Recall • Fundamental group \( \pi_1(X, x_0) \)
  • Invariant under homotopy equiv.
  • \( \pi_1(S^1) \cong \mathbb{Z} \), \( \pi_1(S^n) = 0 \) for \( n \geq 2 \)

Part 1 Application of \( \pi_1(S^1) \cong \mathbb{Z} \)

Thm: (Brouwer fixed pt thm) Every cont. map \( h: \mathbb{D}^2 \to \mathbb{D}^2 \) has a fixed pt.
  i.e. there exists \( x \in \mathbb{D}^2 \) s.t. \( h(x) = x \)

Proof by Contradiction: Assume \( h: \mathbb{D}^2 \to \mathbb{D}^2 \) has no fixed pt. Define
  \[ r: \mathbb{D}^2 \to S^1 \]
  as in the picture:
  If \( x \in S^1 \), \( r(x) = x \) ⇒ \( r \) is a retraction.
  From last semim \( \circ \pi_1(S^1) \to \pi_1(D^2) \) is injective.
  Induced \( \mathbb{Z} \to \mathbb{Z} \) by inclusion.

Thm: (Borsuk-Ulam thm) For any cont. map \( f: S^2 \to \mathbb{R}^2 \), there exists antipodal pts \( x \) and \( -x \) with \( f(x) = f(-x) \).

Proof by Contradiction: Suppose \( f: S^2 \to \mathbb{R}^2 \) is cont. and for any \( x \in S^2 \), \( f(x) \neq f(-x) \).

Thus \( g: S^2 \to S^1 \)
  \[ g(x) = \frac{f(x) - f(-x)}{||f(x) - f(-x)||} \]
  \[ [0,1] \to \mathcal{S} \to S^1 \]
  \[ h = g_1: I \to S^1 \]
  loop in \( S^1 \)

The loop \( \eta \) is homotopically trivial in \( S^2 \) ⇒ \( g_1 \) : homotopically trivial.

\[ g(x) = -g(-x) \Rightarrow h(S(1/2)) = -h(S) \]
  Let \( \tilde{h}: [0,1] \to \mathbb{R} \) be a lifting of \( h \).
  i.e. \( \tilde{h}(0) = h(0) = 0 \) odd number
  \[ \tilde{h}(1) = h(1/2) = h(0) + 1 \equiv 0 \mod 2 \]
  Remk We may use homology to prove Brouwer fixed pt thm and Borsuk-Ulam thm in higher dimension.

Cor: If \( S^2 = \bigcup_{i=1}^{3} A_i \) where \( A_1, A_2, A_3 \) are closed, then at least one \( A_i \) contain a pair of antipodal pts i.e. \( \{x, -x\} \subseteq A_i \) for \( x \in S^2 \).
Define \( f : (d_1, d_2) : S^2 \to \mathbb{R}^2 \)

\[ d_i(x) = \inf_{y \in A_i} |x - y| \text{ distance of } x \text{ from } A_i \]

\[ f(x) = f(-x) \]

\[ d_1(x) = d_1(-x) \]

\[ d_2(x) = d_2(-x) \]

If \( d_1(x) = d_1(-x) = 0 \) \((d_2(x) = d_2(-x) = 0) = \) \( x \perp A_1 \) \( \text{ or } x \perp A_2 \)

If \( d_1(x) = d_1(-x) \neq 0 \)

\[ d_2(x) = d_2(-x) \neq 0 \]

**Thm (Fundamental thm of algebra):** Every polynomial of positive degree and with coefficients in \( \mathbb{C} \) has a root in \( \mathbb{C} \).

**Proof by Contradiction:** Assume \( P(z) = z^n + a_1 z^{n-1} + \cdots + a_n \) with \( n > 0 \) has no root.

\( \Rightarrow P(z) \neq 0 \) for \( z \in \mathbb{C} \).

**Family of loops in \( S^1 \):**

\[ f_r(s) = \frac{P(re^{2\pi is})}{|P(re^{2\pi is})|} \quad f_0(s) = 1 \quad \text{Constant loop} \]

\[ r > 0 \]

For any \( r > 0 \), \( f_r \) is homotopically trivial.

For \( r > \max (|a_1| + \cdots + |a_n|, 1) \)

\[ |z|^n > (|a_1| + \cdots + |a_n|) |z|^{n-1} > |a_1| |z|^{n-1} + |a_2| |z|^{n-2} + \cdots + |a_n| > |a_1 z^{n-1} + \cdots + a_n| \]

\( \Rightarrow P(z) = z^n + a_1 z^{n-1} + \cdots + a_n \) has no root for \( |z| = r \) and \( 0 < |z| < 1 \).

For such \( r \), we use \( P_r \) to construct a homotopy btwn \( f_r \) and \( w_n \):

\[ f_r^+(s) = \frac{P_r(re^{2\pi is})}{|P_r(re^{2\pi is})|} \quad \text{homotopy btwn } f_r^+(s) = f_r(s) \quad \text{and} \]

\[ f_r^-(s) = \frac{r^n e^{2\pi i s}}{|r^n e^{2\pi i s}|} = e^{2\pi is} \]

\[ [f_r] = [w_n] \quad \times \]

**Part 2: Van Kampen’s thm:**

**Free product of group**

Let \( \{ G_\alpha \}_{\alpha \in I} \) be a collection of group.

**Def:** A word in \( \{ G_\alpha \}_{\alpha \in I} \) is a sequence \( g_1 g_2 \cdots g_m \) of finite length s.t.

- \( g_i \in G_{\alpha_i} \) for some \( \alpha_i \in I \).
- A word \( g_1 g_2 \cdots g_m \) is called reduced if for any \( \alpha_i = e \), \( e_a \in G_{\alpha_i} \) and \( \alpha_i \neq \alpha_i \).

New Section 1 Page 2
Any word can be simplified to a reduced words by:
1. Remove \( g_i \) if \( g_i = e_{a_i} \in G_{a_i} \)
2. If \( a_i = a_{i+1} \) then replace the two letter \( g_i g_{i+1} \) by the multi \( g_i g_{i+1} G_{a_i} \)

**Def.** The free group \( *_{a_i} G_{a_i} \) as a set consists of the reduced words in \( \{ G_{a_i} \}_{a_i \in I} \).

Multiplication: \( (g_1 g_2 \ldots g_m) (g'_1 g'_2 \ldots g'_n) = g_1 g_2 \ldots g_m g'_2 g'_3 \ldots g'_n \)
then make it reduced by the above operations 1 and 2.

For example, \( (g_1 g_2 \ldots g_m)^{-1} = g_m^{-1} g_{m-1}^{-1} \ldots g_1^{-1} \), \( (g_1 g_2 \ldots g_m) (g'_1 g'_2 \ldots g'_n) = e \)

identity: empty word

**Properties**
1. Every \( G_{a_i} \) is naturally identified with a subgroup of \( G \) consisting of empty word and one-letter-words \( g \in G_{a_i} \) where \( g \neq e_{a_i} \).
2. For any group \( H \), any collection of homomorphism \( \varphi_{a_i} : G_{a_i} \rightarrow H \) extends to a homomorphism \( \varphi : *_{a_i} G_{a_i} \rightarrow H \) as

\[
\varphi(g_1 \ldots g_n) = \varphi_{a_1}(g_1) \varphi_{a_2}(g_2) \ldots \varphi_{a_n}(g_n)
\]

Example:
- \( Z \times Z \): elements of the form \( a^2 b \) : reduced word \( \begin{array}{c}
\text{set of elements} \\
(a^2) \begin{array}{c}
\text{of the form} \\
(b^4 a^3)
\end{array}
\end{array}
\] \( a^2 = (a^2)^{-1} \)

Example:
- \( \mathbb{Z}_2 \times \mathbb{Z}_2 \): elements of the form \( a^2 \) : reduced word \( \begin{array}{c}
\text{set of elements} \\
(b^4 a^3)
\end{array}
\] \( a^2 = (a^2)^{-1} \)

- \( \mathbb{Z}_2 \times \mathbb{Z}_2 \): infinite cyclic subgroup generated by \( ab : \mathbb{Z} \)

- \( a (ab)^{-1} = ba = (ab)^{-1} \)

A subgroup isomorphic to \( \mathbb{Z}_2 \) generated by \( a \)

**Def.** Free group \( G \) as free product of any number of \( \mathbb{Z} \): i.e. there exists a family \( \{ a_i \} \) of elements of \( G \) such that each \( a_i \) generates an infinite cyclic subgroup \( G_{a_i} \) of \( G \)

\( G = *_{a_i} G_{a_i} \)

\( G = \langle a_i \rangle \_{a_i \in I} \)

**Van Kampen thm**

Let \( X = \bigcup A_{a_i} \) such that each \( A_{a_i} \) is open and path connected. Furthermore, \( \cap A_{a_i} \neq \emptyset \).

Let \( i_{a_i} : \pi_1(A_{a_i}) \rightarrow \pi_1(X) \) be the homomorphism induced by inclusion \( (A_{a_i}) \subset (X, x_0) \).

Then, these maps induce a homomorphism \( \Phi : \pi_1(G_{a_i}) \rightarrow \pi_1(X) \).

1. If for any \( a \) and \( \beta \), \( A_a \cap A_{a \beta} \) is path connected, then \( \Phi \) is surjective.

(Lemma 1.15)
For any $\alpha$ and $\beta$:

$$
\begin{align*}
\pi_1(A_{\alpha} \cap A_{\beta}) & \xrightarrow{i_{\alpha\beta}} \pi_1(A_{\alpha}) \xrightarrow{i_{\alpha}} \pi_1(X) \\
\pi_1(A_{\beta}) & \xrightarrow{i_{\beta}} \pi_1(A_{\beta}) \xrightarrow{i_{\beta}} \pi_1(X)
\end{align*}
$$

Both $i_{\alpha\beta}$ and $i_{\alpha}$ are the induced maps by $A_{\alpha} \cap A_{\beta} \hookrightarrow X$.

$\Rightarrow$ For any $w \in \pi_1(A_{\alpha} \cap A_{\beta})$, $i_{\alpha\beta}(w) i_{\alpha}(w)^{-1} \in \pi_1(A_{\alpha})$

$$
\Phi(w) = i_{\alpha\beta}(w) i_{\alpha}(w)^{-1} = 1
$$

2. If for any $\alpha, \beta, \gamma$, $A_{\alpha} \cap A_{\beta} \cap A_{\gamma}$ is path-connected, then $\text{Ker}(\Phi)$ is the normal subgroup generated by the elements of the form $i_{\alpha\beta}(w) i_{\alpha}(w)^{-1}$ for $w \in \pi_1(A_{\alpha} \cap A_{\beta})$.

$$
\Rightarrow \pi_1(X) \cong \pi_1(A_{\alpha}) / \text{Ker}(\Phi)
$$

$\text{Ex.}$ $X = S^1 \vee S^1$

$A_p = X \setminus \{p\}$

$A_q = X \setminus \{q\}$

$\Rightarrow A_p(A_q)$ deformation retracts on $S^1_2 \cup S^1_1$

$A_p \cap A_q$ deformation retracts on $\{x_0\}$

$\Rightarrow \pi_1(X, x_0) = \pi_1(S^1_1, x_0) \ast \pi_1(S^1_2, x_0) \cong \mathbb{Z} \ast \mathbb{Z}$

In general, $\pi_1(V_\alpha S^1_\alpha) \cong \pi_1 S^1_\alpha$

Cor. Any connected graph is homotopy equiv. to wedge sum of a collection of circles.

$\Rightarrow \pi_1(G)$ is a free group.

More generally, let $\{X_{\alpha}, x_0\}_{\alpha \in I}$ be a collection of based topological spaces such that every $X_{\alpha}$ contains a ball $U_{\alpha}$ of $x_0$, which deformation retracts on $x_0$. Then

$$
X = V_\alpha X_{\alpha} = \bigcup_{\alpha \in I} X_{\alpha}
$$

$\Rightarrow \pi_1(X, x_0) = \pi_1 X_{\alpha, x_0}$

Take $A_{\alpha} = X_{\alpha} \cup \bigcup_{\beta \neq \alpha} U_{\beta} \subset X$.

$A_{\alpha} \cap A_{\beta} \cup U_{\alpha} \cap U_{\beta} \subset X$ deformation retracts on $x_0$.

$\text{Ex.}$ $X = \mathbb{R}^3 \setminus \mathcal{A}$

$\mathbb{R}^3 \setminus D^3$ deformation retracts on $\partial D^3$

$D^3 \setminus \mathcal{A}$ deformation retracts on $\partial D^3 \cup \mathcal{U}^I$

$\Rightarrow X$ deformation retracts $\partial D^3 \cup \mathcal{U}^I \cong S^2 \vee S^1 \Rightarrow \pi_1(X) \cong \pi_1(S^2) \ast \pi_1(S^1) \cong \mathbb{Z} \ast \mathbb{Z}$