of $f$ to $\tilde{X}_H$ based at $\tilde{x}_0$ is $\tilde{f}(1) = \tilde{x}_0$ if $f(1) \sim \tilde{x}_0 \Rightarrow [f] \sim C_{x_0}$ and $[f] \in H$. 

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Def: Action of a group \( G \) on a top space \( Y \) is called free if it has no fixed pt i.e.
\[ \forall y \in Y \text{ and } \forall g \in G \implies gy \neq y. \]

It's called a covering space action, if for any \( y \in Y \), there exists a nbhd \( U \) of \( y \) such that for any \( g \in G \), \( gU \cap U = \emptyset \).

Def: Covering space action \( \implies \) free

Thm: If action of \( G \) on a top space \( Y \) is a covering space action, then \( q: Y \to Y/G \) is a normal covering space.

Def: A covering space \( p: \tilde{X} \to X \) is called normal if for any \( x \in X \) and any \( \tilde{x}_1, \tilde{x}_2 \in p^{-1}(x) \), there exists an isomorphism of covering spaces \( \tilde{x}_1 \to \tilde{x}_2 \) which takes \( \tilde{x}_1 \) to \( \tilde{x}_2 \).

If \( y, y_2 \in Y \) s.t. \( gy_1 = y_2 \) isom. is homo. corresponding to \( g \).

Def: \( p: \tilde{X} \to X \), group of covering space isom \( \tilde{X} \to \tilde{X} \) is called deck trans.

Covering space denoted \( G(\tilde{X}) \).

If \( \tilde{X} \) normal \( \implies \tilde{X}/G(\tilde{X}) \cong X \) and \( p: \tilde{X} \to \tilde{X}/G(\tilde{X}) \).

Suppose \( \tilde{X} \) is path-connected; uniq. lifting property implies that if such an \( f \) exists \( \implies \) it's unique.

\( \implies \) Isom \( f \) is determined by where it sends one pt.

Cor: If \( Y \) path-connected, then \( G(\tilde{Y}) \cong G \).

Thm: Suppose \( X \) and \( \tilde{X} \) are path connected and locally path connected. Then
\[ p: (\tilde{X}, \tilde{x}_0) \to (X, x_0) \text{ normal } \iff p_\ast (\tilde{\pi}_1(\tilde{X}, \tilde{x}_0)) \triangleleft \pi_1(X, x_0) \text{ normal subgroup.} \]

Lem: \( p: (\tilde{X}, \tilde{x}_0) \to (X, x_0) \) Then \( p_\ast(\tilde{\pi}_1(\tilde{X}, \tilde{x}_0)) \) and \( p_\ast(\tilde{\pi}_1(\tilde{X}, \tilde{x}_1)) \) are conjugate.

\[ [g] p_\ast(\tilde{\pi}_1(\tilde{X}, \tilde{x}_1)) [g]^{-1} = p_\ast(\tilde{\pi}_1(\tilde{X}, \tilde{x}_0)) \]

\[ \tilde{x}_0 \overset{\tilde{f}}\longrightarrow \tilde{x}_1 \]
\[ \tilde{x}_0 \overset{\tilde{f}}\longrightarrow \tilde{x} = g \tilde{x}_0 \]

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\[
\begin{align*}
\text{Thm: } & \quad G(\tilde{X}) \cong N(H) \quad \text{where } H = P_*(\pi_1(\tilde{X}, \tilde{x}_0)) \text{ and } N(H) \text{ is normalizer of } H \text{ in } \pi_1(\tilde{X}, \tilde{x}_0) \\
\text{Pf: } & \quad \text{Construct homomorphism } \varphi : N(H) \to G(\tilde{X}) \\
& \quad \text{such that } \varphi([x]) \text{ isomorphisms which map } \tilde{x}_0 \text{ to } \tilde{x}_1 = \tilde{x}(1) \text{ is surjective} \\
& \quad \text{homo } \varphi \text{ of loops } \tilde{y}(1) = \tilde{x}(1) = \tilde{z}(1) \quad \Rightarrow \quad \varphi([\tilde{y}(1)]) = \varphi([\tilde{x}(1)]) = \varphi([\tilde{z}(1)]) \\
& \quad [x] \in \text{Ker}(\varphi) \text{ if } \varphi \text{ lifts to a loop in } \tilde{X} \Leftrightarrow [x] \in H \to \text{Ker}(\varphi) = H \\
\text{Cor. If } \tilde{X} \text{ is normal then } G(\tilde{X}) \cong \pi_1(\tilde{X}, \tilde{x}_0) / H \\
\text{Cor. If } \tilde{X} \text{ is the universal cover of } X \text{, } G(\tilde{X}) \cong \pi_1(X, x_0). \\
\text{Ex. } & \quad p : \mathbb{R} \to S^1 \quad G(\mathbb{R}) \cong \mathbb{Z} \\
& \quad 2 \cdot m \cdot \mathbb{Z} \times n \\
& \quad \text{Cor. If } Y \text{ path connected, locally path connected, } G \cong \pi_1(Y_G) \quad \text{where } q : Y \to Y/G. \\
\end{align*}
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