# Non-algebraicity of hypercomplex nilmanifolds

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# Overview of the talk

1. Plenary session
   - Complex manifolds
   - Nilmanifolds
   - Main results

2. Parallel session
   - Examples
   - Algebraic dimension
   - Subvarieties of hypercomplex nilmanifolds

# Complex structures on manifolds

Let $V$ be a vector space, $I \in \mathrm{End}(V)$, $I^2 = -1$ an **almost complex structure**. Consider the eigenvalue decomposition

$$V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$$

$$Ix = \sqrt{-1}x \text{ for } x \in V^{1,0}, Ix = -\sqrt{-1}x \text{ for } x \in V^{0,1}$$

Consider a smooth manifold $X$ equipped with an **almost complex structure** $I \in \mathrm{End}(TX)$. Then one has the decomposition

$$TX \otimes \mathbb{C} = T^{1,0}X \oplus T^{0,1}X$$

**Definition**

An almost complex structure $I$ on $X$ is called **integrable** or just a **complex structure** if

$$[T^{1,0}X, T^{1,0}X] \subset T^{1,0}X$$

# Newlander-Nirenberg theorem

## Definition

A smooth map $f : X \to Y$ of almost complex manifolds is called holomorphic if $\forall x \in X, v \in T_x X$

$$D_x f(Iv) = I \cdot D_x f(v)$$

## Newlander-Nirenberg theorem

Let $X$ be a smooth manifold with an almost complex structure $I$. Then $I$ is integrable if and only if $X$ is locally biholomorphic to an open ball in $\mathbb{C}^n$.

## Remark

The integrability condition $[T^{1,0}X, T^{1,0}X] \subset T^{1,0}X$ is equivalent to the vanishing of the **Nijenhuis tensor** $N$

$$N(v,u) = [v,u] + I([v,Iu] + [Iv,u]) - [Iv,Iu] = 0 \quad \forall \text{ vector fields } v, u$$

# Kähler manifolds

Let $V$ be a vector space with a complex structure $I$. Let $g$ be an **Hermitian metric** on $V$ i.e. a Euclidean metric on $V$ s.t.

$$g(Iv, Iu) = g(v, u)$$

Then $\omega(v, u) := g(Iv, u)$ is a skew-symmetric 2-form. Let $X$ be a complex manifold, $g$ a Hermitian metric on $X$, $\omega(v, u) := g(Iv, u)$.

### Definition

A complex manifold $X$ is called **Kähler** if $d\omega = 0$.

### Examples

Examples of Kähler manifolds

1. $\mathbb{C}P^n$, all smooth projective varieties $X \subset \mathbb{C}P^n$ (but not all Kähler ones are projective!);
2. Complex tori $\mathbb{C}^n/\Lambda$;
3. A complex submanifold of a Kähler manifold is Kähler.

# Nilpotent Lie algebras and nilmanifolds

Let $\mathfrak{g}$ be a Lie algebra. Define $\mathfrak{g}_1 := [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}_i := [\mathfrak{g}, \mathfrak{g}_{i-1}]$. Then $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 ...$ is called the **lower central series** of $\mathfrak{g}$.

**Definition**

A Lie algebra $\mathfrak{g}$ is called **nilpotent** if $\mathfrak{g}_k = 0$ for some $k$.

If $k$ is the minimal number such that $\mathfrak{g}_k = 0$ then the Lie algebra $\mathfrak{g}$ is called $k$**-step nilpotent**.

**Definition**

Let $G$ be a nilpotent Lie group and $\Gamma \subset G$ a cocompact lattice i.e. a discrete subgroup s.t. $\Gamma \backslash G$ is compact. Then $X := \Gamma \backslash G$ is called a **nilmanifold**.

**Nota bene:** in the definition of a nilmanifold we take the quotient by the **left** action of $\Gamma$. The group $G$ acts on $X = \Gamma \backslash G$ **on the right**.

# Complex structures on Lie groups

Let $G$ be a Lie group, $\mathfrak{g}$ its Lie algebra. Every $v \in \mathfrak{g}$ defines a **left-invariant vector field** $\tilde{v}$ on $G$. The map $v \mapsto \tilde{v}$ is an isomorphism of Lie algebras.

Let $L \in \mathrm{End}(\mathfrak{g})$ be an almost complex structure, $\mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$. It induces a **left-invariant almost complex structure** $\tilde{L}$ on $G$.

**Fact**

The almost complex structure $\tilde{L}$ on $G$ is integrable iff $\mathfrak{g}^{1,0}$ is **a Lie subalgebra of** $\mathfrak{g} \otimes \mathbb{C}$**.**

**Proof:** First, left-invariant vector fields on $G$ generate the space of smooth vector fields on $G$ over the smooth functions. Hence we can check the integrability condition just for them.

# Complex nilmanifolds

## Nota Bene

A left-invariant complex structure $\tilde{L}$ on $G$ makes $G$ into a complex manifold **but in general not into a complex Lie group.** (An example is postponed until the parallel session)

A Lie group $G$ is a complex Lie group iff $\mathfrak{g}^{1,0}$ **is an ideal of** $\mathfrak{g} \otimes \mathbb{C}$**.**

## Definition

Let $G$ be a nilpotent Lie group with a **left-invariant** complex structure $L$ and $\Gamma \subset G$ a cocompact lattice. Then $X := \Gamma \backslash G$ is called a **complex nilmanifold**.

The **right** action of $G$ on $X = \Gamma \backslash G$ need **not** preserve the complex structure.

# Iwasawa manifold

The **complex Heisenberg group** of dimension 3 is

$$H = \left\{ \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} \right\} \quad z_1, z_2, z_3 \in \mathbb{C}$$

An **Iwasawa manifold** is $\Gamma \backslash H$ where

$$\Gamma = \left\{ \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} \right\} \quad z_1, z_2, z_3 \in \mathbb{Z}[\sqrt{-1}]$$

## Remark

Iwasawa manifold is non-Kähler. Actually, **all complex nilmanifolds except of complex tori are non-Kähler.**

# Hypercomplex manifolds

Notation: $\mathbb{H}$ is the quaternion algebra, it is generated by $I, J, K$, $I^2 = J^2 = K^2 = -1, IJ = -JI = K$.

## Fact

An element $L \in \mathbb{H}$ satisfies $L^2 = -1$ iff $L = xI + yJ + zK$, $x^2 + y^2 + z^2 = 1$.

## Definition

A manifold $X$ is called **almost hypercomplex** if $\mathbb{H}$ acts on $TX$. It is called **hypercomplex** if every complex structure on $X$ induced from $\mathbb{H}$ is integrable.

## Definition

Let $G$ be a nilpotent Lie group with a **left-invariant** hypercomplex structure $(I, J, K)$ and $\Gamma \subset G$ a cocompact lattice. Then $X := \Gamma \backslash G$ is called a **hypercomplex nilmanifold**.

# Main theorems: preliminary version

**Notation:** "$\forall\forall L \in \mathbb{H}$" = "for all but a countable number of complex structures $L \in \mathbb{H}$."

Let $X$ be a hypercomplex manifold. We denote by $X_L$ the manifold $X$ considered as a complex manifold with a complex structure $L \in \mathbb{H}$.

**Theorem 1 (A.–Verbitsky)**

Let $X$ be a hypercomplex nilmanifold. Then $\forall\forall L \in \mathbb{H}$ the complex manifold $X_L$ is does not admit a non-trivial meromorphic map onto a Kähler manifold.

**Theorem 2 (A.–Verbitsky)**

Let $X$ be a hypercomplex nilmanifold **admitting an HKT-structure**. Then $\forall\forall L \in \mathbb{H}$ every complex subvariety of $X_L$ is hypercomplex. In particular, every complex subvariety of $X_L$ is even-dimensional.

# Kodaira surface. Part 1

Define

$$G = \left\{ g(z_1, z_2) := \begin{pmatrix} 1 & \bar{z}_1 & z_2 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} \right\} \subset GL(3, \mathbb{C})$$

$(z_1, z_2)$ are complex coordinates on $G$.
The left multiplication by $g(a_1, a_2)$ is given by

$$(z_1, z_2) \mapsto (z_1 + a_1, z_2 + \bar{a}_1 z_1 + a_2) \quad \textbf{It's holomorphic}$$

The right multiplication by $g(a_1, a_2)$ is given by

$$(z_1, z_2) \mapsto (z_1 + a_1, z_2 + a_1 \bar{z}_1 + a_2) \quad \textbf{It's not holomorphic!}$$

The group $G$ is **not a complex Lie group** but admits a left-invariant complex structure.

# Kodaira surface. Part 2

Define $\Gamma := GL(3, \mathbb{Z}[\sqrt{-1}]) \cap G$. Then the complex surface $X = \Gamma \backslash G$ is an example of a **Kodaira surface**. It is **not Kähler**. The map

$$\Gamma \backslash G = X \longrightarrow E = \mathbb{C}/\mathbb{Z}[\sqrt{-1}] \quad (z_1, z_2) \mapsto z_1$$

is a principal elliptic fibration over the elliptic curve $E = \mathbb{C}/\mathbb{Z}[\sqrt{-1}]$.

Kodaira surface **does not** admit a hypercomplex structure.

# "Doubling" construction. Part 1

Let $X$ be a manifold equipped with a flat torsion-free affine connection $\nabla\colon TX \to TX \otimes \Omega^1 X$.

$$[\nabla_v, \nabla_u] = \nabla_{[v,u]} \quad \text{(flat)}$$

$$\nabla_v u - \nabla_u v = [v, u] \quad \text{(torsion-free)}$$

Let $\pi\colon TX \to X$ denote the natural projection. $\nabla$ induces the decomposition $T_x(TX) = H_x \oplus V_x,\ V_x := \ker d\pi$ for any point $x \in TX$.

$$V_x \cong H_x \cong T_{\pi(x)}X \Longrightarrow T_x(TX) \cong T_{\pi(x)}X^{\oplus 2}$$

Define a complex structure $J$ on a manifold $TX$ as

$$J(v, u) := (-u, v) \quad (JV_x = H_x,\ JH_x = V_x)$$

**Fact**

In the assumptions above $J$ is an integrable complex structure on a manifold $TX$

# "Doubling" construction. Part 2

Assume that the monodromy of $\nabla$ preserves a lattice $\Lambda_x \subset T_x X$. Then $\exists$ a lattice $\Lambda \subset TX$ parallel wrt $\nabla$.

**Fact**

The manifold $TX/\Lambda$ is a complex manifold. It is called a **"doubling"** of $X$.

Assume now that $(X, I)$ is a complex manifold and $\nabla I = 0$. Then

$$T_x(\overline{T}X) = T_{\pi(x)}X \oplus \overline{T_{\pi(x)}X}$$

and

$$I(v,u) := (Iv, -Iu) \quad J(v,u) := (-u, v) \quad K(v,u) := (-Iu, -Iv)$$

is an almost hypercomplex structure on the manifold $\overline{T}X$ (and $\overline{T}X/\Lambda$ as well)

**Fact**

The constructed almost hypercomplex structure on $\overline{T}X$ is in fact hypercomplex.

# "Doubling" construction. Part 3

Let's start with a Lie group $G$ with a Lie algebra $\mathfrak{g}$. Left-invariant affine flat connections on $G$ are in one-to-one correspondence with Lie-algebra representations

$$\nabla : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}) \quad v \mapsto \nabla_v$$

Assume also that $\nabla$ is torsion-free.

We define a bracket on $T\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}$ as follows (the first $\mathfrak{g}$ is "horizontal", the second is "vertical")

$$[(x_1, y_1), (x_2, y_2)] = ([x_1, x_2], \nabla_{x_1} y_2 - \nabla_{x_2} y_1)$$

and a complex structure $J$ on $T\mathfrak{g}$ as $J(x, y) = (-y, x)$

**Fact**

The bracket $[-, -]$ makes $T\mathfrak{g}$ into a Lie algebra. The complex structure $J$ is integrable. The hypercomplex analogue of this fact also holds.

# How to measure non-algebraicity?

Let $X$ be a compact complex manifold, $K(X)$ the field of meromorphic functions on $X$

**Definition**

The **algebraic dimension** of $X$ is the transcedence degree of $K(X)$.

**Definition-Proposition**

Consider a projective variety $X^{alg}$ with a dominant rational map $r\colon X \to X^{alg}$. If $r^*\colon K(X^{alg}) \to K(X)$ is an isomorphism then $X^{alg}$ is called an **algebraic reduction of** $X$. An algebraic reduction exists and is unique up to a birational isomorphism.

If $X$ does not contain a divisor then it is of algebraic dimension zero. The opposite does not hold in general (though the opposite is true if $X$ is assumed to be a complex torus).

# Hyperkähler manifolds

Let $X$ be a hypercomplex manifold. Let $g$ be a **hyper-Hermitian metric** on $X$ i.e. Hermitian wrt every complex structure $L \in \mathbb{H}$. Define $\omega_L(x, y) := g(Lx, y)$.

### Definition

A hyper-Hermitian manifold $X$ is called **hyperkähler** if $\forall L \in \mathbb{H}$: $d\omega_L = 0$. A hyper-Hermitian manifold $X$ is called **HKT** if $\partial\Omega_I = 0$.

### Examples

Examples of compact hyperkähler manifolds:

1. Hypercomplex tori $\mathbb{H}^n/\Lambda$;
2. K3-surfaces, their Hilbert schemes of points, etc

Non-example: A doubling of a non-Kähler complex manifold (f.e. Kodaira surface)

# Hyperkähler manifolds are very non-algebraic

**Theorem (Fujiki'87)**

Let $X$ be a compact hyperkähler manifold. Then $\forall\forall L \in \mathbb{H}$ the complex manifold $X_L$ is of algebraic dimension zero.

**Definition**

Let $M$ be a hypercomplex manifold. A subvariety $M \subset X$ is called **trianalytic** if $M$ is complex analytic wrt every complex structure $L \in \mathbb{H}$.

**Theorem (Verbitsky'95)**

Let $X$ be a compact hyperkähler manifold. Then $\forall\forall L \in \mathbb{H}$ every complex subvariety of $X_L$ is trianalytic.

The second theorem implies the first one.

# What about hypercomplex manifolds?

Theorems of Fujiki and Verbitsky do **not** hold for hypercomplex manifolds in general.

## Examples

Let $X = \mathbb{H}^n/\lambda^{\mathbb{Z}}$, $\lambda \in \mathbb{R}_{>1}$. It is an example of a **Hopf manifold**. Then $\forall L \in \mathbb{H}$ there is an isotrivial elliptic fibration $X \longrightarrow \mathbb{C}P^{2n-1}$, hence $\forall L \in \mathbb{H}$, $X_L$ is of algebraic dimension $2n-1$ and contains an elliptic curve.

## Definition-Proposition

Let $X$ be a hypercomplex manifold. Then $\exists!$ torsion-free connection $\nabla$ preserving the hypercomplex structure. It is called the **Obata connection**. If $Hol(\nabla)$ is contained in $SL(n,\mathbb{H})$ then $X$ is called an $SL(n,\mathbb{H})$-**manifold**.

## Theorem (Soldatenkov–Verbitsky'12)

Let $X$ be an $SL(n,\mathbb{H})$-manifold admitting an HKT-metric. Then $\forall\forall X \in \mathbb{H}$ the manifold $X_L$ does not contain divisors and every complex subvariety of $X_L$ of codimension 2 is trianalytic.

# Main theorems

We prove that the theorems of Fujiki, Verbitsky **do** hold (in some sense) for **hypercomplex nilmanifolds**

**Theorem 1 (A.–Verbitsky)**

Let $X$ be a hypercomplex nilmanifold. Then $\forall\forall L \in \mathbb{H}$ the algebraic dimension of $X_L$ is zero.

**Theorem 2 (A.–Verbitsky), preliminary version**

Let $X$ be a hypercomplex nilmanifold **admitting an HKT-structure**. Then $\forall\forall L \in \mathbb{H}$ **every** complex subvariety of $X_L$ is trianalytic.

Hypercomplex nilmanifolds are always $SL(n, \mathbb{H})$-manifolds (Barberis–Dotti–Verbitsky'09).

# Albanese variety

Let $X = \Gamma\backslash G$ be a complex nilmanifold. Then $\Lambda := \log(\Gamma)$ is a lattice in $\mathfrak{g}$ (Mal'cev'51). Consider the *minimal rational L-invariant subspace of* $\mathfrak{g}$ *containing* $[\mathfrak{g},\mathfrak{g}]$. Denote it by $[\mathfrak{g},\mathfrak{g}]_{\mathbb{Q},L}$. The quotient map $\mathfrak{g} \to \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]_{\mathbb{Q},L}$ induces a holomorphic map

$$r\colon \Gamma\backslash G = X \longrightarrow T := (\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]_{\mathbb{Q},L})/\Lambda$$

## Definition

The torus $T$ defined above is called the **Albanese variety** of a nilmanifold $X = \Gamma\backslash G$ and the map $r\colon X \to T$ is called the **Albanese map** of $X$.

## Theorem (Fino–Grantcharov–Verbitsky'18)

Let $X = \Gamma\backslash G$ be a complex nilmanifold and $T$ its Albanese variety. Then every meromorphic map from $X$ to a Kähler manifold is uniquely factorized through the Albanese map $r\colon X \to T$.

The theorem implies that **algebraic dimensions of $X$ and $T$ coincide.**

# Hypercomplex Albanese variety

Let now $X = \Gamma\backslash G$ be a hypercomplex nilmanifold. Consider the *minimal rational* $\mathbb{H}$*-invariant subspace of* $\mathfrak{g}$ *containing* $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$. Denote it by $[\mathfrak{g}, \mathfrak{g}]_{\mathbb{Q},\mathbb{H}}$. Similarly, we obtain a map

$$R\colon \Gamma\backslash G = X \longrightarrow T_{\mathbb{H}} := (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]_{\mathbb{Q},\mathbb{H}})/\Lambda$$

which preserves the hypercomplex structure.

## Definition

The torus $T_{\mathbb{H}}$ defined above is called the **hypercomplex Albanese variety** of a nilmanifold $X = \Gamma\backslash G$ and the map $R\colon X \to T_{\mathbb{H}}$ is called the **hypercomplex Albanese map** of $X$.

## Lemma

Let $X = \Gamma\backslash G$ be a hypercomplex nilmanifold. Then $\forall L \in \mathbb{H}$ the hypercomplex Albanese map is the (complex) Albanese map of $X_L$.

# Hypercomplex Albanese vs Complex Albanese

**Lemma**

Let $X = \Gamma \backslash G$ be a hypercomplex nilmanifold. Then $\forall\forall L \in \mathbb{H}$ the hypercomplex Albanese map is the (complex) Albanese map of $X_L$.

**Proof:**

**Observation**

Let $V$ be an $\mathbb{H}$-vector space with a rational structure. Then $\forall\forall L \in \mathbb{H}$ every rational $L$-invariant space is $\mathbb{H}$-invariant.

Indeed, if an $L$-invariant space is invariant wrt $L' \neq \pm L$ then it is $\mathbb{H}$-invariant. Hence the set of complex structures $L \in \mathbb{H}$ s.t. there exist an $L$- but not $\mathbb{H}$-invariant rational subspace of $V$ is countable.

By applying the observation to $V = \mathfrak{g}$ we obtain that $\forall\forall L \in \mathbb{H}$: $[\mathfrak{g}, \mathfrak{g}]_{\mathbb{Q},L} = [\mathfrak{g}, \mathfrak{g}]_{\mathbb{Q},\mathbb{H}}$.

# Proof of the First theorem

**Theorem 1 (A.–Verbitsky)**

Let $X$ be a hypercomplex nilmanifold. Then $\forall\forall L \in \mathbb{H}$ the algebraic dimension of $X_L$ is zero.

**Proof.** Let $T$ be the hypercomplex Albanese variety. We saw in the previous slides that $\forall\forall L \in \mathbb{H}$ we have

$$\operatorname{alg\,dim} X_L = \operatorname{alg\,dim} T_L$$

The torus $T$ is hyperkähler, hence $\forall\forall L \in \mathbb{H}$ the algebraic dimension of $T_L$ is zero.

# Abelian complex structures

Let $\mathfrak{g}$ be a Lie algebra with a complex structure $L$.

**Definition**

The complex structure $L$ is called **abelian** if $\mathfrak{g}^{1,0}$ is an abelian subalgebra of $\mathfrak{g} \otimes \mathbb{C}$. Equivalently,

$$\forall x, y \in \mathfrak{g}\colon [Lx, y] = -[x, Ly]$$

Suppose that $\mathfrak{g}$ admits a hypercomplex structure $(I, J, K)$. Then $J, K$ are abelian whenever the complex structure $I$ is abelian (Dotti–Fino'03). If one (hence any) complex structure $L \in \mathbb{H}$ is abelian then the hypercomplex structure on $\mathfrak{g}$ is called **abelian**.

**Theorem (Dotti–Fino'01, Barberis–Dotti–Verbitsky'09, also Fino–Grantcharov'03)**

Let $X$ be a hypercomplex nilmanifold. Then $X$ admits an HKT-metric iff the hypercomplex structure is abelian

# Locally homogeneous submanifolds

Let $G$ be a Lie group. We trivialize $TG$ by **left** multiplications. If $G$ is a Lie group with a **left-invariant** complex structure which is not right-invariant then this trivialization is complex but **not holomorphic** because

## Nota bene

The flow of a **left-invariant** vector field $\xi$ is the multiplication **on the right** by $\exp(\xi)$.

This trivialization of $TG$ descends to a, generally speaking, **non-holomorphic** complex trivialization of $TX$ where $X = \Gamma \backslash G$.

## Definition

A submanifold $M \subset X$ is called **locally homogeneous** if $\forall x \in M$ the tangent space $T_x M$ is identified with a fixed subspace $\mathfrak{h} \subset \mathfrak{g}$ via the trivialization of $TX$ above.

The subspace $\mathfrak{h} \subset \mathfrak{g}$ is automatically a rational subalgebra.

# Second theorem. Step 1: case of tori

**Theorem 2 (A.–Verbitsky): Final version**

Let $X = \Gamma \backslash G$ be an abelian hypercomplex nilmanifold. Then $\forall\forall L \in \mathbb{H}$ every complex subvariety of $X_L$ is a trianalytic locally homogeneous submanifold of $X$.

**Sketch of the proof. Step 1.** The claim is known to hold for a hypercomplex torus $T$. Indeed, $\forall\forall L \in \mathbb{H}$ every complex subvariety of $T_L$ is trianalytic. A trianalytic subvariety of a hyperkähler manifold is totally geodesic (Verbitsky'96). Hence every trianalytic subvariety of $T$ is a subtorus.

# Step 2: Principal toric fibration

**Lemma**

Let $\mathfrak{g}$ be a Lie algebra with an abelian Lie structure $L$. Then its center $\mathfrak{z}$ is $L$-invariant.

**Proposition**

Let $X = \Gamma \backslash G$ be an abelian complex nilmanifold. Let $Z$ denote the center of $G$. Then the map

$$\pi \colon \Gamma \backslash G = X \longrightarrow Y := \Gamma \backslash G / Z$$

is a holomorphic principal toric fibration with a fiber $T = Z/(\Gamma \cap Z)$.

**Proof:** The right action of $Z$ on $G$ is holomorphic because it coincides with the right action. Hence the right action of $Z/(\Gamma \cap Z)$ on $X = \Gamma \backslash G$ is also holomorphic.

# Step 3: Induction step. The reduction to the case of a multisection

Let $M \subset X_L$ be a complex subvariety. Consider the principal fibration

$$\pi \colon \Gamma \backslash G = X \longrightarrow Y := \Gamma \backslash G / Z$$

**By induction hypothesis both $\pi(M)$ and the fibers of $\pi|_M$ are trianalytic locally homogeneous submanifolds**. One can use this observation to show that

**Fact**

It is actually enough to assume that $M$ is a multisection of $\pi \colon X \to Y$ i.e. the map $\pi|_M \colon M \to Y$ is surjective and generically finite.

# Step 4: Multisections are étale

Consider the principal fibration $\pi\colon \Gamma\backslash G = X \longrightarrow Y := \Gamma\backslash G/Z$. Let $M \subset X_L$ be a multisection of $\pi$. Consider the Stein factorization of the map $\pi|_M\colon Y$

$$M \xrightarrow{\pi_1} Y' \xrightarrow{\pi_2} Y$$

The map $\pi_1$ is a **birational transformation** and the map $\pi_2$ is **finite**.

## Observation 1

The **branch locus** of $\pi_2$ is a divisor in $Y \implies$ the map $\pi_2$ is **étale** ($Y$ has no divisors by the induction hypothesis).

## Observation 2

The **exceptional locus** $E \subset M$ of $\pi_1$ is a divisor in $M \implies$ if non-empty, $E$ has odd dimension $\implies$ the map $\pi\colon M \to Y$ has an odd-dimensional fiber. But $\forall y \in Y$ all the subvarieties of $\pi^{-1}(y)$ are trianalytic.

**Hence $\pi|_M\colon M \to Y$ is finite étale.**

# The end of the proof

Consider the principal $T$-fiber bundle $\pi\colon \Gamma\backslash G = X \longrightarrow Y := \Gamma\backslash G/Z$. Define $T_k := T/\{k\text{-}torsion\}$. Consider the associated principal $T_k$-bundle $X_k := X \times T_k/T \to Y$. The manifold $X_k$ is a nilmanifold as well.

## Observation 3

A multisection $M \subset X_L$ of degree $k$ gives rise to a section of $X_k \to Y$. Hence $X_k = Y \times T_k$

## Observation 4

By [Maltsev'51] any decomposition $X_k = Y \times T_k$ comes from a Lie algebra decomposition $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{h}$. Here $\mathfrak{z}$ is the center of $\mathfrak{g}$. The existence of such a decomposition contradicts the nilpotency assumption on $\mathfrak{g}$.

# Thanks for your attention! (A nice picture)


Sir Hamilton, the discoverer of quaternions, shows the quaternionic relations to his wife.