Recall
Prop: If $A \subset X$ is a contractible subspace w/ HEP, then $q: X \to X/A$ is a homotopy equivalence.

Def: $A \subset X$. A homotopy relative $A$ is a homotopy $f_t: X \to Y$ s.t. $f_t|_A = f_0|_A$, i.e. independent of $t$.

Ex: $A \subset X$ and a deformation retract, $r_t: X \to X$ of $X$ onto $A \Rightarrow r_t|_A = \mathbb{I}$

Def: $A \subset X$ and $A \subset Y$, then $f: X \to Y$ is called a homotopy equivalence relative $A$ if

if $f|_A = \mathbb{I}$ and there exist $g: Y \to X$ s.t. $g|_A = \mathbb{I}$ and $fg \sim \mathbb{I}$ rel $A$ and $gf \sim \mathbb{I}$ rel $A$

$\Rightarrow$ we say $X \simeq_Y \text{ rel } A$

Ex: $A \subset X$, $r_t: X \to X$ deformation retract on to $A$ s.t. $r_0 = \mathbb{I}$ and $r_t$ is homotopy equiv. relative $A$

$r_1 = \mathbb{I}$

$r_t|_A = \mathbb{I}$ rel $A$

Prop: $A \subset X$, $A \subset Y$, $(X,A)$ and $(Y,A)$ have HEP. Then any homotopy equiv. $f: X \simeq Y$

such that $f|_A = \mathbb{I}$ is a homotopy equivalence relative $A$.

- $r_t: X \to X$ deformation retract of $X$ onto $A \subset X$

If $A$ has HEP

- $i: A \hookrightarrow X$ homotopy equivalence

- $i|_A = \mathbb{I}$

- $X = A$, $Y = X$ $\Rightarrow$ $i: A \hookrightarrow X$ homotopy equiv.

- $i|_A = \mathbb{I}$

- $i$ is homotopy equiv. rel $A$.

- $\tilde{r}: X \to A$ s.t.

- $\tilde{r}|_A = \mathbb{I}$

- $i \tilde{r} = \mathbb{I}$ rel $A$ and $i \tilde{r} = \mathbb{I}$ deformation retract.
Con: \( f: X \to Y \) homotopy equiv \( \iff X \) is a deformation retract of \( M_f \)

(\( X \cong Y \) if \( f \) and \( g \) are both deformation retractions of a third space.)

Pr: \( X \subset M_f \) has HEP, \( i: X \hookrightarrow M_f \) homotopy equiv. \( f \) is homotopy equiv. \( i \circ f \cong 1 \)

Attaching spaces:
\( X, Y \) top spaces, \( A \subset Y \) closed subspace
\( f: A \to X \)
Attaching \( Y \) to \( X \) via \( f \)
\( X \cup_Y Y \)
\( (a \in f(a) \text{ for all } a \in A) \)

Ex: \( f: X \to Y \) \( M_f \). Attaching \( X \times I \) to \( Y \) along \( X \times \{1\} \) via \( f \)

Attaching an \( n \)-cell: \( X, D^n, A = S^{n-1} = \partial D^n \subset D^n \), \( \varphi: S^{n-1} \to X \)
\( X \cup_Q D^n \): attaching an \( n \)-cell to \( X \) via \( \varphi \)

CW complexes: are formed inductively by attaching cells.

Ex: \( X \)
\( X^0 = \{v\} \)
\( X^1 = (D^1_a \cup D^1_b)_q \cup X_0 \)
\( \varphi^1_a: \partial D^1_a \to \{v\} \)
\( \varphi^1_b: \partial D^1_b \to \{v\} \)
\( X^2 = X^1 \cup D^2_\varphi \)

Ex: \( S^2 \)
\( X^0 = \{v\} \)
\( X^1 = X^0 \)
\( X^2 = X^0 \cup D^2_\varphi \)
\( \varphi: S^1 \to \{v\} \)

CW Complex: \( X^0 \): discrete set of pt., \( 0 \)-cell: 0-skeleton
\( X^n = X^{n-1} \cup (\cup D^n_a) \)
\( \varphi^n_a: S^n_a \to X^{n-1} \)
\( e^n_a = \Int D^n_a \subset X \)
$X = U \times X^n \text{ weak topo, i.e. } A \subseteq X \text{ is open (or closed) if } A \cap X^n \text{ is open (or closed)}$

for each $n$.

**Characteristic map**:

$\Phi_\alpha : D^n_\alpha \to X \quad \text{homeom on the interior}$

$D^n_\alpha \hookrightarrow \mathbb{R}^\alpha \sqcup D^n_\alpha \xrightarrow{\text{quotient}} X^n \subseteq X$

$e^n_\alpha := \Phi_\alpha (\text{Int } D^n_\alpha)$

**Ex** $S^n : \text{ Cell complex with 2-cell } e^0 U e^n$

$S^n \times S^n = e^0 U e^m U e^n U e^{m+n}$ (Special Case: $m = n = 1$)

$X, Y \text{ Cell complexes } \implies X \times Y \text{ is a cell complex with cells } e^m_\alpha \times e^m_\beta$

**Ex** Real projective $n$-space $\text{RP}^n = \{1 \text{-dim subspaces of } \mathbb{R}^{n+1} \text{ i.e. all lines in } \mathbb{R}^{n+1} \text{ through origin}\}$

$\simeq \frac{S^n}{(v \sim -v \text{ for all } v \in S^n)} \simeq \frac{D^n}{(p \sim -p \text{ for all } p \in S^{n-1})}$

$\simeq \text{IRP}^{n-1} \cup \frac{D^n}{p_n}$

$\simeq \text{IRP}^n \simeq e^0 U e^1 U e^2 U \cdots U e^n$

**Def** $\text{RP}^n = U \text{IRP}^n \simeq e^0 U e^1 U \cdots U e^n U \cdots$

**Ex** Similarly, complex proj. $n$-space $\text{CP}^n : \text{Complex lines through origin in } \mathbb{C}^n$

$\simeq \text{CP}^{n-1} \cup \frac{D^n}{p_n} \simeq e^0 U e^2 U \cdots U e^n$

**Def** A subcomplex of $X$ is a closed subspace $A \subseteq X$ s.t. it’s a union of cells in $X$.

**Properties of CW Complexes**

* CW Complexes are Haudorff

* If $A \subseteq X$ is compact, then $A \subseteq \text{ finite subcomplex of } X$.

* If $A \subseteq X$ is a subcomplex, then it satisfies HEP.

**Cor** $A \subseteq X$, subcomplex $\Rightarrow X \simeq A$ contrable
**Ex.** $X = \text{Theta graph}$

$A = a \cup \{v, w\}$

$\Rightarrow X \simeq X^1_A$

$S^1 \cup S^1$

In general, $X$: connected graph with finitely many vertices and edges

$X \simeq S' \cup S^1 \cup \ldots \cup S^1$

$n$: # of edges

**Prop.** Assume $A \simeq Y$ and has HEP. If $f, g: A \to X$ s.t. $f \simeq g$

Then $X \cup Y \simeq X \cup Y$. 

**Cor.** To determine homotopy type of CW Complex, we only need attaching maps up to homotopy.

Take $A = X^n \subset X$. Homotopy type of $X^{n+1}$ doesn't change attaching map up to homotopy.