# From Feix–Kaledin metric to algebraic geometry

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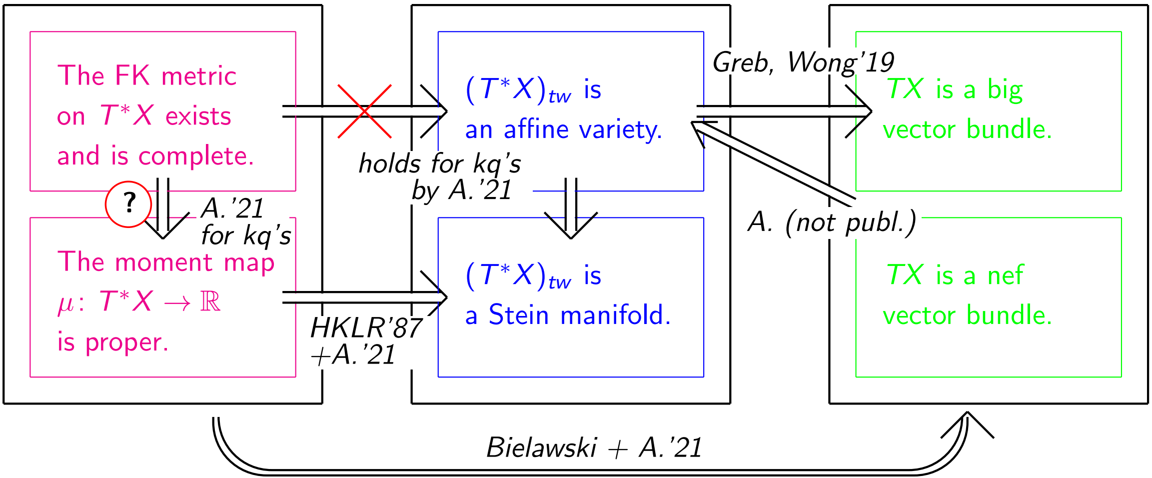
# Overview of the talk

1. Hyperkähler geometry
2. Feix–Kaledin metric and twisted cotangent bundles
3. When are twisted cotangent bundles Stein/affine?
4. Big and nef tangent bundles

Most results mentioned in the talk are contained in

[A.'21] A. Abasheva. *Feix–Kaledin metric on the total spaces of cotangent bundles to Kähler quotients*, Int. Math. Res. Not., 2021, rnab047, https://doi.org/10.1093/imrn/rnab047, arXiv:2007.05773

# The whole picture



# Complex structures on manifolds

Let $V$ be a vector space, $I \in \mathrm{End}(V)$, $I^2 = -1$ an **almost complex structure**. Consider the eigenvalue decomposition

$$V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$$

$$Ix = \sqrt{-1}x \text{ for } x \in V^{1,0}, Ix = -\sqrt{-1}x \text{ for } x \in V^{0,1}$$

Consider a smooth manifold $X$ equipped with an **almost complex structure** $I \in \mathrm{End}(TX)$. Then one has the decomposition

$$TX \otimes \mathbb{C} = T^{1,0}X \oplus T^{0,1}X$$

### Definition

An almost complex structure $I$ on $X$ is called **integrable** or just a **complex structure** if

$$[T^{1,0}X, T^{1,0}X] \subset T^{1,0}X$$

# Kähler manifolds

Let $V$ be a vector space with a complex structure $I$. Let $g$ be an **Hermitian metric** on $V$ i.e. a Euclidean metric on $V$ s.t.

$$g(Iv, Iu) = g(v, u)$$

Then $\omega(v, u) := g(Iv, u)$ is a skew-symmetric 2-form. Let $X$ be a complex manifold, $g$ a Hermitian metric on $X$, $\omega(v, u) := g(Iv, u)$.

## Definition

A complex manifold $X$ is called **Kähler** if $d\omega = 0$.

## Examples

Examples of Kähler manifolds

1. $\mathbb{C}P^n$, all smooth projective varieties $X \subset \mathbb{C}P^n$;
2. Complex tori $\mathbb{C}^n/\Lambda$;
3. A complex submanifold of a Kähler manifold is Kähler.

# Hypercomplex manifolds

Notation: $\mathbb{H}$ is the quaternion algebra, it is generated by $I, J, K$,
$I^2 = J^2 = K^2 = -1, IJ = -JI = K$.

**Fact**

An element $L \in \mathbb{H}$ satisfies $L^2 = -1$ iff $L = xI + yJ + zK$, $x^2 + y^2 + z^2 = 1$.

**Definition**

A manifold $X$ is called **almost hypercomplex** if $\mathbb{H}$ acts on $TX$. It is called **hypercomplex** if every complex structure on $X$ induced from $\mathbb{H}$ is integrable.

# Hyperkähler manifolds

**Definition**

Let $(X, g)$ be a hypercomplex manifold, $g$ a metric on $X$ which is Hermitian wrt $I, J, K$. Define $\omega_L(x, y) := g(Lx, y)$ for $L = I, J, K$. If $d\omega_L = 0$ for $L = I, J, K$ then $X$ is called **hyperkähler**.

Consider the 2-form $\Omega := \omega_J + \sqrt{-1}\omega_K$

**Fact**

The form $\Omega$ is **holomorphic symplectic** i.e. it is a closed non-degenerate holomorphic $(2, 0)$-form wrt the complex structure $I$.

**Remark**

If $X$ is compact then **Kähler + holomorphic symplectic** implies **hyperkähler** by Calabi–Yau. If $X$ is non-compact then Calabi–Yau does not work.

# Twistor spaces

Let $X$ be a hyperkähler manifold. We are going to construct the **twistor space** $\mathrm{Tw}(X)$ of $X$.

$$\mathrm{Tw}(X) = X \times S^2$$

as a real manifold. An (a priori almost) complex structure $I$ on $\mathrm{Tw}(X)$ is

$$I_{(x,t)}(v,u) := (I_t v, I_{\mathbb{C}P^1} u)$$

where $I_t$ is the complex structure corresponding to $t \in S^2 \subset \mathbb{H}$.

**Fact**

The almost complex structure $I$ on $\mathrm{Tw}(X)$ is integrable.

There is a natural holomorphic map

$$\mathrm{Tw}(X) \to \mathbb{C}P^1 \quad (x,t) \mapsto t$$

# Cotangent bundles

Let $X$ be a complex manifold. Consider the total space $T^*X$ of the cotangent bundle to $X$, $\pi\colon T^*X \to X$. The **tautological holomorphic** 1**-form on** $T^*X$ is defined as

$$\tau_{x,\alpha}(v) := \alpha(\pi_* v)$$

where $x \in X$, $\alpha \in T^*_x X$. Or in holomorphic coordinates $(z^1, ...z^n, w^1 := \frac{\partial}{\partial z^1}, ...w_n := \frac{\partial}{\partial z^n})$ on $T^*X$

$$\tau = \sum_{i=1}^{n} w^i dz^i$$

Define

$$\Omega := -d\tau = \sum_{i=1}^{n} dz^i \wedge dw^i$$

### Fact

$\Omega$ is a holomorphic symplectic form on $T^*X$. It is called the **standard holomorphic symplectic form on** $T^*X$**.**

# Feix–Kaledin theorem

$T^*X$ comes with the holomorphic action of $\mathrm{U}(1) \subset \mathbb{C}^\times$ by fiberwise multiplications. This action satisfies $\lambda^*\Omega = \lambda\Omega$, $\forall \lambda \in \mathrm{U}(1)$.

**Theorem (B. Feix'01, D. Kaledin'99)**

Let $X$ be a Kähler manifold with a real analytic metric $g$. Then there exists a $\mathrm{U}(1)$-invariant neighbourhood of the zero section $X \subset T^*X$ and a hyperkähler metric $(h, I, J, K)$ on $T^*X$ s.t.

- The corresponding holomorphic symplectic form $\Omega := \omega_J + \sqrt{-1}\omega_K$ is the standard one.
- The action of $\mathrm{U}(1)$ preserves $h$ and "rotates the complex structures" i.e.

$$\mathrm{L}_\xi I = 0, \quad \mathrm{L}_\xi J = K, \quad \mathrm{L}_\xi K = -J$$

  where $\xi$ is the vector field tangent to the $\mathrm{U}(1)$-action.
- The metric $h$ restricts to the given Kähler metric on $X$ embedded as the zero section.

Both existing proofs are non-trivial.

# Twisted cotangent bundles. Part 1

Let $X$ be a complex manifold, $V, W$ two vector bundles on $X$.

**Fact**

The isomorphism classes of short exact sequences of the form

$$0 \to V \to E \to W \to 0$$

are classified by the group $H^1(X, W^* \otimes V))$.

Assume that $(X, \omega)$ is compact Kähler. Consider $W = \mathcal{O}_X$, $V = \Omega_X$. Then the extensions of the form

$$0 \to \Omega_X \to \mathcal{E} \to \mathcal{O}_X \to 0$$

are classified by $H^1(X, \Omega_X) = H^{1,1}(X)$. **There is the natural extension $\mathcal{E}$ corresponding to the Kähler class $[\omega] \in H^{1,1}(X)$.**

# Twisted cotangent bundles. Part 2

Consider the total space $E$ of the vector bundle $\mathcal{E}$ constructed in the previous slide as an extension

$$0 \to \Omega_X \to \mathcal{E} \to \mathcal{O}_X \to 0$$

The complex manifold $E$ comes together with the holomorphic map

$$P\colon E \to X \times \mathbb{C} \to \mathbb{C} \subset \mathbb{C}P^1$$

The fiber of $P$ over $0 \in \mathbb{C}$ is $T^*X$.

**Definition**

The fiber of $P$ over $1 \in \mathbb{C}$ is called the **twisted cotangent bundle** of $X$ and is denoted by $(T^*X)_{tw}$

# Twisted cotangent bundles. Part 3

**Theorem(Feix, Kaledin + A.'21)**

Let $X$ be a Kähler manifold. Assume that the Feix–Kaledin metric is defined on $T^*X$. Let $P_1 : \mathrm{Tw}(T^*X) \to \mathbb{C}P^1$ be the twistor projection. Then there exists an isomorphism $E \cong P_1^{-1}(\mathbb{C})$ s.t. the following diagram is commutative.

$$\begin{array}{ccc} E & \longrightarrow & \mathrm{Tw}(X) \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle P_1} \\ \mathbb{C} & \longrightarrow & \mathbb{C}P^1 \end{array}$$

**Corollary**

In the assumptions of the theorem above the twisted cotangent bundle $(T^*X)_{tw}$ is isomorphic to $(T^*X)_J$ as a complex manifold.

# Example: $\mathbb{C}P^n$

Consider the **Euler exact sequence** of vector bundles on $\mathbb{P}^n = \mathbb{P}(V)$ (the set of lines $l \subset V$).

$$0 \to \Omega_{\mathbb{P}(V)} \to \mathcal{O}_{\mathbb{P}(V)}(-1) \otimes V^* \to \mathcal{O}_{\mathbb{P}(V)} \to 0$$

This short exact sequence is non-trivial and $h^{1,1}(\mathbb{P}(V)) = 1 \Longrightarrow E = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}(V)}(-1) \otimes V^*)$

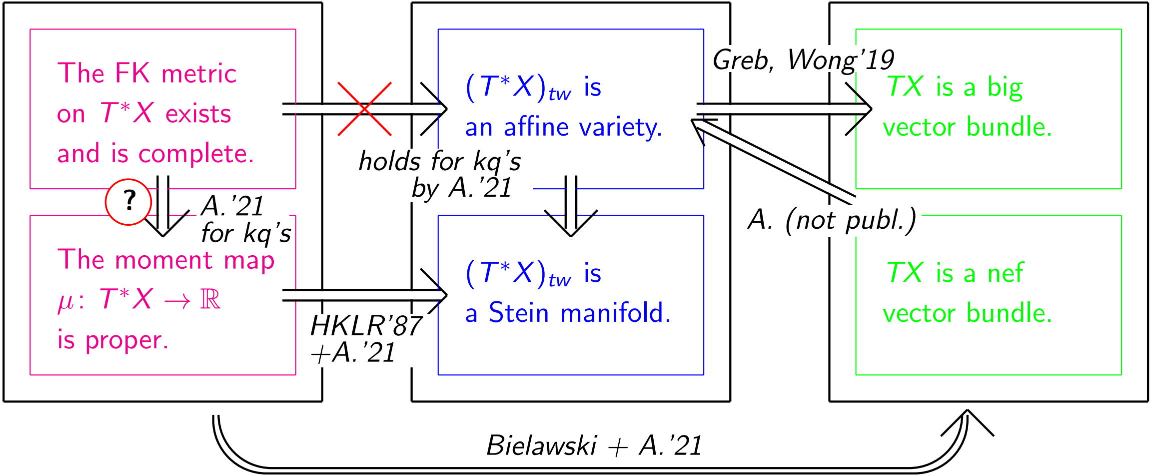
The fiber of $E$ over $[l] \in \mathbb{P}(V)$ is $l \otimes V^* \subset V \otimes V^* = \mathrm{End}(V)$. We obtain a map

$$F \colon E \longrightarrow \{A \in \mathrm{End}(V) \mid \mathrm{rk}\, A \leq 1\}$$

The map $F$ is a **birational morphism** and is an **isomorphism outside of the zero section** $\mathbb{P}(V) \subset E$. This map identifies $(T^*\mathbb{P}(V))_{tw}$ with the affine variety

$$\{A \in \mathrm{End(V)} \mid \mathrm{rk}\, A = 1,\ \mathrm{tr}\, A = 1\}$$

# The whole picture



# The moment for the $\mathrm{U}(1)$-action

Assume that the Feix–Kaledin metric is defined on $T^*X$. Let $\xi$ be the vector field tangent to the $\mathrm{U}(1)$-action. One can show that there exists a **moment map** $\mu\colon T^*X \to \mathbb{R}_{\geq 0}$ for the $\mathrm{U}(1)$-action i.e. the function s.t.

$$d\mu = -\iota_\xi \omega_I$$

**Fact (HKLR'87)**

The function $\mu$ is a **Kähler potential** on $(T^*X)_J$ i.e.

$$dd^c_J \mu = \omega_J, \quad d^c_J = JdJ$$

**Proof:**

$$dd^c_J \mu = dJd\mu = -dJ(\omega_I(\xi, -)) = -d\omega_I(\xi, J-) = dh(\xi, IJ-) = -d\omega_K(\xi, -) = -\mathrm{L}_\xi \omega_K = \omega_J$$

# When $(T^*X)_{tw}$ is Stein? Part 1

**Definition**

A complex manifold $Y$ is called **Stein** if it is a complex submanifold of $\mathbb{C}^N$.

**(solution to) Levi's problem**

A complex manifold $Y$ is Stein iff $\exists$ a proper function $\mu\colon Y \to \mathbb{R}_{\geq 0}$ s.t. $dd^c\mu$ is a Kähler form.

**Corollary**

Assume that the FK metric is defined on $T^*X$. Suppose that the moment map $\mu\colon T^*X \to \mathbb{R}_{\geq 0}$ is proper. Then $(T^*X)_{tw}$ is a Stein manifold.

**Proof:** According to the previous slide $dd^c_J\mu = \omega_J$ is a Kähler form on $(T^*X)_J$. By Levi's problem $(T^*X)_J$ is Stein. The complex manifold $(T^*X)_{tw}$ is isomorphic to $(T^*X)_J$.

# When $(T^*X)_{tw}$ is Stein? Part 2

**Open question**

Assume that the FK metric on $T^*X$ exists and is complete. Does it follow that the moment map $\mu\colon T^*X \to \mathbb{R}_{\geq 0}$ is proper?

**Remark 1**

The question above has an affirmative answer when $X$ is a **Kähler quotient** by [A.'21] (see the next slide for the definition of a Kähler quotient).

**Remark 2**

Stein manifolds are affine. But there exists a **smooth non-affine algebraic variety** $Y$ **whose analytification is Stein**. In addition to that, such a manifold $Y$ might be **biholomorphic to an affine variety**. An example is given by $(T^*X)_{tw}$ for $X$ a torus (see f.e. [Greb, Wong'19]).

# A digression on Kähler quotients

Let $G$ be a connected compact Lie group acting holomorphically on a Kähler manifold $(X, \omega)$. Assume that $G$ preserves $\omega$. Then $\forall \xi \in \mathfrak{g}$

$$0 = \mathrm{L}_\xi \omega = d\iota_\xi \omega + \iota_\xi d\omega = d\iota_\xi \omega$$

Suppose that $\exists$ a $G$-equivariant smooth function $\mu \colon X \to \mathfrak{g}^*$ s.t.

$$d\mu_\xi = -\iota_\xi \omega, \quad \mu_\xi(x) := \langle \mu(x), \xi \rangle$$

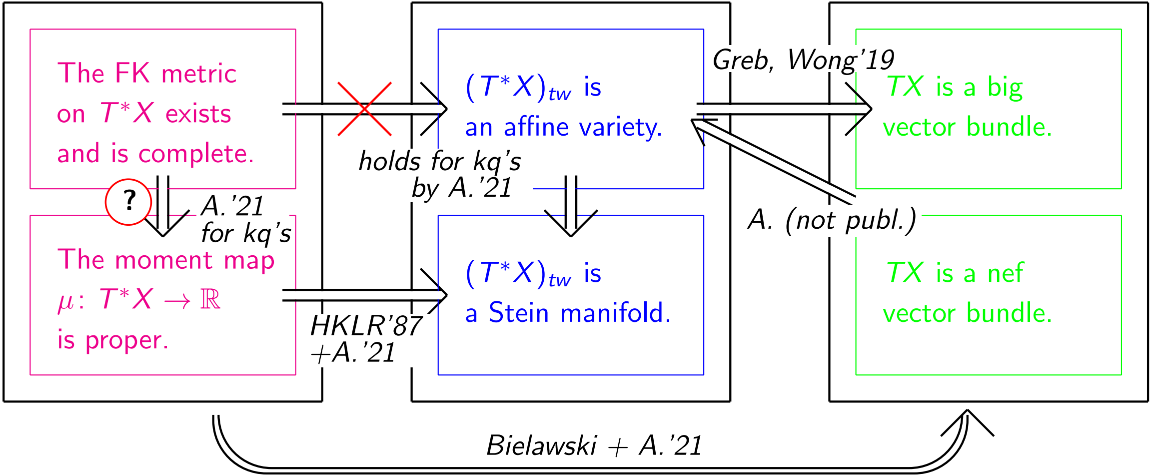
Then $\mu$ is called a **moment map**.

**Theorem (Marsden–Weinstein)**

The subset $\mu^{-1}(t)$ is $G$-invariant for every $t \in \mathfrak{g}^*$. If $t$ is a regular value of $\mu \colon X \to \mathfrak{g}^*$ then the quotient $\mu^{-1}(t)/G$ is naturally a Kähler manifold.

Let $X$ is be an Hermitian vector space, $G \to \mathrm{U}(V)$ a unitary representation. In this case quotients $\mu^{-1}(t)/G$ are called **Kähler quotients**. They include toric varieties, grassmanians, moduli of hypersurfaces etc

# The whole picture



# Big and nef vector bundles

**Definition 1**

Let $L$ be a line bundle on a compact Kähler manifold $X$. It is called

- **nef = numerically effective** if $c_1(L) \in H^{1,1}(X) \cap H^2(X, \mathbb{Z})$ can be realized as a limit of Kähler classes;
- **big** if $h^0(L^d) = O(d^n)$ where $n = \dim X$.

**Ample** line bundles can be defined as those for which $c_1(L)$ is a Kähler class. They are big and nef. The opposite is not true.

Let $V$ be a vector bundle on a Kähler manifold $X$. Let $\mathbb{P}(V)$ denote the **projectivization** of $V$ (the set of **hyperplanes** in $V$). The manifold $\mathbb{P}(V)$ is equipped with the natural line bundle $\mathcal{O}_{\mathbb{P}(V)}(1)$.

**Definition 2**

A vector bundle $V$ is called **big/nef/ample** if $\mathcal{O}_{\mathbb{P}(V)}(1)$ a big/nef/ample line bundle on $\mathbb{P}(V)$.

# From Feix–Kaledin metric towards Campana–Peternell

## (a version of the) Campana–Peternell conjecture

Let $X$ be a compact Kähler manifold whose tangent bundle $TX$ is **big and nef**. Then $X$ is a **complex homogeneous variety**.

## Remark

In the assumptions of the conjecture above the manifold $X$ is necessarily **Fano** (in particular, projective).

## Theorem (Greb,Wong'19)

Assume that $(T^*X)_{tw}$ is **affine**. Then the vector bundle $TX$ is **big**.

## Theorem (Bielawski + A.'21)

Assume that the FK metric exists on $T^*X$ and is **complete**. Assume also that the moment map $\mu\colon T^*X \to \mathbb{R}_{\geq 0}$ is **proper**. Then the vector bundle $TX$ is **nef**.

# Nef tangent bundle 1

**Theorem (Bielawski + A.'21)**

Assume that the FK metric exists on $T^*X$ and is **complete**. Assume also that the moment map $\mu\colon T^*X \to \mathbb{R}_{\geq 0}$ is **proper**. Then the vector bundle $TX$ is **nef**.

**A sketch of the proof: Step 1.** Choose $\mu\colon T^*X \to \mathbb{R}_{\geq 0}$ in such a way that $X = \mu^{-1}(0)$. For any $t > 0$ one has the following diagram

$$\begin{array}{ccc} \mu^{-1}(t) & \xrightarrow{\iota_t} & T^*X \setminus \{0\} \\ {\scriptstyle p_t}\downarrow & & \downarrow \\ \mu^{-1}(t)/\mathrm{U}(1) & \longrightarrow & (T^*X \setminus \{0\})/\mathbb{C}^\times = \mathbb{P}(TX) \end{array}$$

One can show that the map $\mu^{-1}(t)/\mathrm{U}(1) \longrightarrow \mathbb{P}(TX)$ is an isomorphism of complex manifolds $\forall t > 0$. Thus we obtain a family $\omega_t$ of Kähler forms on $\mathbb{P}(TX)$ s.t. $p_t^*\omega_t = \iota_t^*\omega_I$.

# Nef tangent bundle 2

**Theorem (Bielawski + A.'21)**

Assume that the FK metric exists on $T^*X$ and is **complete**. Assume also that the moment map $\mu\colon T^*X \to \mathbb{R}_{\geq 0}$ is **proper**. Then the vector bundle $TX$ is **nef**.

**Step 2.** In [Pedersen, Poon'91] the authors study Kähler Ricci-flat manifolds with a $\mathrm{U}(1)$-action. It follows from their results that the forms $\omega_t$ on $\mathbb{P}(TX)$ satisfy the following differential equation:

$$C\frac{d}{dt}\omega_t = \rho_t + dd^c f_t$$

where $C$ is a constant, $\rho_t$ is the Ricci form of $\omega_t$ and $f_t\colon \mathbb{P}(TX) \to \mathbb{R}$ is a smooth family of functions. In [Bielawski'02] the author proves that $C = \dim_{\mathbb{C}} X =: n$. As $[\rho_t] = c_1(\mathbb{P}(TX))$ we obtain

$$n\frac{d}{dt}[\omega_t] = c_1(\mathbb{P}(TX))$$

# Nef tangent bundle 3

**Theorem (Bielawski + A.'21)**

Assume that the FK metric exists on $T^*X$ and is **complete**. Assume also that the moment map $\mu\colon T^*X \to \mathbb{R}_{\geq 0}$ is **proper**. Then the vector bundle $TX$ is **nef**.

**Step 3:** Let $\pi\colon Z := \mathbb{P}(X) \to X$ denote the natural projection, $n := \dim X$. By using the relative Euler exact sequence for $\mathbb{P}(TX)$

$$0 \to \Omega_{Z/X} \to \pi^* TX(-1) \to \mathcal{O}_Z \to 0$$

and the short exact sequence

$$0 \to \pi^*\Omega_X \to \Omega_Z \to \Omega_{Z/X} \to 0$$

one can show that the canonical line bundle $\mathcal{K}_X$ of $X$ is $\mathcal{K}_X = \mathcal{O}(-n)$. Hence $c_1(Z) = -c_1(\mathcal{K}_X) = nc_1(\mathcal{O}(1))$. From the previous step

$$c_1(\mathcal{O}(1)) = \frac{d}{dt}[\omega_t] = \lim_{t\to\infty} \frac{1}{t}[\omega_t] \Longrightarrow TX \text{ is nef} \qquad \blacksquare$$

# The case of Kähler quotients

**Theorem (A.'21)**

Let $X$ be a Kähler quotient. Assume that the Feix–Kaledin metric on $T^*X$ is **complete**. Then

- The moment map $\mu \colon T^*X \to \mathbb{R}$ is proper.
- The twisted cotangent bundle $(T^*X)_{tw}$ is **affine** (and not only Stein).
- The tangent bundle $TX$ is **big and nef.**

**Proof:** For the first two assertions see [A.'21]. The fact that $TX$ is **nef** follows form the first one, while the fact that $TX$ is **big** follows from the second.

# What about the opposite direction?

**Theorem (A., unpublished)**

Let $X$ be a projective manifold whose tangent bundle $TX$ is nef and big. Then the twisted cotangent bundle $(T^*X)_{tw}$ is an affine variety.

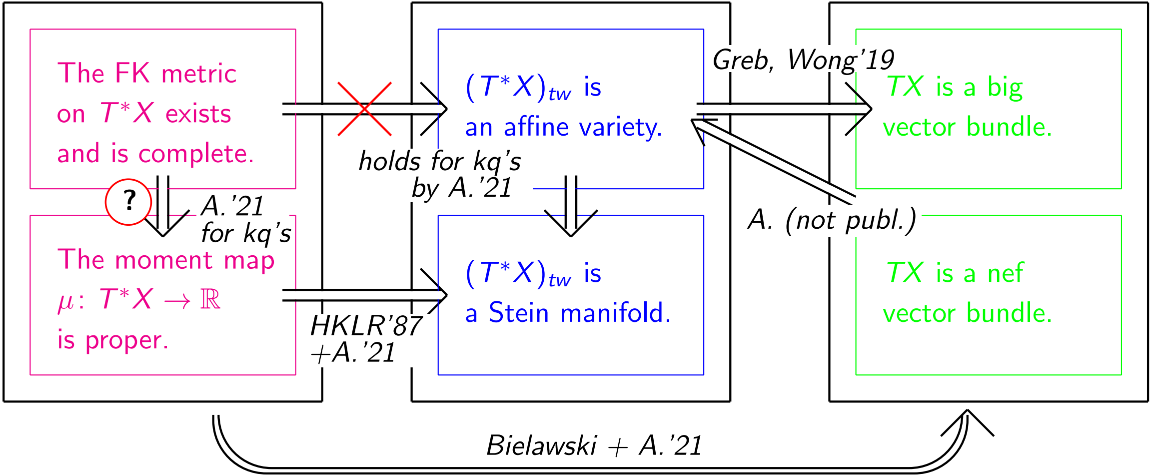
**Sketch of the proof: Step 1** Consider the extension $0 \to \Omega_X \to E \to \mathcal{O}_X \to 0$. We compute that $\mathcal{K}_{\mathbb{P}(E^*)} = \mathcal{O}_{\mathbb{P}(E^*)}(-n-1)$.

**Step 2:** We use Kawamata–Shokurov theorem to conclude that $\mathcal{O}_{\mathbb{P}(E^*)}(r)$ is generated by global sections for $r$ big enough.

**Step 3:** We can realize $(T^*X)_{tw}$ as the complement of the divisor $D := \mathbb{P}(TX) \subset \mathbb{P}(E^*)$. Next, we show that $(T^*X)_{tw}$ does not contain compact curves.

**Step 4:** Let $D \subset Y$ be an effective divisor whose multiple is generated by global sections. Assume that $X \setminus D$ does not contain compact curves. Then $X \setminus D$ is affine.

# The whole picture



# Thanks for your attention! (A nice picture)


Sir Hamilton, the discoverer of quaternions, shows the quaternionic relations to his wife.