Recall: Covering Space \( P : (\tilde{X}, \tilde{x}_0) \to (X, x_0) \) implies Subgroup \( P_* (\pi_1 (\tilde{X}, \tilde{x}_0)) \subset \pi_1 (X, x_0) \) is invariant under isom.

Prop: Suppose \( X \) is path connected and locally path connected. Then any two path connected Covering Space \( p_1 : (\tilde{X}_1, \tilde{x}_1) \to (X, x_0) \) with \( p_1_* (\pi_1 (\tilde{X}_1, \tilde{x}_1)) = p_2_* (\pi_1 (\tilde{X}_2, \tilde{x}_2)) \)

are iso via an isom. \( f : \tilde{X}_1 \to \tilde{X}_2 \) s.t. \( f(\tilde{x}_1) = \tilde{x}_2 \).

Prop (Lifting property)

\[ \begin{array}{ccc}
\tilde{X} & \xrightarrow{f} & \tilde{X} \\
P: \text{Covering Space} & & \text{Covering Space} \\
(\tilde{Y}, \tilde{y}_0) & \xrightarrow{f} & (X, x_0) \\
\end{array} \]

0. A lift \( \tilde{f} \) exists \iff \( f_* (\pi_1 (\tilde{Y}, \tilde{y}_0)) \subset P_* (\pi_1 (\tilde{X}, \tilde{x}_0)) \)

0. Lift \( \tilde{f} \) is unique.

Proof: (\( \Rightarrow \)) \checkmark

(\( \Leftarrow \)) To prove a lift \( \tilde{f} \),

\[ \tilde{f}(\tilde{y}_0) = \tilde{x}_0. \]

\[ \begin{array}{ccc}
\tilde{y}_0 & \xrightarrow{\tilde{f}} & \tilde{x}_0 \\
\tilde{f}(\tilde{y}_0) = \tilde{x}_0 & \Rightarrow & \tilde{f}(\tilde{y}) = \tilde{f}(\tilde{y}(1)) \\
\tilde{f}(\tilde{y}) & \xrightarrow{P} & x_0 \\
\end{array} \]

Well-defined: \checkmark

0. Continuous: Let \( \tilde{U} \) be an open nbd of \( \tilde{f}(y) \). We need to show that there exists an open nbd \( V \) of \( y \) s.t. \( \tilde{f}(V) \subset \tilde{U} \). Let \( U' \) be an evenly covered nbd of \( f(y) \) and \( \tilde{U}' \) be a lift of \( U' \) containing \( \tilde{f}(y) \) s.t. \( p: \tilde{U}' \to U' \) is a homeo.

Then \( \tilde{U} \cap \tilde{U}' \) is an open nbd of \( \tilde{f}(y) \) and \( P(\tilde{U} \cap \tilde{U}') \) is a homeo. \( f \) is conti.

0. Let \( V \) be a path connected open nbd of \( y \) s.t. \( f(V) \subset P(\tilde{U} \cap \tilde{U}') \).

For any \( y' \in V \), take a path \( \eta \) from \( y \) to \( y' \). Then \( \tilde{f}(y') = \tilde{f}(\eta(1)) = \tilde{f}(\eta) \), so \( \tilde{f}(y') \in \tilde{U} \cap \tilde{U}' \) is a lift of \( f(y') \) starting at \( \tilde{f}(y) \) lying in \( \tilde{U} \cap \tilde{U}' \Rightarrow \tilde{f}(y') \in \tilde{U} \cap \tilde{U}' \).

0. Uniqueness: Let \( \tilde{f}_1 \) and \( \tilde{f}_2 \) be lifts of \( f \). Suppose \( \tilde{f}(y) = \tilde{f}_2(y) \). Then take an evenly covered nbd \( U \) of \( f(y) \) and suppose \( \tilde{f}(y) \in \tilde{U} \) and \( \tilde{f}_2(y) \in \tilde{U}_2 \) are lifts.
of $U$ such that $p: \tilde{U}_1 \rightarrow U$ and $p: \tilde{U}_2 \rightarrow U$ is a homeo.

If $f_1$ and $f_2$ are Conti $\Rightarrow$ there is a nbd $V$ of $y$ st. $f_1(V) \subset \tilde{U}_1$
and $f_2(V) \subset \tilde{U}_2$ $\Rightarrow$ $f_1(y) = f_2(y)$ for any $y \in V$ $\Rightarrow$ The set of pts $\tilde{y}$ where $f_1(y) = f_2(y)$ is open.

Suppose $y \in Y$ be a pt s.t. $\tilde{f}_1(y) = \tilde{f}_2(y)$. Take an evenly covered nbd $U$ of $f_1(y)$ and
a lift $\tilde{U}$ of $U$ containing $\tilde{f}_1(y) = \tilde{f}_2(y)$ as above. Then for a nbd $V$ of $y$ we have
$\tilde{f}_1(V), \tilde{f}_2(V) \subset \tilde{U}$. $\tilde{f}_1 = \tilde{f}_2$ and $p: \tilde{U} \rightarrow U$ is a homeo. $\Rightarrow$ $p$ is injective on $\tilde{U}$.

$\Rightarrow$ $f_1 = f_2$ on $V$. The set of pts where $\tilde{f}_1 = \tilde{f}_2$ is open.

$\Rightarrow$ $Y$ is connected and $\tilde{f}_1(y_0) = \tilde{f}_2(y_0) = \tilde{x}_0$ implies that $\tilde{f}_1 = \tilde{f}_2$.

Prop. $p: (X, x_0) \rightarrow (X, x_0)$ is a path connected covering space of a path connected space $X$.

The cardinality of the fiber $p^{-1}(x_0)$ is equal to the index of $p_*(\pi_1(X, x_0))$ in
$\pi_1(X, x_0)$.

PF. Let $H = p_*(\pi_1(X, x_0))$, let $\Phi: \text{Cosets of } H \rightarrow p^{-1}(x_0)$

$H[g] \rightarrow \Phi_p([g]) = \tilde{g}(1)$

Well-defined: $H[g_1] = H[g_2] \Rightarrow [g_1][g_2]^{-1} \in H$

$\Rightarrow [g_1][g_2]^{-1} \in H \Rightarrow \tilde{g}_1 \cdot \tilde{g}_2^{-1}$ is a loop.

$\Rightarrow \tilde{g}_1 \cdot \tilde{g}_2^{-1}(1) = \tilde{x}_0 \Rightarrow \tilde{g}_1(1) = \tilde{g}_2(1)$

Surjective: $\hat{X}$ is path connected. For $\tilde{x} \in p^{-1}(x_0)$ take a path $\gamma$ connecting $\tilde{x}_0$
to $\tilde{x}$ $\Rightarrow$ let $g = p(\gamma) \Rightarrow \Phi(H[g]) = \hat{\gamma}$.

Injective: Suppose $\Phi(H[g_1]) = \Phi(H[g_2])$ $\Rightarrow$ $\tilde{g}_1(1) = \tilde{g}_2(1)$

$\Rightarrow \tilde{g}_1 \cdot \tilde{g}_2$ is the lift of $g_1 \cdot g_2$. $\Rightarrow [g_1, g_2] \in H$

$\Rightarrow [g_1][g_2]^{-1} \in H \Rightarrow H[g_1] = H[g_2]$.

Q: Suppose $X$ is path connected and locally path connected. Let $H < \pi_1(X, x_0)$. Is there
a path-connected covering space $p: \hat{X} \rightarrow (X, x_0)$ s.t. $p_*(\pi_1(\hat{X}, \hat{x}_0)) = H$?

NO

Def. A top. space $X$ is semi-locally simply connected if for any $x \in X$ there exists
a nbd $U$ of $x$ such that $\pi_1(U, x_2) \rightarrow \pi_1(X, x)$ is trivial.

Why thin Condition is necessary? Let $H$ be the trivial subgroup. Then $p_*(\pi_1(\hat{X}, \hat{x}_0)) = H$
$\Rightarrow \pi_1(\hat{X}, \hat{x}_0)$ is trivial $\Rightarrow \hat{X}$ is simply connected. Take an evenly covered nbd $U$ of $x$ and
let $\tilde{U}$ be a lift of $U$ st. $p: \tilde{U} \rightarrow U$ is a homeo. For any $\tilde{f} \in \pi_1(\tilde{U}, \tilde{x})$, $p^{-1}[\tilde{U}]$. 

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is homotopically trivial in $\tilde{X}$. Compare null homotopy with $p$, shows $f$ is null hom.

in $X \xrightarrow{\pi_1(U, x)} \pi_1(X, z)$ is trivial.

**Prop:** Suppose $X$ is path connected, locally path connected and semi-locally simply connected. Then for any subgroup $H < \pi_1(X, x_0)$ there is a path connected covering space $p: (\tilde{X}_H, \tilde{x}_0) \to (X, x_0)$ such that $p_*(\pi_1(\tilde{X}_H, \tilde{x}_0)) = H$.

**Def:** The simply connected covering space $p: \tilde{X} \to X$ is called the universal cover of $X$.

why? $\tilde{X}$ is a covering space for every other covering space of $X$.

**EX:** $p: \mathbb{R} \to S^1$

$t \mapsto (\cos(2\pi t), \sin(2\pi t))$

$p_n: S^1 \to S^1$

$z \mapsto z^n$

**Covering map**

**Ex:** Möbius band

Universal Cover:

Simply Connected

2-sheeted

cylinder

**Ex:** Torus

Universal Cover: $\mathbb{R}^2$

Similarly: $\mathbb{R}^2$ with an app. map $p$ is the universal cover of the Klein bottle.

**EX:** $\mathbb{R}^2$

Universal Cover
Ex. \( \mathbb{R}P^n \cong S^n_{(x \sim -x)} \Rightarrow q: S^n \to \mathbb{R}P^n \) is a covering space. For \( n \geq 2 \), \( S^n \) is simply connected \( \Rightarrow (S^n, q) \) is the universal cover of \( \mathbb{R}P^n \) for \( n \geq 2 \).

\((\mathbb{R}P^1 \cong S^1 \Rightarrow \text{it's universal cover is } \mathbb{R}^1).\)

**Idea of proof for H = trivial subgroup:**

Suppose \( p: (\tilde{X}, \tilde{x}_0) \to (X, x_0) \) be a simply connected covering space.

- \( \tilde{X} \) is simply connected \( \Rightarrow \) for any \( \tilde{x} \in \tilde{X} \) there is a unique homotopy class of path connecting \( \tilde{x}_0 \) to \( \tilde{x} \).

\[ \tilde{x}_0 \sim \tilde{x}, \quad \tilde{y} \]

- \( \gamma_1 \tilde{y} \) is null homotopic \( \Rightarrow \gamma_1 \tilde{y} \tilde{y}_2 \sim C_{\tilde{y}_2} \Rightarrow \gamma_1 \tilde{y} \tilde{y}_2 \sim C. \gamma_2 \)

- \( \Rightarrow \) any \( \tilde{x} \in \tilde{X} \) \( \iff \) a homotopy class of paths starting at \( \tilde{x}_0 \)

- \( \iff \) a homotopy class of paths starting at \( x_0 \).

\[ \tilde{X} = \{ [x] \mid x \text{ is a path in } X \text{ starting at } x_0 \} \quad p: \tilde{X} \to X \]

**Basis for topology:** \( U = \{ U \subseteq X \mid U \text{ is path connected, open}, \pi_1(U) \to \pi_1(X), \text{ trivial} \} \)

(Ex: \( U \) is a basis of topology for \( X \))

For any \( U \in \mathcal{U} \), and a path \( \gamma \) from \( x_0 \) to a pt in \( U \)

\[ U_{[\gamma]} = \{ [\gamma_0 \eta] \mid \eta \subseteq U \text{ s.t. } \eta(0) = \gamma(1) \} \]

- \( \{ U_{[\gamma]} \} \) form a basis for topology on \( \tilde{X} \)

- \( p: \tilde{X} \to X \) is a covering space \( (p: U_{[\gamma]} \to \gamma(1)) \)

* \( \tilde{X} \) is simply connected.

1. path connected : \([\gamma] \in \tilde{X}\)

\[ \gamma_t(s) = \begin{cases} \gamma(s) & 0 \leq s \leq t \\ \gamma(t) & 0 \leq t \leq 1 \end{cases} \]

\( t \to [\gamma_t] : \text{path in } \tilde{X} \text{ connecting } [\gamma_0] = [x_0] \text{ to } [\gamma_1] = [\gamma]. \)

It's invalid the lift of \( \gamma \) starting at \([\tilde{x}_0]\), because \( p[\gamma_t] = \gamma(t) \).

\( \Rightarrow \) For any \( [\gamma] \in \pi_1(X, x_0) \) lift of \( \gamma \) starting at \([\tilde{x}_0]\) is a path connecting \([\tilde{x}_0]\) to \([\gamma]\). Thus if \( \gamma \) is null homotopic my lift of \( \gamma \) is not a loop.

\( \Rightarrow \text{im}(p_*) \) is the trivial subgroup of \( \pi_1(X, x_0). \)