$P = W$ Conjecture Seminar Notes

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1 Wednesday, October 5

There are now two proofs of the $P = W$ conjecture that appeared a few days apart.

1.1 Betti space

Let C be a a smooth projective curve over $\mathbf{C}$. Considering it in the analytic topology, we have

$$\pi_1(C^{\text{an}}) = \langle \alpha_1, \beta_1, ..., \alpha_g, \beta_g \rangle / (\prod [\alpha_i, \beta_i] = 1)$$

Definition 1.1 (Betti space = character variety)**.**

$$\mathcal{M}_B(n, 0) := \text{Rep}(n, C) // \text{GL}_n(\mathbf{C})$$

It is a GIT quotient of $\text{Rep}(n, C) := \{\rho : \pi_1(C^{\text{an}}) \to \text{GL}_n(\mathbf{C})\}$.

Fact 1.2. *$\mathcal{M}_B(n, 0)$ is an affine variety.*

Indeed, $\text{Rep}(n, C)$ is an affine variety given as

$$\{(A_1, B_1, ..., A_g, B_g) \in \text{GL}_n(\mathbf{C})^{2g} : \prod [A_i, B_i] = 1\}$$

and a GIT-quotient of an affine variety is affine.

Definition 1.3 (Twisted character variety)**.**

$$\text{Rep}(n, \ell, C) := \{(A_1, B_1, ..., A_g, B_g) : \prod [A_i, B_i] = \mu^\ell\}$$

where μ is a primitive n-th root of unity. The twisted character variety is $\mathcal{M}_B(n, \ell) := \text{Rep}(n, \ell, C) // \text{GL}_n(\mathbf{C})$.

Fact 1.4. *$\mathcal{M}_B(n, \ell)$ is smooth iff $(n, \ell) = 1$.*

For $n = 1$, $\text{GL}_1 = \mathbf{C}^\times$, so

$$\mathcal{M}_B(n, 0) \simeq \text{Hom}(\pi_1(C^{\text{an}}), \mathbf{C}^\times) \simeq \text{Hom}(H_1(C^{\text{an}}), \mathbf{C}^\times) \simeq (\mathbf{C}^\times)^{2g}$$

1.2 De Rham space

Definition 1.5.

- Let $X/\mathbf{C}$ be a complex manifold, E be a complex vector bundle on X. A *connection* is a differential operator

$$\nabla : C^\infty(E) \to C^\infty(E \otimes \Omega^1)$$

  with the property that

$$\nabla(fs) = f\nabla s + df \otimes s$$

  for a smooth function f and a section s.

- The *curvature* of ∇ is $F_\nabla := \nabla^2 \in C^\infty(\mathrm{End}\ E \otimes \Omega^1)$.

- The connection is *flat* if $F_\nabla = 0$.

Theorem 1.6. *There is a one to one correspondence between Rep(n, X), local systems on X^{an} of rank n, and vector bundles on X with a flat connection.*

Local systems yield a representation via the monodromy representation, and a local system L yields a vector bundle $L \otimes_{\mathbf{C}} \mathcal{O}_X^{\mathrm{an}}$ with $\nabla(L \otimes 1) = 0$.

Let E be a holomorphic vector bundle on X. One can define an operator

$$\overline{\partial}_E : C^\infty(E) \to C^\infty(E \otimes \Omega^{0,1})$$

as follows. If s_1,...,s_n a local basis of holomorphic sections of E then

$$\overline{\partial}_E(\sum f_i s_i) := \sum \overline{\partial} f_i \otimes s_i$$

The operator $\overline{\partial}_E$ satisfies $\overline{\partial}_E(fs) = f\overline{\partial}_E s + \overline{\partial} f \otimes s$ and $\overline{\partial}_E^2 = 0$.

Theorem 1.7. *Holomorphic vector bundles correspond to complex vector bundles with an operator $\overline{\partial}$ satisfying the above requirements.*

Consider a connection

$$\nabla\colon C^\infty \to C^\infty(E \otimes \Omega^1) \simeq C^\infty(E \otimes \Omega^{1,0}) \oplus C^\infty(E \otimes \Omega^{0,1})$$

The *(0,1)-part* $\nabla^{0,1}$ of ∇ is defined to be the composition of ∇ with the projection $C^\infty(E \otimes \Omega^1) \to C^\infty(E \otimes \Omega^{0,1})$. The flatness of ∇ implies that $(\nabla^{0,1})^2 = 0$. We thus obtain a holomorphic structure on E and ∇ becomes a holomorphic connection. By definition, a *holomorphic connection* ∇ on a holomorphic vector bundle E is a connection such that $\nabla^{0,1} = \overline{\partial}_E$.

Fact 1.8. *Let C be a projective curve. Complex vector bundles with a flat connection on C are in one-to-one correspondence with holomorphic vector bundles E with a holomorphic connection. In their turn, they are in one-to-one correspondence with algebraic vector bundles with an algebraic connection (by GAGA).*

Definition 1.9. The *de Rham moduli space* $\mathcal{M}_{dR}(n)$ parametrizes algebraic vector bundles on C with an algebraic connection.

A twisted version also exists, which we use here:

Theorem 1.10. *There exists a biholomorphism between $\mathcal{M}_B(n,\ell)$ and $\mathcal{M}_{dR}(n,\ell)$. This biholomorphism is not algebraic.*

Example. $n=1$. A point in $\mathcal{M}_{dR}(1)$ corresponds to a pair $\{L, D\}$ with L a line bundle and D a connection. By forgetting the connection, we get a map $\mathcal{M}_{dR}(1) \to \mathrm{Jac}(C)$.

If ∇, ∇' are algebraic connections, then $\nabla' - \nabla \in H^0(K_C)$ is a holomorphic form. Conversely, a sum of a connection and a 1-form is again a connection. Hence $\mathcal{M}_{dR}(1)$ is an affine bundle over $\mathrm{Jac}(C)$.

Affine bundles over $\Omega^1_{\mathrm{Jac}(C)}$ correspond to elements in $H^1(\mathrm{Jac}(C), \Omega^1_{\mathrm{Jac}(C)})$. The class of $\mathcal{M}_{dR}(1)$ is the class of theta divisor, which we will not prove. We have a biholomorphism

$$(\mathbf{C}^\times)^{2g} \simeq \mathcal{M}_{dR}(1)$$

It cannot preserve algebraic structures because $(\mathbf{C}^\times)^{2g}$ does not admit a non-trivial map to an abelian variety.

1.3 Dolbeault space

Definition 1.11. Let E be a vector bundle. A *Higgs field* is $\psi \in \mathrm{End}(E) \otimes K_C$. A pair (E, ψ) is called a *Higgs bundle.*

The pair (E, ψ) is *(semi)-stable* if for every ψ-stable sub-bundle $F \subset E$,

$$\frac{\deg F}{\mathrm{rk}\ F} \leq \frac{\deg E}{\mathrm{rk}\ E}$$

($<$ for stable).

Theorem 1.12. *There exists an algebraic variety $\mathcal{M}_{Dol}(n,d)$ called* Dolbeault moduli space *parameterizing semi-stable Higgs bundles.*

1. *It's quasi-projective*
2. *The subspace corresponding to stable bundles $\mathcal{M}^s_{Dol}(n,d) \subset \mathcal{M}_{Dol}(n,d)$ is smooth and Zariski open*
3. *If $(n,d)=1$ then $\mathcal{M}_{Dol}(n,d)$ is smooth.*
4. *The moduli of semistable vector bundles embeds $M(n,d) \hookrightarrow \mathcal{M}_{Dol}(n,d)$ as a subset of Higgs bundles with a trivial Higgs field.*
5. *The variety $T^*M^s(n,d)$ embeds into $\mathcal{M}_{Dol}(n,d)$ as a Zariski open dense. Indeed,*

$$T_{[E]}M(n,d) \simeq H^1(C, End\ E) \simeq H^0(End(E) \otimes K_C)^*$$

6. *There exists a holomorphic symplectic form on $\mathcal{M}_{Dol}(n,d)$ which is an extension of the standard holomorphic symplectic form on $T^*M^s(n,d)$.*

Theorem 1.13 (Non-abelian Hodge correspondence)**.** *There exists a real analytic isomorphism*

$$\mathcal{M}_{Dol}(n,d) \simeq \mathcal{M}_{dR}(n,\ell).$$

where d is a function of l. Moreover, the following holds.

- *There exists a family $\pi : \mathcal{M}_{Hdg}(n,d) \to \mathbf{A}^1$ with $\pi^{-1}(0) \simeq \mathcal{M}_{Dol}$ and $\pi^{-1}(1) \simeq \mathcal{M}_{dR}$;*
- *$\overline{\mathcal{M}}_{Hdg} \to \mathbf{A}^1$ is a real analytically trivial deformation.*

Hence we have isomorphisms of three moduli spaces

$$\mathcal{M}_{\mathrm{Dol}} \xrightarrow[\text{diffeo}]{\simeq} \mathcal{M}_{dR}(n,\ell) \xrightarrow[\text{biholo}]{\simeq} \mathcal{M}_B(n,\ell).$$

They give us isomorphisms of cohomology groups:

$$H^\bullet(\mathcal{M}_{\mathrm{Dol}}) \simeq H(\mathcal{M}_{dR}) \simeq H^\bullet(\mathcal{M}_B)$$

We will soon endow the leftmost group with a filtration called *P-filtration*, and the rightmost one with *W-filtration.*

1.4 Perverse filtration

Let $\psi \in \mathrm{End}(E) \otimes K_C$ be a Higgs field. Send

$$\psi \mapsto (\mathrm{Tr}\psi, \mathrm{Tr} \wedge^2 \psi, ..., \mathrm{Tr} \wedge^n \psi)$$

where $\mathrm{Tr} \wedge^k \psi \in H^0(K_C^{\otimes n})$. Locally $\psi = \varphi \otimes dz$ where $\varphi \in \mathrm{End}(E)$ and ψ is mapped to $(\mathrm{Tr}\varphi \otimes dz, ...\mathrm{Tr} \wedge^n \otimes dz)$.

Definition 1.14. The *Hitchin map* is the map

$$\mathcal{M}_{\mathrm{Dol}} \to \mathbf{A}^n, \quad (E,\psi) \mapsto (\mathrm{Tr}\psi, ..., \mathrm{Tr} \wedge^n \psi)$$

Fact 1.15. *The Hitchin map is a projective morphism. Moreover, it is a Lagrangian fibration. Hence a generic fiber is an abelian variety.*

Theorem 1.16 (Deligne)**.** *For $f\colon X \to Y$ smooth proper morphism with X smooth there exists a direct sum decomposition*

$$Rf_*\mathbf{Q}_X \simeq \bigoplus_{i=0}^{2d} R^i f_* \mathbf{Q}_x[-i]$$

Theorem 1.17 (BBDG)**.** *For $f : X \to Y$ proper with X smooth we have a decomposition*

$$Rf_*\mathbf{Q}_X[\dim X] \simeq \bigoplus_{i=0}^{2d} {}^P\mathcal{H}^i(Rf_*\mathbf{Q}_X[\dim X])$$

where ${}^P\mathcal{H}^i$ are perverse cohomology groups (not defined today).

It follows that there is a decomposition

$$H^\bullet(X, \mathbf{Q}_X[\dim X]) \simeq \bigoplus R\Gamma({}^P\mathcal{H}^i(Rf_*\mathbf{Q}_X[\dim X]))$$

Direct sum is not canonical, but the following filtration is:

Definition 1.18. *Perverse filtration* on $H^\bullet(X, \mathbf{Q})$ is defined as

$$P_\ell(H^\bullet(X, \mathbf{Q}[\dim X])) = \bigoplus_{i=0}^{\ell} R\Gamma({}^P\mathcal{H}^i[\dim X])$$

Theorem 1.19 (de Cataldo, Migliorini)**.** *Let $f : X \to Y$ be a proper map, X smooth, Y affine. Embed Y into $\mathbf{A}^N$. Then*

$$P_\ell H^K(X) = \ker[H^K(X) \to H^K(f^{-1}\mathbf{A}^{N-K+\ell+1} \cap Y)]$$

The Hitchin map $h : \mathcal{M}_{\mathrm{Dol}} \to \mathbf{A}^N$ induces a filtration $P_\ell H^\bullet(\mathcal{M}_{\mathrm{Dol}})$ for $(n, d) = 1$.

Example. Let $\psi \in H^0(\mathrm{End}(E) \otimes K_C)$ be a Higgs field. When $n = \mathrm{rk}\, E = 1$ we have $\psi \in H^0(K_C)$. Hence

$$\mathcal{M}_{\mathrm{Dol}} = \mathrm{Jac}(C) \times H^0(K_C) \simeq T^*\mathrm{Jac}(C)$$

The Hitchin map is identified with the projection $h : \mathcal{M}_{\mathrm{Dol}} \to H^0(K_C)$ to the second factor. The perverse filtration is trivial i.e.

$$P_\ell H^\ell(\mathcal{M}_{\mathrm{Dol}}) = H^\ell(\mathcal{M}_{\mathrm{Dol}})$$

Definition 1.20. A *mixed Hodge structure* is a $\mathbf{Q}$-vector space $V_\mathbf{Q}$ with two filtrations

- an increasing weight filtration $W^\bullet V_\mathbf{Q}$;
- a decreasing Hodge filtration $F_\bullet V_\mathbf{C}$;

such that $\mathrm{Gr}_i^W V_\mathbf{Q}$ is a pure Hodge structure $\forall i$.

Theorem 1.21 (Deligne)**.**

- *The cohomology $H^\bullet(X)$ of an algebraic variety X has a natural mixed Hodge structure.*
- *If X is smooth, weights on $H^i(X)$ are $[i, ..., 2i]$*
- *Let $X \subset \overline{X}$ be a compactification of X. Then*

  $$W^i H^i(X) = \mathrm{im}(H^i(\overline{X}) \to H^i(X))$$

- *MHS depends on an algebraic structure!*

Example. Let us compute the MHS on the first cohomology of $\mathcal{M}_B(1) = (\mathbf{C}^\times)^{2g}$. First, let us embed $\mathbf{C}^\times$ in $\mathbf{P}^1$. The previous theorem implies that $W^1H^1(\mathbf{C}^\times) = \mathrm{im}\, H^1(\mathbf{P}^1) = 0$. Then $W^2H^1 = H^2$. The mixed Hodge structure on $H^1(\mathbf{C}^\times)$ is pure of weight 2. Kunneth formula implies that the same is true for $H^1(\mathcal{M}_B(1))$.

Consider the projection $\mathcal{M}_{dR}(1) \to \mathrm{Jac}(C)$. It induces an isomorphism $H^1(\mathrm{Jac}(C)) \simeq H^1(\mathcal{M}_{dR}(1))$, which is an isomorphism of mixed Hodge structures. Hence $H^1(\mathcal{M}_{dR}(1))$ is a pure Hodge structure of weight 1. We see that the biholomorphism between $\mathcal{M}_{dR}(1)$ and $\mathcal{M}_B(1)$ does not preserve mixed Hodge structures.

Theorem 1.22 (Curious Hard Lefschetz)**.** *Consider the variety $\mathcal{M}_B(n,\ell)$, $(n,\ell)=1$.*

1. *There exists a class $\alpha \in H^2(\mathcal{M}_B(n,\ell))$ such that $\forall K, i$ the multiplication by α^K*

$$\alpha^K : Gr^W_{\dim \mathcal{M}_B - 2K} H^i \to Gr^W_{\dim \mathcal{M}_B + 2K} H^{i+2K}$$

is an isomorphism.

2. *$W_{2K-1}H^\bullet = W_{2K}H^\bullet$ and all classes in $Gr^W_{2K}H^i$ are of type (K,K).*

Recall that we have a diffeomorphism $\mathcal{M}_{\mathrm{Dol}} \simeq \mathcal{M}_B$ so $H^0(\mathcal{M}_{\mathrm{Dol}}) \simeq H^0(\mathcal{M}_B)$.

Theorem 1.23 (Maulik-Shen'22 Hausel-Mellit-Minetz-Shiffman'22)**.** *The perverse and weight filtrations essentially coincide i.e.*

$$P_K H^\bullet(\mathcal{M}_{Dol}(n,d)) = W_{2K} H^\bullet(\mathcal{M}_B(n,d)).$$

Consider the rank 1 case. On $\mathcal{M}_B$ we have

$$W_1(H^1(\mathcal{M}_B)) = 0$$

$$W_2(H^1(\mathcal{M}_B)) = H^1(\mathcal{M}_B) = \mathbf{Q}^{2g}$$

$$P_1 H^1(\mathcal{M}_{\mathrm{Dol}}) \simeq H^1(\mathcal{M}_{\mathrm{Dol}}) = H^1(\mathrm{Jac}(C)) \simeq \mathbf{Q}^{2g}$$

so $W_2(H^1(\mathcal{M}_B)) = P_1 H^1(\mathcal{M}_{\mathrm{Dol}})$ as the theorem predicts.