Homework 1

1. \( f(x) = x^x \) on interval \([0, 5]\)

In \( f(x) = x \ln x \) since this transformation will not change \( x^x \)

Since \( f(x) \) is continuous on \([0, 5]\), a closed and bounded interval, there will be an absolute max and min on \([0, 5]\), and they will occur at a critical point, where either the derivative \( = 0 \) or at the boundary.

\[
\frac{d}{dx} (\ln f(x) = x \ln x)
\]

\[
\frac{1}{f(x)} f'(x) = \ln x + \frac{1}{x} (x)
\]

\[
f'(x) = f(x) (\ln x + 1)
\]

\[
f'(x) = x^x (\ln x + 1)
\]

\[
f'(x) = 0 \text{ when } \ln x = -1
\]

\[
x = e^{-1}
\]

\[
f(x) = (e^{-1})^{e^{-1}} = e^{-\frac{1}{e}} = \frac{1}{e^{\frac{1}{e}}}
\]

\[
f(0) = 1
\]

\[
f(\frac{1}{e}) = \frac{1}{e^{\frac{1}{e}}} = .69 \Rightarrow \text{minimum}
\]

\[
f(5) = 5^5 = 3125 \Rightarrow \text{maximum}
\]

2. \( f(x) = \frac{x^3}{3} + 3x^2 + 8x + 5 \) on interval \([-3, 2]\)

\( f(x) \) is continuous on a compact domain \( \Rightarrow \) abs. min/\( \max \) will occur at a critical point or at boundary.

\[
f'(x) = \frac{1}{3} (3)x^2 + 6x + 8 = x^2 + 2x + \frac{8}{3} = (x+2)(x+2')
\]

\[
f'(x) = 0 \text{ when } x = -1, -2
\]

\[
f(-3) = -\frac{8}{3} + 24 - 18 + 5 = -1
\]

\[
f(-2) = \frac{8}{3} + 12 - 16 + 5 = \frac{5}{3} \Rightarrow \text{min}
\]

\[
f(2) = \frac{8}{3} + 12 + 16 + 5 = \frac{109}{3} \Rightarrow \text{max}
\]
bounded \( \frac{1}{2} \) closed \( \Rightarrow \) min and max will exist and will occur at vertex.

Obj fn: \( z = 4x + 3y \)

Vertices:
- \( (0,4) \Rightarrow z(0,4) = 12 \)
- \( (0,2) \Rightarrow z(0,2) = 6 \)
- \( (3,0) \Rightarrow z(3,0) = 12 \)
- \( (5,3) \Rightarrow z(5,3) = 29 \)

Given the constraints, \( z = 4x + 3y \) achieves its max value of 29 at \((5,3)\) and its min value of 6 at \((0,2)\).

4. min and max will exist, and will occur at vertex if bounded and closed domain.

Obj fn: \( z = x + by \)

Vertices:
- \( (0,4) \Rightarrow z(0,4) = 24 \)
- \( (0,2) \Rightarrow z(0,2) = 12 \)
- \( (3,0) \Rightarrow z(3,0) = 2 \)
- \( (5,3) \Rightarrow z(5,3) = 23 \)

Given the constraints, \( z = x + by \) achieves its max value of 24 at \((0,4)\) and its min value of 3 at \((3,0)\).

6. bounded and closed domain \( \Rightarrow \) min and max will exist and will occur at vertex.

Obj fn: \( z = 50x + 35y \)

Vertices:
- \( (0,600) \Rightarrow z(0,600) = 21,000 \)
- \( (0,800) \Rightarrow z(0,800) = 28,000 \)
- \( (675,0) \Rightarrow z(675,0) = 33,750 \)
- \( (900,0) \Rightarrow z(900,0) = 45,000 \)

Given the constraints, \( z = 50x + 35y \) achieves its max value of 45,000 at \((900,0)\) and its min value of 21,000 at \((0,600)\).

18. Obj fn: \( z = 5x_1 + x_2 \)

Constraints:
- \( 3x_1 + x_2 \leq 15 \)
- \( 4x_1 + 3x_2 \leq 30 \)
- \( x_1 \geq 0, x_2 \geq 0 \)

\( P_1: 4(x_1) + 3(15 - 3x_1) = 30 \)
- \( 4x_1 + 45 - 9x_1 = 30 \)
- \( -5x_1 = -15 \)
- \( x_1 = 3 \)
- \( x_2 = 6 \)

Vertices:
- \( (0,10) \Rightarrow z(0,10) = 10 \)
- \( (3,6) \Rightarrow z(3,6) = 21 \)
- \( (5,0) \Rightarrow z(5,0) = 25 \)

Given the constraints, \( z \) achieves its max of 25 at \((5,0)\).
$x$ is fraction per litre of A, $y$ is fraction per litre of B, $z$ is per litre cost

Obj. fn. $z = 0.83x + 0.98y$

Constraints $80x + 92y \geq 90$, $x + y = 1$

$x \geq 0, y \geq 0,$

$$p_1: 80x + 92(1-x) = 90$$

$$80x + 92 - 92x = 90.$$  

$$-12x = -2$$

$$x = \frac{1}{6}, y = \frac{5}{6}.$$

Vertices: $(0,1) = z(0,1) = 0.98$  

$$\left(\frac{1}{6}, \frac{5}{6}\right) = z\left(\frac{1}{6}, \frac{5}{6}\right) = 0.955$$

The blend that minimises cost is $\frac{1}{6}$ of A and $\frac{5}{6}$ of B.

The cost of this blend is 0.955.

26. Obj. fn.: $z = x + y$

Constraints: $x \geq 0, y \geq 0$

$-x + y = 1$  

$x + 2y = 4$

28. Obj. fn.: $z = x + y$

Constraints: $x \geq 0, y \geq 0$

$-x + y \leq 0$  

$-3x + y \geq 3$

29. Obj. fn.: $3x + 4y = z$

Constraints: $x \geq 0, y \geq 0$

$x + y \leq 1$  

$2x + y \geq 4$

30. Obj. fn.: $z = x + 2y$

Constraints: $x \geq 0, y \geq 0$

$x + 2y \leq 4$  

$2x + y \leq 4$

$P_1: 2(4-2y) + y = 4$  

Vertex:

$(0,2) = z(0,2) = 4$  

$\left(\frac{4}{3}, \frac{4}{3}\right) = z\left(\frac{4}{3}, \frac{4}{3}\right) = 4$  

$y = \frac{4}{3}, x = \frac{4}{3}$  

$(2,0) = z(2,0) = 2$

Solutions are all points on the line segment $x + 2y = 4$ between $(0,2)$ and $\left(\frac{4}{3}, \frac{4}{3}\right)$. 