1. Let $V$ be a vector space over $\mathbb{R}$ and $B : V \otimes V \to \mathbb{R}$ a bilinear form. Suppose that there is a half-dimensional subspace $L$ of $V$ so that $B(v, w) = 0$ for all $v, w \in L$. Prove that the signature of $B$ is zero.

2. Show that if $X$ is closed, oriented, simply connected $n$-manifold with $H_i(X) = 0$ for $0 < i < n$, then it is homotopy equivalent to $S^n$.

3. Let $M$ be a smooth, closed, oriented $n$-manifold and $f : M \to S^{n-4i}$ be a smooth map. Show that for every regular value $y$ of $f$,

$$\langle L_i(M) \cup f^*(u), \mu_M \rangle = \sigma(f^{-1}(y)),$$

where $u$ and $\mu_M$ denote the fundamental cohomology class of $S^{n-4i}$ and fundamental homology class of $M$.

4. a. Let $\xi$ be the underlying oriented 4-plane bundle of the quaternion line bundle over $\mathbb{H}P^m$. Use the Gysin sequence for $\xi$ to compute $H^*(\mathbb{H}P^m)$.

b. For $m = 1$ i.e. $\mathbb{H}P^1 \cong S^4$, show that $p_1(\xi) = -2u$, and $e(\xi) = u$.

c. Milnor-Stasheff 20-A.

5. Milnor-Stasheff 19-A.

6. a. Suppose $X$ be a smooth, closed 4-manifold, and $J$ be an almost complex structure on $X$, i.e. $J$ makes tangent space of $X$ into a complex vector bundle. Show that, considering $J$, $c_1^2[X] = 3\sigma(X) + 2\chi(X)$.

b. Prove that $S^4$ do not admit any almost-complex structure.

7. Fix a closed, smooth, oriented manifold $M^7$ with $H^3(M) = H^4(M) = 0$. Let $B_1$ and $B_2$ be oriented 8-manifolds with boundary $\partial B_i = M$. Let $C = B_1 \cup_M (-B_2)$. Let $i : H^4(B_j, M) \to H^4(B_j)$ be the map from the long exact sequence of a pair. Define $\sigma(B_j)$ to be the signature of the bilinear form

$$H^4(B_j, M; \mathbb{Q}) \otimes H^4(B_j, M; \mathbb{Q}) \to \mathbb{Q}$$

$$(a, b) \mapsto \langle a \cup b, [B_j] \rangle$$

Prove that

$$\langle p_1(TC)^2, [C] \rangle = \langle i^{-1}p_1(TB_1)^2, [B_1] \rangle - \langle i^{-1}p_1(TB_2)^2, [B_2] \rangle$$

$$\sigma(C) = \sigma(B_1) - \sigma(B_2)$$