SHAFAREVICH–TATE GROUPS OF HOLOMORPHIC LAGRANGIAN FIBRATIONS

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Abstract

Consider a Lagrangian fibration $\pi\colon X \to \mathbb{P}^n$ on a compact hyperkähler manifold X. There are two ways to construct a holomorphic family of deformations of π over $\mathbb{C}$. The first one is known under the name **Shafarevich–Tate family** while the second one is the **degenerate twistor family** constructed by Verbitsky. We show that both families coincide. We prove that for a very general X all members of the Shafarevich–Tate family are Kähler. There is a related notion of the **Shafarevich–Tate group** $Ш$ associated to a Lagrangian fibration. The connected component of unity of $Ш$ can be shown to be isomorphic to $\mathbb{C}/\Lambda$ where Λ is a finitely generated subgroup of $\mathbb{C}$ and $\mathbb{C}$ is thought of as the base of the Shafarevich–Tate family. We show that for a very general X, projective deformations in the Shafarevich–Tate family correspond to the torsion points in $Ш^0$. This poster is based on the paper **[AR]**.

Main definitions

DEFINITION 1: A compact connected Kähler manifold X is called a **hyperkähler manifold** if it is simply connected and $H^0(X, \Omega^2_X)$ is generated by a holomorphic symplectic form σ.

DEFINITION 2: Let X be a hyperkähler manifold of complex dimension $2n$. A map $\pi\colon X \to \mathbb{P}^n$ is called a **Lagrangian fibration** if it is surjective with connected fibers and the restriction of σ to every smooth fiber vanishes.

DEFINITION 3: The **Shafarevich–Tate group** $Ш$ of a Lagrangian fibration is the abelian group $H^1(\mathbb{P}^n, Aut^0_{X/\mathbb{P}^n})$ where $Aut^0_{X/\mathbb{P}^n}$ is the connected component of unity of the sheaf of vertical automorphisms of X over $\mathbb{P}^n$.

Shafarevich–Tate twists

Choose an affine open cover $\mathbb{P}^n = \bigcup U_i$. Denote $U_i \cap U_j$ by U_{ij}. A class $s \in Ш$ can be represented by a Čech cocycle with coefficients in $Aut^0_{X/\mathbb{P}^n}$. That is to say, for every pair of indices i, j we are given an automorphism s_{ij} of $\pi^{-1}(U_{ij})$ that commutes with π. Let us reglue the manifolds $\pi^{-1}(U_i)$ by the automorphisms s_{ij}. We obtain a complex manifold X^s equipped with a holomorphic projection $\pi^s\colon X^s \to \mathbb{P}^n$. The fibers of π^s are isomorphic to the fibers of $\pi\colon X \to \mathbb{P}^n$. This new manifold is called the **Shafarevich–Tate twist** of X by $s \in Ш$.

Degenerate twistor deformations

Let α be a closed $(1,1)$-form on $\mathbb{P}^n$. There exists a unique complex structure I_α on X such that the form $\sigma + \pi^*\alpha$ is a holomorphic 2-form on (X, I_α). The manifold (X, I_α) is called the **degenerate twistor deformation** of X with respect to the form α.

The two constructions give the same result!

The holomorphic symplectic form σ on X induces an isomorphism $\sigma\colon \Omega^1_X \xrightarrow{\sim} T_X$. The map σ sends $\pi^*\Omega^1_{\mathbb{P}^n} \subset \Omega^1_X$ into $T_{X/\mathbb{P}^n} := \ker(d\pi\colon T_X \to \pi^*T_{\mathbb{P}^n})$. Assume that the fibration $\pi\colon X \to \mathbb{P}^n$ does not have multiple fibers. One can show that **the sheaves $\Omega^1_{\mathbb{P}^n}$ and $\pi_*T_{X/\mathbb{P}^n}$ are isomorphic.**

The exponential map $\pi_*T_{X/\mathbb{P}^n} \to Aut^0_{X/\mathbb{P}^n}$ induces a map $H^1(\mathbb{P}^n, \pi_*T_{X/\mathbb{P}^n}) \to Ш$. Since the sheaf $\pi_*T_{X/\mathbb{P}^n}$ is isomorphic to $\Omega^1_{\mathbb{P}^n}$ we obtain a natural map

$$H^{1,1}(\mathbb{P}^n) \cong \mathbb{C} \to Ш \qquad (1)$$

The kernel and cokernel of this map are finitely generated abelian groups. Define $Ш^0$ to be the image of $\mathbb{C}$ in $Ш$. We will refer to $Ш^0$ as the **connected component of unity of $Ш$**. Let Γ denote the kernel of the exponential map $\pi_*T_{X/\mathbb{P}^n} \to Aut^0_{X/\mathbb{P}^n}$. **The kernel of the map $\mathbb{C} \to Ш$ can be identified with the image of the group $H^1(\mathbb{P}^n, \Gamma)$ in $H^1(\mathbb{P}^n, \pi_*T_{X/\mathbb{P}^n}) \cong \mathbb{C}$.**

THEOREM A [AR]. Pick a class $s \in H^1(\pi_*T_{X/\mathbb{P}^n})$. Consider the twist X^s of $\pi\colon X \to \mathbb{P}^n$ by the image of s in $Ш$. Let α be a closed $(1,1)$-form on $\mathbb{P}^n$ representing the same class in $H^{1,1}(\mathbb{P}^n) \cong H^1(\mathbb{P}^n, \pi_*T_{X/\mathbb{P}^n})$ as s. Then the complex manifolds X^s and (X, I_α) are isomorphic as fibrations over $\mathbb{P}^n$.

Some important isomorphisms of sheaves

Let ω be a Kähler form on X. The contraction of ω with holomorphic vector fields defines a map

$$\tilde{\omega}\colon \pi_*T_{X/\mathbb{P}^n} \to R^1\pi_*\mathcal{O}_X$$

For every small open $U \subset \mathbb{P}^n$ the map $\tilde{\omega}$ sends a holomorphic vector field $v \in \pi_*T_{X/\mathbb{P}^n}(U)$ to the class of the $(0,1)$-form $\iota_v\omega$ in $R^1\pi_*\mathcal{O}_X(U) = H^{0,1}(\pi^{-1}(U))$. **The map $\tilde{\omega}$ is an isomorphism** by **[Mats]**. Therefore,

$$\Omega^1_{\mathbb{P}^n} \cong \pi_*T_{X/\mathbb{P}^n} \cong R^1\pi_*\mathcal{O}_X. \qquad (2)$$

The embedding of sheaves $\mathbb{Z}_X \subset \mathcal{O}_X$ induces an embedding $R^1\pi_*\mathbb{Z}_X \to R^1\pi_*\mathcal{O}_X$.

PROPOSITION 1 [AR]. The isomorphism $\tilde{\omega}^{-1}\colon R^1\pi_*\mathcal{O}_X \to \pi_*T_{X/\mathbb{P}^n}$ sends the subsheaf $R^1\pi_*\mathbb{Q}_X$ isomorphically to $\Gamma \otimes \mathbb{Q}$.

The group $Ш^0$

Leray spectral sequence of the fibration $\pi\colon X \to \mathbb{P}^n$ enable us to compute cohomology groups of the sheaves $R^1\pi_*\mathbb{Z}_X$ and $R^1\pi_*\mathcal{O}_X$ in terms of $H^2(X)$:

$$H^1(\mathbb{P}^n, R^1\pi_*\mathcal{O}_X) \cong H^{0,2}(X)$$
$$H^1(\mathbb{P}^n, R^1\pi_*\mathbb{Z}_X) \cong W/\pi^*[H]$$

where $[H] \in H^2(\mathbb{P}^n, \mathbb{Z})$ is the class of a hyperplane section, $\pi^*[H] \in H^2(X, \mathbb{Z})$ its pullback to X and

$$W := \bigcap_{b \in \mathbb{P}^n} \ker(H^2(X, \mathbb{Z}) \to H^2(\pi^{-1}(b), \mathbb{Z}))$$

Therefore

$$H^1(\mathbb{P}^n, \pi_*T_{X/\mathbb{P}^n}) \cong H^{0,2}(X) \qquad (3)$$
$$H^1(\mathbb{P}^n, \Gamma) \otimes \mathbb{Q} \cong W \otimes \mathbb{Q}/\pi^*[H] \qquad (4)$$

We can derive the following description of $Ш^0$.

THEOREM B [AR]. The connected component of unity $Ш^0$ of the Shafarevich–Tate group $Ш$ is a quotient of the group

$$H^{0,2}(X)/W$$

by a finite subgroup.

Kählerness and projectivity

DEFINITION 4 Let X be a projective hyperkähler manifold. It is called **M-special [Mar]** if the subspace $H^{2,0}(X) + H^{0,2}(X)$ of $H^2(X, \mathbb{C})$ contains a rational class.

A very general hyperkähler manifold is not M-special.

THEOREM C [AR]. Let $\pi\colon X \to \mathbb{P}^n$ be a Lagrangian fibration on a not M-special projective hyperkähler manifold X. Then every Shafarevich–Tate twist X^s of X by an element $s \in Ш^0$ is a hyperkähler manifold, in particular it is Kähler.

SKETCH OF THE PROOF: One can show that **a projective hyperkähler manifold X is not M-special if and only if the image of the group W in $H^{2,0}(X)$ is dense.** By **Theorem B** this is equivalent to saying that the image of $H^1(\mathbb{P}^n, \Gamma)$ in $H^1(\mathbb{P}^n, \pi_*T_{X/\mathbb{P}^n}) \cong \mathbb{C}$ is dense. Consider the set of $s \in H^1(\mathbb{P}^n, \pi_*T_{X/\mathbb{P}^n}) \cong \mathbb{C}$ such that the twist X^s of X by the image of s in $Ш$ is a Kähler manifold. **This set is open, non-empty and invariant with respect to a dense subgroup. Hence it is the whole set $\mathbb{C}$.**

We finish by stating which Shafarevich–Tate twists of X are projective manifolds.

THEOREM D [AR]. Let $\pi\colon X \to \mathbb{P}^n$ be a Lagrangian fibration on a not M-special projective hyperkähler manifold X. Then the set of $s \in Ш^0$ such that the twist X^s is a projective manifold forms a torsor over the group of torsion points of $Ш^0$.

References

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